Chapter 3

Fluid mechanics of droplets

In this chapter the principles of fluid dynamics will be used to describe the trajectory of the droplet under the influence of drag and gravitation force.

3.1 Trajectories of fluid particles

For a small particle in an ideal fluid, within region $D \subseteq \mathbb{R}^2$ or $D \subseteq \mathbb{R}^3$, the position in $D$ is

$$x = (x, y, z) \in D$$

and the velocity field

$$v(x, t) = (u, v, w)$$

(Malham, 2010).

The velocity at time $t$ is given by the simultaneous solution of the three coupled ordinary differential equations
\[
\frac{dx}{dt} = u(x(t), y(t), z(t), t) \\
\frac{dy}{dt} = v(x(t), y(t), z(t), t) \\
\frac{dz}{dt} = w(x(t), y(t), z(t), t)
\]
(Malham, 2010).

In vector notation

\[
\frac{d}{dt} \mathbf{x}(t) = \mathbf{v}(\mathbf{x}(t), t)
\]
(Malham, 2010).

The solution of this ordinary differential equation gives the trajectory or path of a moving fluid particle. In order to solve this equation the velocity function in space and time must be known.

The acceleration of the particle is

\[
\frac{d^2}{dt^2} \mathbf{x}(t) = \frac{d}{dt} \mathbf{v}(\mathbf{x}(t), t) \\
= \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t} \\
= [\frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dz}{dt} \frac{\partial}{\partial z}] \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \\
= \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}
\]
(Malham, 2010).

### 3.2 Non-dimensional fluid flow properties

For a spherical droplet the Weber, Reynolds and Froude numbers are important dimensionless quantities respectively given by
\[ W e = \frac{\rho v^2 D_0}{\gamma}, \]  
(3.1)  
\[ Re = \frac{\rho D_0 v}{\mu}, \]  
and  
\[ Fr = \frac{v^2}{g D_0}, \]  
with \( \rho \) the density, \( v \) the velocity, \( D_0 \) the diameter of the droplet, \( \gamma \) the surface tension and \( \mu \) the viscosity (Sellier, Baechtel & Taylor, 2010).

The Weber number is the ratio of fluid inertia to surface tension, the Reynolds number is the ratio of inertia force to viscous force and the Froude number is the ratio of a body’s inertia to gravitational force.

If the Weber number is small, then the surface tension is greater than the kinetic energy of spreading and if the Weber number is large, the kinetic energy of spreading is larger than the surface tension. If the Weber number is greater than the square root of the Reynolds number, the diameter of a droplet can be determined by

\[ D_0 = \frac{2D_1}{C_d Re^{\frac{3}{2}}}, \]  
(3.2)  
with \( D_1 \) the diameter of the stain and \( C_d \) some experimental constant (Adam, 2010; Hulse-Smith, Mehdizadeh & Chandra, 2005; Knock & Davison, 2007). This formula has been experimentally proven to hold for any Reynolds number and in the case of blood \( C_d = 1.11 \). This formula is also very useful for forensic experts at a crime scene where they only have the diameter of the stain to determine the diameter of the droplet.

The spines of a droplet, as illustrated in figure 3.1, are the parts of the droplet outside of the ellipse forming points.

Thus in figure 3.1 there are thirteen spines.

The number of spines of a droplet can be determined experimentally in terms of the Weber number

\[ N = 1.14 C_n \sqrt{We} \]  
(3.3)
with $C_n$ an experimental constant (Adam, 2010; Hulse-Smith et al., 2005; Knock & Davison, 2007).

Using the definition of the Reynolds number, equation 3.2 can be rewritten as

$$D_0 = \left( \frac{\mu (2D_1)^4}{\rho v (C_d)^4} \right)^{\frac{1}{5}}.$$

Rewrite equation 3.3 in terms of the Weber number

$$We = \left( \frac{N}{1.14C_n} \right)^2 \quad (3.4)$$

Now equate equations 3.1 and 3.4

$$\left( \frac{N}{1.14C_n} \right)^2 = \frac{\rho v^2 D_0}{\gamma},$$

which results in

$$\frac{\gamma}{\rho D_0} \left( \frac{N}{1.14C_n} \right)^2 = v^2.$$
3.3 Laminar and turbulent flow

The flow of air around a droplet is turbulent for high Reynolds numbers, while it is laminar for low Reynolds number (Holterman, 2003).

