Non-local and Non-autonomous Fragmentation-Coagulation Processes in Moving Media

By

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PREFACE

This study was carried out in the School of Mathematical and Physical Sciences, North-west University, Mafikeng campus, South Africa, from January 2012 to March 2014, under the supervision of Doctor Suares Clovis OUKOUOMI NOUTCHIE. This study is the original work of the researcher and has not been submitted in any form for any degree or diploma at any tertiary institution. Where use has been made of works by other authors, they have been duly acknowledged.

Abstract

It is well known that fragmentation and aggregation are not the only processes occurring in population grouping dynamics. The latter also includes other processes, like advection, convection, diffusion, direction changing, flow (transport). Existence of global solutions to discrete models and continuous non-local convection-fragmentation equations are investigated in spaces of distributions with finite higher moments. Assuming that the velocity field is divergence free, the method of characteristics and Friedrichs lemma [56] are used to show that the transport operator generates a stochastic dynamical system. This allows for the use of substochastic methods and Kato- Voigt perturbation theorem [12] to ensure that the combined transport-fragmentation operator is the infinitesimal generator of a strongly continuous semigroup. In particular, it is shown that the solution represented by this semigroup is conservative.

A double approximation technique is used to show existence result for a non-local and non-autonomous fragmentation dynamics occurring in a moving process. The case where sizes of clusters are discrete and fragmentation rate is time, position and size dependent is considered. The system involving transport and non-autonomous fragmentation processes, where in addition, new particles are spatially randomly distributed according to some probabilistic law, is investigated by means of forward propagators associated to evolution semigroup theory and perturbation theory. The full generator is considered as a perturbation of the pure non-autonomous fragmentation operator. One can therefore make use of the truncation technique [57], the resolvent approximation [88], Duhamel formula [39] and Dyson-Phillips series [76] to establish the existence of a solution for this model, hereby, bringing a contribution that may lead to the proof of uniqueness of strong solutions to this type of transport and non-autonomous fragmentation problem which remains unresolved. After that, the solution of the same model is approximated by a sequence of solutions of cut-off problems of a similar form. Then, the classical argument of Dini [50, Lemma 4] is used to show existence of strong solutions in the class of Banach spaces (of functions with finite higher moments) $\mathcal{X}_r := \{g : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g(x,n),\ \|g\|_r := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r |g(x,n)| dx < \infty\}$. Finally, the equivalent norm approach and semigroup perturbation theory are used to state and set conditions for a non-autonomous fragmentation system to be conservative. Generally, it is assumed that the generators are of class $\mathcal{G}(1,0)$ [33, 50], but this condition is modified in this study. Instead, it is assumed that the renormalisable generators involved in the perturbation process are in the class of quasi-contractive semigroups. This, henceforth, allows the use of admissibility with respect to the involved operators, Hermitian conjugate [74], Hille-Yosida's condition [12, 88] and the uniform boundedness [50] to show that the operator sum is closable, its closure generates a propagator (evolution system) and, therefore, a C_0 semigroup, leading to the existence result and conservativeness of the model.

Existence of a global solution to continuous, non-common and non-linear convection-

coagulation equations are investigated in the space $L_1(\mathbb{R}^3 \times \mathbb{R}_+, mdmdx)$. This is done by showing first that the transport operator generates a stochastic dynamical system, making use, as mentioned above, of the method of characteristics and Friedrichs lemma. Next, substochastic methods and Kato-Voigt perturbation theorem are used to ensure that the linear operator (transport-coagulation) is the infinitesimal generator of a strongly continuous semigroup. Once the existence for the linear problem has been established, the solution of the full non-linear problem follows by showing that the coagulation term is globally Lipschitz. In particular, it is shown that the solution of the full model is unique.

Laplace transform techniques and the method of characteristics are used to solve fragmentation equations explicitly. The result is a breakthrough in the analysis of pure fragmentation equations as this is the first instance where by, an exact solution is provided for the fragmentation evolution equation with arbitrary fragmentation rates. This provides a key for resolving most of the open problems in fragmentation theory including 'shattering' and the sudden appearance of infinitely many particles in some systems with initial finite particles number. In another concrete application, the effects of ocean iron fertilisation on the evolution of the phytoplankton biomass are investigated, using the theory of semilinear dynamical systems and numerical simulations are performed. The results demonstrate the validity of the iron hypothesis in fighting climate change.

In the process of discrete and non-local aggregation, the major problem arises when each fragmentation rate becomes infinite at infinity. A discrete Cauchy problem describing multiple fragmentation processes is investigated by means of parameter-dependent operators together with the theory of substochastic semigroups with a parameter. Focus is on the case where fragmentation rates are size and position dependent and where new particles are spatially randomly distributed according to a certain probabilistic law. Discrete models with both bounded and unbounded fragmentation rates are treated. The existence of semigroups is established for each parameter and "glued"together in order to obtain a semigroup to the full space. The dominated convergence theorem [21] is used to show existence of the infinitesimal generator of a positive semigroup of contractions and give sufficient conditions for honesty. Essentially, it was demonstrated that even in discrete and non-local case, the process is conservative if at infinity daughter particles tend to go back into the system with a high probability.

DECLARATION 1 - PLAGIARISM

I, Emile Franc DOUNGMO GOUFO declare that

1. The information contained in this thesis, except where otherwise indicated, is my original research;
2. This thesis has not been submitted for any degree or examination at any other university;
3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons;
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   - Their words are re-written but the general information attributed to the authors are acknowledged; and
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DECLARATION 2 - PUBLICATIONS

Papers published

- Global Solvability of a Continuous Model for Non-local Fragmentation Dynamics in a Moving Medium (Mathematical Problems in Engineering)
- Honesty in discrete, non-local and randomly position structured fragmentation model with unbounded rates (Comptes Rendus Mathematique, C.R Acad. Sci, Paris)
- Global solvability of a discrete non-local and non-autonomous fragmentation dynamics occurring in a moving process (Abstract and Applied Analysis)
- On the Honesty in Non-local and Discrete Fragmentation Dynamics in Size and Random Position (ISRN Mathematical Analysis)
- Analysis of the effects of large scale marine iron fertilisation (Journal of Pure and Applied Mathematics: Advances and Applications).

Papers submitted

- On conservativeness of evolution family by equivalent norm analysis for a non-autonomous fragmentation model
- Analysis by approximation technique for discrete, non-local and non-autonomous fragmentation models
- Global solvability of a continuous and special model for coagulation process in a moving medium.

Signed: ....

DEDICATION

I dedicate this thesis to the memories of my father,

Jean-Paul GOUFO and grand-mother, Pauline TSAYEM.

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I wish to thank God, the Almighty Father, for his guidance and for seeing through this journey. A special thought to the memory of my younger sister Marianne TSAYEM GOUFO who has become a guardian angel for our family.
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Contents

Chapter 1

Introduction

This study explores some important aspects of fragmentation and coagulation processes evolving in moving areas which are not properly discussed in the literature. In fact, fragmentation-coagulation models which combine other processes like transport or direction changing and where the rate at which particles coalesce or fragment depends on time, is still sketchy in the domain of applied analysis. In this chapter several concepts are introduced, such as mathematical models and spaces and will be examined using various techniques including the theory of substochastic semigroups, Kato- Voigt perturbation, equivalent norm approach, the theory of evolution systems, Laplace transform techniques and the method of characteristics.

1.1 Transport, direction changing, fragmentation and coagulation processes

An **Organism's (Population) Grouping** refers to a phenomenon in which a number of living beings are involved in movement as a group (cluster). For example, one count the swarms of locust, mosquitoes, flies or midges, a herd of elephants or sheep, a school of fish, marine zooplankton or phytoplankton cluster. A group size can change due to splitting (fission or fragmentation) into smaller sizes or combining (aggregation, fusion or coagulation) to form a bigger group size. The dynamics in population grouping is not limited only to fragmentation and aggregation. There are other processes like advection, diffusion, direction changing and flow (transport). It is obvious that some clustering and direction change act on a faster time scale (school of fish) or a slower time scale (herd of elephants). Theses processes combined in the same model are still barely investigated and pose a challenge for this study. **(Pure) Fragmentation processes** can be observed in natural sciences and engineering. A few examples include the study of stellar fragments in astrophysics, rock fracture, degradation of large polymer chains, DNA fragmentation, evolution of phytoplankton aggregates, liquid droplet breakup or breakup of solid drugs in organisms. **Coagulation-fragmentation processes** describe the evolution of systems in which particles react in either fusing together or breaking apart while the **transport and direction changing processes** add movement to all of it.

In concrete applications, the mathematics of an evolution dynamical system is represented by a concentration function $(t, \eta) \to p(t, \eta)$, where t is the time and η is an element of some state space Ω which identifies an individual uniquely. The function p is then interpreted as the probability (density) of finding an individual which at the time t enjoys the property η. An intrinsic property of the dynamical process is that all the particles must be accounted for or, in other words:

$$\int_{\Omega} p(t, \eta) d\mu_{\eta} = \int_{\Omega} p(0, \xi) d\mu_{\eta}, \tag{1.1}$$

for any time t, where $d\mu_{\eta}$ is an appropriate measure in the state space. Therefore, from the physical point of view, the natural spaces for studying such problems are L_1 spaces. In fragmentation-coagulation theory, η could be for example, the mass or the size of a particle, its spatial location or a combination of all of them.

The general discrete model of the dynamics as described above and which is a spatially explicit group-size distribution model as presented in [67] reads as follows:

$$\begin{aligned} \frac{\partial}{\partial t} p_n + div(\omega p_n) &= -\xi(n) p_n + \xi(n) \int_V p_n K(\omega, \omega', n) d\omega' \\ &+ \frac{1}{2} \sum_{m=1}^{n-1} c(m, n-m) p_m p_{n-m} - \sum_{m=1}^{\infty} c(n, m) p_n p_m \\ &- \frac{1}{2} \sum_{m=1}^{n-1} h(n, m) p_n + \sum_{m=n+1}^{\infty} h(m, n) p_m, \end{aligned} \tag{1.2}$$

where the velocity $\omega = \omega(x, n)$ of the transport is supposed to be a known quantity, depending on the size n of aggregates and their position x. More details about this model are given in equation (3.1). It should be noted that the term $-\xi(n)p_n + \xi(n) \int_V p_n K(\omega, \omega', n) d\omega'$, which is the Boltzmann part of the equation, describes the change of the direction of motion. This study is interested in solving the problem (1.2) with the transport and fragmentation processes only. The following Cauchy Problem (the model with an initial condition) is considered:

$$\frac{\partial}{\partial t} p(t, x, n) = -div(\omega(x, n) p(t, x, n)) - a_n p(t, x, n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t, x, m)$$

$$p(0, x, n) = \overset{\circ}{p}_n(x), \qquad n = 1, 2, 3, ... \tag{1.3}$$

where $a_n = \frac{1}{2} \sum_{m=1}^{n-1} h(n, m)$ is the average fragmentation rate, that is the average number at which clusters of size n undergo splitting, $b_{n,m} \geq 0$ is the average number of n-groups produced upon the splitting of m-groups and given by $h(m, n) = b_{n,m} a_m =$

$\frac{1}{2}\sum_{k=1}^{m-1} b_{n,m}h(m,k)$. The coefficients a_n and $b_{n,m}$ give a randomly spatial distribution and are better to analyse than the previous ones c and h which describe a binary process. The space variable x is supposed to vary in the whole of $\mathbb{R}^3 = \Omega$. The function $\overset{o}{p}_n$ represents the density of n-groups at the beginning of observation ($t = 0$) and is integrable with respect to x over the full space $\mathbb{R}^3$, this integral multiplied by n is summable so that the total initial population is finite.

Because the total number of individuals in a population is not modified by interactions among groups, the following conservation law is supposed to be satisfied:

$$\frac{d}{dt}U(t) = 0 \tag{1.4}$$

where $U(t) = \int_{\mathbb{R}^3}\sum_{n=1}^{\infty} np(t,x,n)dx = \sum_{n=1}^{\infty} n\int_{\mathbb{R}^3} p(t,x,n)dx$ is the total number of individuals in the space. Since $p_n = p(t,x,n)$ is the density of groups of size n at the position x and time t and that mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$\begin{aligned} \mathcal{X}_1 :=& \{\mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \\ & \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n|g_n(x)|dx < \infty\}. \end{aligned} \tag{1.5}$$

Work is dine in this space because they have many desirable properties, like controlling the norm of their elements which, in this case, represents the total mass (or total number of individuals) of the system and must be finite. Because uniqueness of solutions of (1.3) proved to be a more difficult problem [15], the analysis is limited to a smaller class of functions, then, the following class of Banach spaces (of distributions with finite higher moments) is introduced:

$$\begin{aligned} \mathcal{X}_r :=& \{\mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \\ & \|\mathbf{g}\|_r := \int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n^r|g_n(x)|dx < \infty\}, \end{aligned} \tag{1.6}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$.