If a spherical droplet smaller than 5 mm falls through the air, the flow around the droplet is laminar. Laminar flow is the flow of a fluid in layers of laminae (Van Boxel, 1997). For laminar flow there are only two forces acting on a droplet, namely the force due to the gravitation acting on the droplet

\[
F_g = mg = \left(\frac{4\pi}{3}\rho \left(\frac{D_0}{2}\right)^3\right) g = \frac{g\rho\pi D_0^3}{6}
\]

and the drag force \( F_d \) as illustrated in figure 3.2.

![Figure 3.2: Forces on a droplet (Cecchetto & Heidrich, 2011)](image)

The drag force on a spherical droplet moving through the air depends on the radius of the sphere and the velocity of the droplet (Van Boxel, 1997). The total drag force is given by

\[
F_d = Avr + Bv^2r^2,
\]

where the first term is applicable for laminar flow with \( A = 6\pi\mu \) and the second term is applicable for turbulent flow. For turbulent flow the layers or
laminas intersect. The Reynolds number is the ratio between laminar and turbulent forces for the transition from laminar and turbulence flow. The friction increases due to turbulence with a factor

$$C_t = 1 + 0.16 Re^{\frac{2}{3}}$$

(Van Boxel, 1997).

For laminar flow the drag force is

$$F_d = 6\pi \mu vr$$

$$= 3\pi \mu D_0 v,$$

(3.5)

with $D_0 = 2r$, $D_0$ the diameter of the droplet, $r$ the radius of the droplet, $v$ velocity and $\mu$ viscosity (Shearer & Hudson, 2011; Van Boxel, 1997).

A general expression for the drag force in flow is given by

$$F_d = \frac{d\rho Av^2}{2}$$

$$= \frac{d\rho \pi D_0^2 v^2}{8},$$

(3.6)

where $A$ refers to the cross-sectional area (Holterman, 2003).

For a low Reynolds number the general drag force can be equated to the laminar flow drag force and therefore by equations 3.5 and 3.6 (Holterman, 2003).

$$\frac{d\rho \pi D_0^2 v^2}{8} = 3\pi \mu D_0 v.$$ 

(3.7)

Simplifying gives
\[ d = \frac{24\pi\mu D_0 v}{\rho \pi D_0^3 v^2} \]
\[ = \frac{24\mu}{\rho D_0 v} \]
\[ = \frac{24}{Re} \quad (3.8) \]


If the droplet reaches terminal velocity, the acceleration is zero. According to Newton’s second law the total force will be equal to zero and therefore

\[ F_g - F_d = 0, \]

so that

\[ \frac{g \rho \pi D_0^3}{6} = 3\pi \mu D_0 v. \]

We thus obtain

\[ v_t = \frac{\rho g D_0^2}{18\mu} \]

for the terminal velocity \( v_t \) with \( \mu \) the viscosity. This is known as Stokes’ law (Van Boxel, 1997).

For a droplet 0.1 mm and larger the flow will be turbulent and Stokes’ law will no longer be applicable.

### 3.4 Modelling of falling object

Newton’s second law states that

\[ \mathbf{F} = m\mathbf{a} \quad (3.9) \]
with \( F \) the sum of forces, \( m \) mass and \( a \) acceleration.

### 3.4.1 No drag

For no drag

\[
my'' = mg \tag{3.10}
\]

with \( y'' = a \) and \( g = 9.81 \text{ m} \cdot \text{s}^{-2} \).

### 3.4.2 Linear drag

The height of a spherical droplet can be described by

\[
y(t) = \frac{mg}{d} t + \frac{m}{d} \left[ \left( v_0 - \frac{mg}{d} \right) \left( 1 - e^{-\frac{dt}{m}} \right) \right] + y_0
\]

(Cecchetto & Heidrich, 2011).

**Proof**

Using Newton’s second law in the form

\[
my'' + dy' = mg
\]

the equation can be derived.