Mathematical expression of the non autonomous model: The dynamical behaviour of a system that can break up to produce smaller particles is given by the integro-differential system:

$$\begin{cases} \frac{\partial}{\partial t}p(t,x) = -a(t,x)p(t,x) + \int_x^{\infty} a(t,y)b(x|y)p(t,y)dy \\ p(\tau,x) = p_\tau(x) \qquad\qquad 0 \leq \tau < t \leq T,\ x > 0 \end{cases} \tag{1.7}$$

where p is the particle mass distribution function ($p(\tau,x) = p_\tau(x)$ is the mass distribution at some fixed time $\tau \geq 0$) with respect to the mass x, $b(x|y)$ is the distribution of particle masses x spawned by the fragmentation of a particle of mass y, $T \in \mathbb{R}$, $a(t,x)$ is the evolutionary time-dependent fragmentation rate that is the rate at which mass x

particles break up at a time t. The first term on the right-hand side of (1.7) describes the reduction in the number of particles in the mass range $(x; x+dx)$ due to the fragmentation of particles in the same range. The second term describes the increase in the number of particles in the range due to fragmentation of larger particles.
The idea here is to analyse the equation (1.7) in the Banach space $L_1(\mathfrak{I}, X_1)$ where $\mathfrak{I} = [0, T]$ and

$$X_1 = L_1([0,\infty), xdx) = \left\{\psi : \quad \|\psi\|_{X_1} := \int_0^\infty x|\psi(x)|\, dx < \infty\right\},$$

using the theory of evolution semigroup. The model (1.7) is recast as the non-autonomous abstract Cauchy problem in X_1:

$$\begin{cases} \frac{d}{dt}u(t) = Q(t)u(t) & 0 \le \tau < t \le T \\ u(\tau) = u_\tau \end{cases} \tag{1.8}$$

where $Q(t)$ is defined by $Q(t) = \mathcal{Q}(t)$ and which represents the realisation of $\mathcal{Q}(t)$ on the domain $D(Q(t)) = \{u \in X_1;\ \mathcal{Q}(t)u(t) \in X_1\}$, with $(\mathcal{Q}u)$ defined as follows: $(\mathcal{Q}u)(t,x) = (\mathcal{Q}u)(t)(x) = -a(t,x)u(t,x) + \int_x^\infty a(t,y)b(x|y)u(t,y)dy$
$\mathcal{Q}(t)$ is seen as the pointwise operation

$$\psi(t,x) \longmapsto -a(t,x)\psi(t,x) + \int_x^\infty a(t,y)b(x|y)\psi(t,x)dy$$

defined on the set of measurable functions. $\mathcal{Q}(t)$ indeed defines various operators. To analyse this system, a two parameter family called ***(Evolution system [74] or propagator [64])*** is needed.

The analysis of such models required the researcher to proceed step by step as presented in this study. Important results, definitions and theorems which lie at the core of dynamical systems are used.

1.2 Outline of the thesis

This study explores less known aspects characterising the multiple combination of forms arising in fragmentation-coagulation-transport (non-local or non-autonomous) theory. It is the outcome of the researcher's three years PhD research at the North-West University. Most of the materials contained in this study are based on the following published articles:

1. E.F. Doungmo Goufo (co-published with S.C. Oukouomi Noutchie), "Global Solvability of a Continuous Model for Non-local Fragmentation Dynamics in a Moving Medium," Mathematical Problems in Engineering, vol. 2013, Article ID 320750, 8 pages, 2013. http://dx.doi.org/10.1155/2013/320750;

2. E.F. Doungmo Goufo (co-published with S.C. Oukouomi Noutchie), Honesty in discrete, nonlocal and randomly position structured fragmentation model with unbounded rates, Comptes Rendus Mathematique, C.R Acad. Sci, Paris, Ser, I, 2013, http://dx.doi.org/10.1016/j.crma.2013.09.023;

3. E.F. Doungmo Goufo (co-published with S.C. Oukouomi Noutchie), Global solvability of a discrete non-local and non-autonomous fragmentation dynamics occurring in a moving process, Abstract and Applied Analysis, vol. 2013, Article ID 484391, 9 pages, 2013. doi:10.1155/2013/484391;

4. E.F. Doungmo Goufo (co-published with S.C. Oukouomi Noutchie), "On the Honesty in Non-local and Discrete Fragmentation Dynamics in Size and Random Position," ISRN Mathematical Analysis, vol. 2013, Article ID 908753, 7 pages, 2013. http://dx.doi.org/10.1155/2013/908753;

5. E.F. Doungmo Goufo (co-published with S.C. Oukouomi Noutchie), *Analysis of the effects of large scale marine iron fertilisation*, Journal of Pure and Applied Mathematics: Advances and Applications, 2012 Scientific Advances Publishers.

And the following submitted papers (still under review):

1. E.F. Doungmo Goufo (with S.C. Oukouomi Noutchie), On conservativeness of evolution family by equivalent norm analysis for a non-autonomous fragmentation model;

2. E.F. Doungmo Goufo (with S.C. Oukouomi Noutchie), Analysis by approximation technique for discrete, non-local and non-autonomous fragmentation models;

3. E.F. Doungmo Goufo (with S.C. Oukouomi Noutchie), Global solvability of a continuous and special model for coagulation process in a moving medium.

Despite the fact that most of the methods and techniques used in the study are relatively well known, the investigation and analysis often required some possibly less familiar results and considerations. Hence, Chapter 2 discusses these subsidiary results and considerations.

The aim of Chapter 3 is to combine and analyse fragmentation models with the transport (streaming) operator in order to model fragmentation processes in moving media. The streaming operator arises in many models of mathematical physics (e.g. Boltzmann equation, radiative transfer, neutron transport theory) and mathematical biology (population dynamics etc.) dealing with the time evolution of the distribution function $p(t; x; n)$ of individuals of some population (particles in the Boltzmann kinetic theory, population of cells in biomathematics) having the state $(x; n)$ at time $t \geq 0$. Commonly, x stands for the position of a particle and n for its size. Because uniqueness of solutions of the model under investigation proved to be a more difficult problem [15], the researcher in this study distances himself from previous works [12, 31, 68] by restricting the analysis to the spaces of distributions with finite higher moments. The analysis consists of considering at first, the model only with the transport process, and later, gradually add the loss and the gain parts of fragmentation operator with the hope that it will make a significant contribution to the analysis of the full problem (with transport, direction changing, fragmentation and coagulation processes) which remains an open problem. Assuming that the velocity field is divergence free, the researcher succeeded in using the method of characteristics and Friedrichs lemma [56] to show existence of global solutions to discrete models and continuous non-local convection-fragmentation equations. In particular, it is shown that the solutions represented by these semigroups are conservative.

Chapter 4 deals with non-autonomous fragmentation dynamics. In the first part of the chapter, a global analysis of the discrete non-local and non-autonomous fragmentation dynamics occurring in a moving process is performed. Use of a double approximation technique together with the truncation technique [57], the resolvent approximation [88], Duhamel formula [39] and Dyson-Phillips series [76] is made to show that the solution for the model exists. Various factors, such as temperature and viscosity, influence the rate at which particles coalesce and fragment. These factors, and the kernels which model their effects, are discussed in the survey article by Drake [37]. Most mathematical investigations have concentrated on time-independent coalescence and breakdown rates and a number of existence and uniqueness results have been produced for the autonomous version of the fragmentation and coagulation equations. It should be noted that local non-autonomous cases have been examined by McLaughlin *et al* [59]. An investigation into the non-local non-autonomous fragmentation equations is therefore a natural extension as this allows the factors which determine breakdown to be time-dependent and spatially non-homogeneous. In this study, a special focus is placed on the particle distribution kernel represented by a time dependent probabilistic density function as well as the fragmentation rate that will be time and position dependent. The investigation is done by means of forward propagators associated to evolution semigroup theory and perturbation theory. The analysis in the second part of Chapter 4 consists of approximating the solution of the same model by a sequence of solutions of cut-off problems of a similar form. The classical argument of Dini [50, Lemma 4] is then used to show existence of strong solutions of the model in the class of Banach spaces of functions with finite higher moments. The chapter concludes by applying the equivalent norm approach to non-autonomous fragmentation systems. In the common literature, it is assumed that the generators are of class $\mathcal{G}(1,0)$ [33, 50], but this condition is modified by assuming that the renormalisable generators involved in the perturbation process are in the class of quasi-contractive semigroups. Nevertheless, it is possible to show that, thanks to admissibility with respect to the involved operators, Hermitian conjugate [74], Hille-Yosida's condition [12, 88] and the uniform boundedness [50], that the operator sum is closable leading to the existence result and conservativeness of the model.

In Chapter 5, a continuous and less known model for coagulation process evolving in a moving medium is globally solved in the space $L_1(\mathbb{R}^3 \times \mathbb{R}_+, mdmdx)$. The first part of the analysis resembles the one performed in Chapter 3. Once the existence for the linear problem is established, the solution of the full non-linear problem follows by showing that the coagulation term is globally Lipschitz. In particular, it is shown that the solution of the full model is unique. The coagulation model considered here differs from the classical one and it is assumed that any individual in the populations is viewed as a collection of joined cells.

The aim of Chapter 6 is to establish a better understanding concerning some real phenomena occurring in applied sciences, and which are shattering and marine iron fertilisation. In the first part of the chapter, exact solutions of fragmentation equations with arbitrary fragmentation rates and separable particles distribution kernels are found, using Laplace transform techniques. Since fragmentation processes are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it, that is why, though linear, they display non-linear features such as phase transition which, in this case, is called shattering and consists in the formation of a 'dust' of particles of zero size carrying, nevertheless, a non-zero mass. Quantitatively, one can identify this process by disappearance of mass from the system even though it is conserved in each fragmentation event. Probabilistically, shattering is an example of an explosive, or dishonest Markov process, see e.g. [3, 66]. So the analysis yields a key for resolving most of the open problems in fragmentation theory including shattering and the sudden appearance of infinitely many particles in some systems with initial finite particles number. In the second part of the chapter, the theory of semilinear dynamical systems is exploited in order to investigate the effects of ocean iron fertilisation on the evolution of the phytoplankton biomass and provide numerical simulations of the results. The results demonstrate the validity of the iron hypothesis in fighting climate change.

Chapter 7 focuses on conservativeness in discrete, non-local and randomly positioned structured fragmentation model with unbounded rates. The major problem arises when each fragmentation rate becomes infinite at infinity so the dominated convergence theorem [21] is used to show existence of the infinitesimal generator of a positive semigroup of contractions and to give sufficient conditions for honesty in the case of unbounded fragmentation rates. Essentially, it is demonstrated that even in a discrete and non-local case, the process is conservative if at infinity, daughter particles tend to go back into the system with a high probability.

Chapter 2

Preliminary and Auxiliary Results

In this chapter, results, definitions and theorems to be used later in the analysis were collected. For the most part of this study, techniques from the calculus of vector-valued functions are applied and a brief introduction to some functional analysis concepts used in subsequent chapters is given.

2.1 Calculus of vector-valued functions and Banach lattices

2.1.1 Spaces and operators

Definition 2.1.1. *A vector-valued function u from some abstract set I to a Banach space X is a mapping $t \to u(t)$ from I into X, where to each point $t \in I$ there corresponds a unique vector $u(t) \in X$.*

In the case where the Banach space is the space of bounded linear operators from X into Y, denoted by $\mathcal{L}(X,Y)$ with norm $\|\cdot\|_{\mathcal{L}(X,Y)}$, the function is referred to as an operator valued function. (When $X = Y$, $\mathcal{L}(X)$ with norm $\|\cdot\|_{\mathcal{L}(X)}$ is written.)