Define \( y' = v \), which is the velocity, then

\[
v' + \lambda v = g
\]

which is a non-homogeneous linear ordinary differential equation with \( \lambda = \frac{d}{m} \).
By integration

\[ \begin{align*}
    v &= e^{-\lambda t} \left( \int e^{\lambda t} g dt + c_1 \right) \\
    &= e^{-\lambda t} \left( \frac{g}{\lambda} e^{\lambda t} + c_1 \right) \\
    &= c_1 e^{-\lambda t} + \frac{g}{\lambda}
\end{align*} \]

With the assumption \( v(0) = v_0 \) the integration constant \( c_1 \) can be determined and yields

\[ c_1 = v_0 - \frac{g}{\lambda} \]

Then by integrating again

\[ \begin{align*}
    y &= \int \left[ \left( v_0 - \frac{g}{\lambda} \right) e^{-\lambda t} + \frac{g}{\lambda} \right] dt \\
    &= - \left( \frac{v_0}{\lambda} - \frac{g}{\lambda^2} \right) e^{-\lambda t} + \frac{g}{\lambda} t + c_2.
\end{align*} \]

With the assumption that \( y(0) = y_0 \) the integration constant can be determined and yields

\[ c_2 = y_0 + \frac{v_0}{\lambda} - \frac{g}{\lambda^2}. \]

Therefore

\[ \begin{align*}
    y &= - \left( \frac{v_0}{\lambda} - \frac{g}{\lambda^2} \right) e^{-\lambda t} + \frac{g}{\lambda} t + \frac{v_0}{\lambda} - \frac{g}{\lambda^2} + y_0 \\
    &= \frac{mg}{d} t + \frac{m}{d} \left( v_0 - \frac{mg}{d^2} \right) \left( 1 - e^{-\frac{d}{m} t} \right) + y_0
\end{align*} \] (3.11)

which is similar to the result found by Cecchetto & Heidrich (2011).

### 3.4.3 Quadratic drag

For quadratic drag the distance is

\[ y(t) = \frac{m}{d_1} \ln \left( \frac{e^{2t \sqrt{\frac{d_1}{m}}} - 1}{2} \right) \]
Proof

The equation can be derived as follows: Based on Newton’s second law the motion of the droplet will be governed by the equation

\[ F_g - F_d = ma \]

with \( a \) the acceleration if terminal velocity has not been reached. The drag and gravitation forces are given by

\[ F_d = \frac{d \rho A v^2}{2} = \frac{d \rho \pi D_0^2 v^2}{8} \]

and

\[ F_g = mg \]

respectively (Lynch & Lommatsch, 2011).

Therefore

\[ mg - \frac{d \rho \pi D_0^2 v^2}{8} = ma \]

\[ g - \frac{d \rho v^2 \pi D_0^2}{8m} = a. \]

The acceleration before terminal velocity is reached, is thus

\[ a(t) = \frac{dv}{dt} = g - \frac{d_1}{m} v^2 \]

with \( m \) the mass of the droplet and \( d_1 = \frac{\rho \pi D_0^2}{8} \) (Ng 2010).

Using separation of variables, the velocity can be determined as follows
\[
\left( g - \frac{d_1}{m} v^2 \right)^{-1} dv = dt
\]

which on simplification gives

\[
\frac{dv}{1 - \frac{d_1}{mg} v^2} = gdt.
\]

Let \( \frac{d_1}{mg} = \alpha^2 \) then

\[
\frac{dv}{1 - \alpha^2 v^2} = gdt.
\]

Using partial fractions

\[
\frac{1}{(1 - \alpha v)(1 + \alpha v)} = \frac{1}{2} \frac{1}{1 - \alpha v} + \frac{1}{2} \frac{1}{1 + \alpha v}
\]

then

\[
-\frac{1}{2\alpha} \ln(1 - \alpha v) + \frac{1}{2\alpha} \ln(1 + \alpha v) = tg + k
\]

\[
\frac{1}{2\alpha} \ln \left( \frac{1 + \alpha v}{1 - \alpha v} \right) = tg + k
\]

\[
\frac{1 + \alpha v}{1 - \alpha v} = e^{2tg\alpha}
\]

\[
1 + \alpha v = e^{2tg\alpha} (1 - \alpha v)
\]

\[
v (\alpha + ce^{2tg\alpha}) = ce^{2tg\alpha} - 1
\]

\[
v = \frac{1}{\alpha} \left( \frac{ce^{2tg\alpha} - 1}{1 + ce^{2tg\alpha}} \right)
\]

where \( c = 1 \) under the assumption that \( v(0) = 0 \).