Definition 2.1.2. (Strong Derivative) *Let X be a Banach space with norm $\|\cdot\|_X$ and let the function u be an X-valued function of $t \in [0,\infty)$. Then the strong derivative $\dfrac{du(t)}{dt}$ of u at $t > 0$ is defined to be an element $\overline{u}(t)$ such that*

$$\lim_{h\to 0} \|h^{-1}[u(t+h) - u(t)] - \overline{u}(t)\|_X = 0 \tag{2.1}$$

provided that the limit exists.

Definition 2.1.3. *Let Π denote any partition $a = t_0 < t_1 < t_2 ... < t_n = b$ of the closed interval $[a,b]$ together with the arbitrary points $s_k \in [t_{\varsigma-1}, t_\varsigma]$, $\varsigma = 1, 2, ..., n$ and let the* *norm* $|\Pi| = \max_{\varsigma}(t_\varsigma - t_{\varsigma-1})$. *If for a vector-valued function* $u : [a,b] \to X$, *there exists* $v \in X$ *(independently of the manner in which* $|\Pi| \to 0^+$*) such that*

$$\lim_{|\Pi|\to 0^+} \left\| \sum_{\varsigma=1}^{n} u(s_\varsigma)(t_\varsigma - t_{\varsigma-1}) - v \right\|_X = 0,$$

then v *is the strong Riemann integral and is denoted by*

$$\int_a^b u(t)dt.$$

Theorem 2.1.4. *If* u *is a strongly continuous vector-valued function on* $[a,b]$ *to* X, *then the strong Riemann integral over* $[a,b]$ *exists. Moreover, if* $A : X \supseteq D(A) \to Y$ *is a closed linear operator,* $u(t) \in D(A)$ *for each* $t \in [a,b]$ *and if* Au *is strongly continuous on* $[a,b]$, *then*

$$A\left[\int_a^b u(t)dt\right] = \int_a^b [Au](t)dt.$$

Proof. [46, Theorem 3.3.2]. □

Definition 2.1.5. *A Banach space* X *is of type* L *if it consists of equivalence classes of numerically-valued functions defined on a set* Ω *and if it has the following two properties: (i) If* u *is a continuous* X*-valued function defined on* $I = [\alpha, \beta]$, *then there exists a function* ψ *measurable on the product* $I \times \Omega$ *such that* $u(t) = \phi(t,\cdot)$ *for each* $t \in [\alpha,\beta]$. *Note* $u(t) = \psi(t,\cdot)$ *means equality in* X. *(ii) If* u *is continuous on* $I = [\alpha,\beta]$ *and* ψ *is any function that is measurable on* $I \times \Omega$ *and satisfies* $u(t) = \psi(t,\cdot)$ *for each* $t \in [\alpha,\beta]$, *then*

$$\left[\int_\alpha^\beta u(t)dt\right](\cdot) = \int_\alpha^\beta \psi(t,\cdot)dt, \tag{2.2}$$

where the integral on the left-hand side is the abstract Riemann integral and the integral on the right-hand side is the Lebesgue integral of numerically-valued functions.

Theorem 2.1.6. *Any space* $L_p(\Omega)$, $1 \le p < \infty$ *is of type* L.

Proof. [12, Theorem 2.39]. □

Theorem 2.1.7. *Let* X *be a Banach space of type* L. *If* u *is a vector-valued function on* $I = [a,b]$ *to* X *and if* u *is* n*-times continuously strongly differentiable, then there exists a numerically-valued function* v *measurable on* $I \times \Omega$ *such that (i) for* $0 \le s \le n-1$, $\frac{\partial^s}{\partial t^s} v(t,x)$ *is absolutely continuous for each* $x \in \Omega$ *and*

$$\frac{\partial^s}{\partial t^s} v(t,\cdot) = \left[\frac{d^s}{dt^s} u(t)\right](\cdot)$$

for each $t \in I$*; (ii)* $\frac{\partial^n}{\partial t^n} v(t,x)$ *exists almost everywhere in* $I \times \Omega$ *and*

$$\frac{\partial^n}{\partial t^n} v(t,\cdot) = \left[\frac{d^n}{dt^n} u(t)\right](\cdot)$$

for almost all $t \in I$.

Proof. See [46, Theorem 3.4.2]. □

Note that in case the Banach space X is a space of numerically-valued functions defined on some abstract set Ω, the relation between the differential equation $\frac{d}{dt}u(t) = g(t, u(t))$ (in strong sense) and the partial differential equation $\frac{\partial}{\partial t}u(t,x) = g(t, u(t,x))$ depends on the nature of X.

Theorem 2.1.8. *Let* $\{\psi_n\}$ *be a Cauchy sequence in* $L_p(\Omega)$ *that converges strongly to* ψ*. Then there exists a subsequence* $\{\psi_{n_\varsigma}\}$ *that converges pointwise almost everywhere on* Ω *to a limit function* ψ*.*

Proof. See [75, Corollary 5.11]. □

Theorem 2.1.9. *Let* $\{\psi_n\}$ *be a sequence of Lebesgue-integrable functions over* $\Omega \subseteq \mathbb{R}^n$ *such that (i)* $\{\psi_n\}$ *increases almost everywhere on* Ω*; (ii)* $\lim_{n\to\infty} \int_\Omega \psi_n(x)dx$ *exists. Then* $\{\psi_n\}$ *converges almost everywhere to a limit function* $\psi \in L_1(\Omega)$ *and*

$$\lim_{n\to\infty} \int_\Omega \psi_n(x)dx = \int_\Omega \psi(x)dx.$$

Proof. See [4, Theorem 10.24]. □

2.1.2 Banach lattices and positive operators

Definition 2.1.10. *Let* X *be an arbitrary set. A partial order (or simply, an order) on* X *is a binary relation, denoted here by '* $\geq$ *', which is reflexive, transitive, and antisymmetric, that is, (1)* $x \geq x$ *for each* $x \in X$*; (2)* $x \geq y$ *and* $y \geq x$ *imply* $x = y$ *for any* $x, y \in X$*; (3)* $x \geq y$ *and* $y \geq z$ *imply* $x \geq z$ *for any* $x, y, z \in X$*.*