The velocity found, corroborates with the results of Ng (2010), which are
\[ v(t) = \sqrt{\frac{mg}{d_1}} \tanh \left( t \sqrt{\frac{d_1g}{m}} \right) \]  \hspace{1cm} (3.12)

or

\[ v(t) = \sqrt{\frac{mg}{d_1}} \left( \frac{e^{2t\sqrt{\frac{d_1g}{m}}} - 1}{e^{2t\sqrt{\frac{d_1g}{m}}} + 1} \right) \]  \hspace{1cm} (3.13)

(\text{Ng, 2010; Lynch & Lommatsch, 2011}).

Also, the distance can be determined using

\[ v(t) = \frac{dy}{dt} \]

and by integrating once more to obtain

\[
y(t) = \sqrt{\frac{mg}{d_1}} \ln \left( \cosh \left( t \sqrt{\frac{d_1g}{m}} \right) \right) \sqrt{\frac{m}{d_1g}} + k
\]

\[
= \frac{m}{d_1} \ln \left( \cosh \left( t \sqrt{\frac{d_1g}{m}} \right) \right) + k
\]

\[
= \frac{m}{d_1} \ln \left( \frac{e^{t\sqrt{\frac{d_1g}{m}}} + e^{-t\sqrt{\frac{d_1g}{m}}}}{2} \right) + k
\]

(\text{Ng, 2010}).

Using \( y(0) = 0, k = 0 \), then the height is

\[
y(t) = \frac{m}{d_1} \ln \left( \frac{e^{t\sqrt{\frac{d_1g}{m}}} + e^{-t\sqrt{\frac{d_1g}{m}}}}{2} \right)
\]

\[
= \frac{m}{d_1} \ln \left( \frac{e^{2t\sqrt{\frac{d_1g}{m}}} + 1}{2} \right) \hspace{1cm} (3.14)
\]
which corroborate with Ng (2010).

Equations 3.12 and 3.14 denote the vertical velocity and position of the droplet as time progresses (Hughes & Brighton, 1999).

3.5 Conclusion

In this chapter we showed that drag and gravitation have an influence on the vertical movement of a droplet. In addition, the argument above yields an equation describing the vertical movement of a droplet more accurately.

The graphic representation of the equation for linear drag is as follows: Equation 3.11 is plotted in figure 3.3 for initial velocity and position $v_0 = 6\text{ms}^{-1}$ and $y_0 = 2\text{m}$ and constant values $g = 9.81\text{ms}^{-2}$, $d = 0.47$ and $m = 0.0026671\text{g}$. A time interval of 0.000001s starting at $t = 0\text{s}$ and ending at $t = 1.5\text{s}$ was used.

Further, if we let initial velocity vary between $v_0 = 0\text{ms}^{-1}$ and $v_0 = 5\text{ms}^{-1}$ with an interval 0.5ms$^{-1}$ and the rest of the parameter values are kept the same then the plot of equation 3.11 is as illustrated in figure 3.4.

Figure 3.4 represents the variation of the height of droplets with different initial velocities over time.

The graphic representation of the equation for quadratic drag is as follows: Equation 3.14 is plotted in figure 3.5 for constant values $v = 6\text{ms}^{-1}$, $g = 9.81\text{ms}^{-2}$, $d = 0.47$, $D_0 = 7.83 \times 10^{-4}\text{m}$, $\rho = 1060$ and $m = 0.0026671\text{g}$. A time interval of 0.000001s starting at $t = 0\text{s}$ and ending at $t = 1.5\text{s}$ was used.

Further, if we let velocity vary between $v = 1\text{ms}^{-1}$ and $v = 5\text{ms}^{-1}$ with an interval 0.5ms$^{-1}$, then $D_0$ will change in each calculation. The rest of the parameter values are kept the same, then the plot of equation 3.14 is as illustrated in figure 3.6.

Figure 3.6 represents the variation in distance of droplets with different velocities over time.

This is helpful to describe the displacement of a single droplet. For the case of multiple droplets an alternative approach is needed and therefore the next chapter studies multi-target tracking to describe multi-droplet trajectories.
Figure 3.3: Height of droplet over time
Figure 3.4: Height of droplet over time
Figure 3.5: Distance fall over time
Figure 3.6: Distance fall over time