Definition 2.1.11. *An ordered vector space is a vector space* X *equipped with partial order which is compatible with its vector structure in the sense that (4)* $x \geq y$ *implies* $x + z \geq y + z$ *for all* $x, y, z \in X$*; (5)* $x \geq y$ *implies* $\alpha x \geq \alpha y$ *for any* $x, y \in X$ *and* $\alpha \geq 0$*.*

The set $X_+ = \{x \in X; \quad x \geq 0\}$ is referred to as the positive cone of X. It is considered that X is a lattice if every pair of elements (and so every finite collection of them) has both supremum and infimum. If an ordered vector space X is also a lattice, then it is called a vector lattice or a Riesz space. Typical examples of Riesz spaces are provided by spaces of functions. If X is a vector space of real-valued functions on a set Ω, then a pointwise order in X can be introduced by saying that $f \leq g$ in X if $f(x) \leq g(x)$ for any $x \in S$. Equipped with such an order, X becomes an ordered vector space. It should be recalled that if Ω is a measure space, then all considerations above are valid when the pointwise order is replaced by $f \leq g$ if $f(x) \leq g(x)$ almost everywhere. With this in mind, $L_0(\Omega)$ and $L_p(\Omega)$ spaces with $1 \leq p \leq \infty$ become function spaces and are thus Riesz spaces. For an element x in a Riesz space X, its positive and negative part, and its absolute value could be defined, respectively, by

$$x_+ = \sup\{x, 0\}, \quad x_- = \sup\{-x, 0\}, \quad |x| = \sup\{x, -x\}.$$

Proposition 2.1.12. *If x is an element of a Riesz space, then*

$$x = x_+ - x_-, \quad |x| = x_+ + x_-$$

Thus, in particular, the positive cone in a Riesz space is generating.

Proof. See [12, Proposition 2.46]. □

In the next step, the relation between the lattice structure and the norm is investigated when X is both a normed and an ordered vector space.

Definition 2.1.13. *A norm on a vector lattice X is called a lattice norm if*

$$|x| \leq |y| \quad \textit{implies} \quad \|x\| \leq \|y\|. \tag{2.3}$$

A Riesz space X complete under a lattice norm is called a Banach lattice.

Property (2.3) gives the important identity:

$$\|x\| = \||x|\|, \quad x \in X. \tag{2.4}$$

Proposition 2.1.14. *If X is a normed lattice, then all lattice operations are uniformly continuous in the norm of X with respect to all variables involved.*

Proof. [12, Proposition 2.55]. □

2.1.3 Positive operators

Definition 2.1.15. *A linear operator A from a Banach lattice X into a Banach lattice Y is called positive, denoted by $A \geq 0$, if $Ax \geq 0$ for any $x \geq 0$.*

Positive operators are fully determined by their behaviour on the positive cone. Precisely speaking, the following theorem is obtained.

Theorem 2.1.16. *If $A : X_+ \to Y_+$ is additive, then A extends uniquely to a positive linear operator from X to Y. Keeping the notation A for the extension, we have, for each $x \in X$,*

$$Ax = Ax_+ - Ax_-. \tag{2.5}$$

Proof. [12, Theorem 2.64] □

An easy and often used result on positive operators could be pointed out here.

Proposition 2.1.17. *If A is positive, then*

$$\|A\| = \sup_{x \geq 0,\, \|x\| \leq 1} \|Ax\|. \tag{2.6}$$

Proof. [12, Theorem 2.67] □

Definition 2.1.18. *A Banach lattice X is said to be a KB-space (Kantorovic Banach space) if every increasing norm bounded sequence of elements of X_+ converges in norm in X.*

The next theorem characterises the KB-spaces and is very useful in applications.

Theorem 2.1.19. *Assume that X is a weakly sequentially complete Banach lattice. If $(x_n)_{n \in \mathbf{N}}$ is increasing and $(\|x_n\|)_{n \in \mathbf{N}}$ is bounded, then there is $x \in X$ such that*

$$\lim_{n \to \infty} x_n = x \tag{2.7}$$

in X. In other words, weakly sequentially complete and, in particular, reflexive Banach lattices are KB-spaces.

Proof. [12, Theorem 2.82]. □

2.2 Linear semigroups

In this section methods of finding solutions of a Cauchy problem are examined.

Definition 2.2.1. *Given a Banach space X and a linear operator $\mathcal{A}$ with domain $D(\mathcal{A})$ and range $Im\mathcal{A}$ contained in X and also given an element $u_0 \in X$, find a function $u(t) = u(t, u_0)$ such that (1) $u(t)$ is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$, (2) for each $t > 0$, $u(t) \in D(\mathcal{A})$ and*

$$\frac{du}{dt}(t) = \mathcal{A}u(t), \quad t > 0, \tag{2.8}$$

(3)

$$\lim_{t \to 0} u(t) = u_0 \tag{2.9}$$

in the norm of X. A function satisfying all the conditions above is called the classical (or strict) solution of (2.8), (2.9).

Definition 2.2.2. *A family $(S(t))_{t \geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup if (i) $S(0) = I$; (ii) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$; (iii) $\lim_{t \to 0^+} S(t)x = x$ for any $x \in X$. A linear operator A is called the (infinitesimal) generator of $(S(t))_{t \geq 0}$ if*

$$Ax = \lim_{h \to 0^+} \frac{S(h)x - x}{h}, \tag{2.10}$$

where the domain of A, $D(A)$, is defined as the set of all $x \in X$ for which this limit exists. Typically, the semigroup generated by A is denoted by $(S_A(t))_{t \geq 0}$.

it should be noted that if A is the generator of $(S(t))_{t \geq 0}$, then for $x \in D(A)$ the function $t \to S(t)x$ is a classical solution of the following Cauchy problem,

$$\begin{aligned} \frac{du}{dt}(t) &= A(u(t)) \\ \lim_{t \to 0^+} u(t) &= x \end{aligned} \quad t \geq 0 \tag{2.11}$$

For $x \in X \setminus D(A)$, however, the function $u(t) = S(t)x$ is continuous but, in general, not differentiable, nor $D(A)$-valued, and, therefore, not a classical solution. Nevertheless, the integral $v(t) = \int_0^t u(s)ds \in D(A)$ and it is a strict solution of the integrated version of (2.11):

$$\begin{aligned} \frac{dv}{dt}(t) &= A(v(t)) + x \\ \lim_{t \to 0^+} v(t) &= 0 \end{aligned} \quad t \geq 0 \tag{2.12}$$

or equivalently,

$$u(t) = A \int_0^t u(s)ds + x. \tag{2.13}$$

It is said that a function u satisfying (2.12) (or, equivalently, (2.13)) is a mild solution or integral solution of (2.11).

Proposition 2.2.3. *Let $(S(t))_{t\geq 0}$ be the semigroup generated by $(A, D(A))$. Then $t \to S(t)x$, $x \in D(A)$, is the only solution of (2.11) taking values in $D(A)$. Similarly, for $x \in X$, the function $t \to S(t)x$ is the only mild solution to (2.11).*

Proof. [12, Proposition 3.4] □

Thus, if there is a semigroup, the Cauchy problem of which it is a solution can be identified. Usually, however, the interest is in the reverse question, that is, in finding the semigroup for a given equation. The answer is given by the Hille-Yosida theorem.

Theorem 2.2.4. (Hille-Yosida Theorem) *$A \in \mathcal{G}(M, \omega)$ if and only if (a) A is closed and densely defined, (b) there exists $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(A)$ and for all $n \geq 1$, $\lambda > \omega$,*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{2.14}$$

where $\rho(A)$ is the resolvent set of the operator A and is defined as follows:

$$\rho(A) = \{\lambda \in \mathbb{R}; \quad \lambda I - A : D(A) \to X \text{ is invertible and } (\lambda I - A)^{-1} \in \mathcal{L}(X)\}. \tag{2.15}$$

Proof. [12, Theorem 3.5] □

Theorem 2.2.5. *Assume that the closure $(\overline{A}, D(\overline{A}))$ of an operator $(A, D(A))$ generates a C_0-semigroup in X. If $(B, D(B))$ is also a generator, such that $B|_{D(A)} = A$, then $(B, D(B)) = (\overline{A}, D(\overline{A}))$.*

Proof. [12, Proposition 3.8] □

The Lumer-Phillips Theorem gives an alternative characterisation of the infinitesimal generator of a C_0-semigroup of contractions. Before stating the theorem a definition of the term dissipative is given.

Definition 2.2.6. *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in X. The operator $\mathcal{A}$ is dissipative if $\|(\lambda I - \mathcal{A})\psi\|_X \geq \lambda\|\psi\|_X$ for all $\psi \in D(\mathcal{A})$ and $\lambda > 0$.*

Theorem 2.2.7. (Lumer-Phillips) *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in X. (i) If $\mathcal{A}$ is dissipative and if there exists $\lambda_0 \in \mathbb{C}$, such that the range $Im(\lambda_0 I - \mathcal{A})$ of $\lambda_0 I - \mathcal{A}$ is X, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X. (ii) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X, then $\mathcal{A}$ is dissipative and for all $\lambda > 0$, $Im(\lambda I - \mathcal{A}) = X$.*

Proof. [74, Theorem 4.3, p14]. □

It is not always necessary to know the infinitesimal generator on its whole domain.

Definition 2.2.8. *Let A be a closed operator in a Banach space X. A core of A is a dense subspace D of X such that A is the closure of its restriction to D i.e. $\overline{A|_D} = A$.*

Theorem 2.2.9. (Core) *Let A be the generator of the semigroup $(S_A(t))_{t\geq 0}$ on a Banach space X and let D be a dense set contained within the domain of A, i.e. $D \subset D(A)$. If the set D is invariant under the semigroup $(S_A(t))_{t\geq 0}$, then D is a core for A.*

Proof. [61, Theorem 2.1.1]. □

Next, a case of restrictions of $(S(t))_{t\geq 0}$, acting in a Banach space X, to a subspace Y which is continuously embedded in X and which is invariant under $(S(t))_{t\geq 0}$, is considered. The restriction $(S_Y(t))_{t\geq 0}$ of $(S(t))_{t\geq 0}$ to Y is obviously a semigroup but not necessarily a C_0-semigroup. If, however, it is strongly continuous, then the generator of $(S_Y(t))_{t\geq 0}$ can be identified as the part in Y of the generator A of $(S(t))_{t\geq 0}$.

Proposition 2.2.10. *Let $(A, D(A))$ generate a C_0-semigroup $(S(t))_{t\geq 0}$ in a Banach space X and let Y, be a subspace continuously embedded in X, invariant under $(S(t))_{t\geq 0}$. If the restricted semigroup $(S_Y(t))_{t\geq 0}$ is strongly continuous in Y then its generator is the part A_Y of A in Y . Moreover, if Y is closed in X, then $(S_Y(t))_{t\geq 0}$ is automatically strongly continuous and A_Y is the restriction of A to the domain $D(A) \cap Y$.*

Proof. [12, Proposition 3.12] □

Next, resolvent positive operators are considered.

Definition 2.2.11. *Let X be a Banach lattice. It is said that the semigroup $(S(t))_{t\geq 0}$ on X is positive if for any $x \in X_+$ and $t \geq 0$,*

$$S(t)x \geq 0.$$

It is said that an operator $(A, D(A))$ is resolvent positive if there is ω such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

It should be noted that a strongly continuous semigroup is positive if and only if its generator is resolvent positive. Let A be a resolvent positive operator. The following notation is introduced:

$$s(A) = \inf\{\omega \in \mathbb{R} : \quad (\omega, \infty) \subset \rho(A) \text{ and } R(\lambda, A) \geq 0 \text{ for all } \lambda > \omega\},$$

where $\rho(A)$ is the resolvent set of A.

Theorem 2.2.12. (Arendt-Robinson-Batty) *Let A be a densely defined resolvent positive operator. If there exists $\lambda_0 > s(A)$, $c > 0$ such that for all $\psi \geq 0$,*

$$\|R(\lambda_0, A)\psi\| \geq c\|\psi\|, \tag{2.16}$$

then A generates a positive semigroup $(S_A(t))_{t\geq 0}$ on X and $s(A) = \omega_0(S_A)$, where $\omega_0(S_A)$ is the uniform growth bound of the semigroup $(S_A(t))_{t\geq 0}$.

Proof. [12, Theorem 3.39] □

2.3 Some classical perturbation results

Let $(A, D(A))$ be a generator of a C_0-semigroup on a Banach space X and $(B, D(B))$ be another operator in X. The purpose of the perturbation theory is to find conditions that ensure that there is an extension G of $A + B$ that generates a C_0-semigroup on X and characterise this extension.

2.3.1 Bounded perturbation theorem

The simplest and possibly the most often used perturbation result can be obtained if the operator B is bounded. The following theorem holds:

Theorem 2.3.1. (Bounded perturbation) *Let $(A, D(A)) \in \mathcal{G}(M,\omega)$; that is, it generates a C_0-semigroup $(S_A(t))_{t\geq 0}$ satisfying $\|S_A(t)\| \leq Me^{\omega t}$ for some $\omega \in \mathbb{R}$ and $M \geq 1$. If $B \in \mathcal{L}(X)$, then*

$$(A + B, D(A)) \in \mathcal{G}(M, \omega + M\|B\|).$$

Proof. [12, Theorem 4.9] □

2.3.2 Kato-Voigt perturbations

The Kato-Voigt theorem is useful in the sense that, it allows the establishment of the existence of a *smallest* substochastic semigroup associated with a specific Cauchy problem. The definitions of the terms *stochastic* and *substochastic* semigroups introduce this section.

Definition 2.3.2. *The strongly continuous semigroup of operators $(S(t))_{t\geq 0}$ on the Banach space $X = L_1(\Omega, \mu)$ is said to be (i) substochastic if $S(t) \geq 0$ and $\|S(t)\| \leq 1$ for all $t \geq 0$, (ii) stochastic if, in addition, it satisfies $\|S(t)\psi\| = \|\psi\|$ for all non-negative $\psi \in X$.*

Theorem 2.3.3. *Let A be the generator of a C_0-semigroup in $X = L_1(\Omega)$ and let $B \in \mathcal{L}(D(A), X)$ be a positive operator. If for some $\lambda > s(A)$ the operator $\lambda I - A - B$ is resolvent positive, then $(A + B, D(A))$ generates a positive C_0-semigroup on X.*

Proof. [12, Theorem 5.13] □

Corollary 2.3.4. *Let $(S(t))_{t\geq 0}$ be the semigroup generated by $(A + B, D(A))$. Then $(S(t))_{t\geq 0}$ satisfies the Duhamel equation*

$$S(t)x = S_A(t)x + \int_0^t S(t-s)BS_A(s)x ds, \quad x \in D(A). \tag{2.17}$$

Proof. [12, Corollary 5.15] □

Theorem 2.3.5. *Let $X = L_1(\Omega)$ and suppose that the operators A and B satisfy: (1) $(A, D(A))$ generates a substochastic semigroup $(S_A(t))_{t\geq 0}$; (2) $D(B) \supset D(A)$ and $Bu \geq 0$ for $u \in D(B)_+$; (3) For all $u \in D(A)_+$,*

$$\int_\Omega (Au + Bu)d\mu \leq 0. \tag{2.18}$$

Then there exists a smallest substochastic semigroup, $(S_G(t))_{t\geq 0}$, generated by an extension, G, of $A + B$. Moreover, G is characterised by

$$(I - G)^{-1}\psi = \sum_{n=0}^{\infty}(I - A)^{-1}[B(I - A)^{-1}]^n\psi, \quad \forall\psi \in X. \tag{2.19}$$

Proof. [12, Corollary 5.17] □

Proposition 2.3.6. *Let D be a core of A. If $(S(t))_{t\geq 0}$ is another semigroup generated by an extension of $(A + B, D)$, then $S(t) \geq S_G(t)$.*

Proof. [12, Proposition 5.7] □

2.4 Semilinear semigroups

The success of linear semigroup theory in solving linear evolution equations has stimulated extensions of the linear ideas, which provide an opportunity for the examination of semilinear problems. Unlike the linear case, semilinear semigroup theory is not complete, yet it remains a useful and powerful method of analysing more difficult evolution equations.

Definition 2.4.1. (Semilinear Abstract Cauchy Problem) *Let X be a Banach space and let $(G, D(G))$ be an operator in X with associated semigroup $(S_G(t))_{t\geq 0}$. Furthermore, let N be a non-linear operator which maps a subset D of X into X where $D(G)\cap D$ is not empty. Then the abstract problem*

$$\frac{du}{dt}(t) = Gu(t) + Nu(t), \quad (t > 0); \quad u(0) = u_0 \in D(G) \cap D, \tag{2.20}$$

is called a semilinear *abstract Cauchy problem (ACP).*

Definition 2.4.2. *A function u is said to be a strong solution to the semilinear ACP* (2.20) *on $[0, t_0)$ if u is continuous on $[0, t_0)$, differentiable on $(0, t_0)$ and is such that $u(t) \in D(G) \cap D$ for all $t \in [0, t_0)$ and u satisfies* (2.20).

Proposition 2.4.3. *Let u be a strong solution on $[0, t_0)$ of the semilinear ACP* (2.20). *Then u satisfies the integral equation*

$$u(t) = S_G(t)u_0 + \int_0^t S_G(t-s)N(u(s))ds, \quad 0 \le t < t_0, \tag{2.21}$$

where $(S_G(t))_{t\ge 0}$ is the semigroup associated with the linear operator G.

Proof. [25, p. 108]. □

Definition 2.4.4. *$u : [0, t_0) \to X$ is said to be a mild solution to the semilinear ACP* (2.20) *if*

1. *u is continuous on $[0, t_0)$,*
2. *$u(t) \in D$ for all $t \in [0, t_0)$,*
3. *u satisfies* (2.21).

Some of the definitions required in the theorems are as follows:

Definition 2.4.5. (Local Lipschitz Condition) *An operator N on a Banach space X is said to satisfy a local Lipschitz condition if for any given $u_0 \in X$, there exists a closed ball,*

$$\overline{B}(u_0, r) = \{f \in X : \quad \|f - u_0\| \le r\},$$

such that $\|Nf - Ng\| \le C\|f - g\|$ for all $f, g \in \overline{B}(u_0, r)$ where C depends on u_0 and r.

Definition 2.4.6. (Fréchet Derivative) *If a linear operator $N_f \in \mathcal{L}(X)$ exists such that $N(f + \delta) = Nf + N_f\delta + \mathcal{H}(f, \delta)$ where $\mathcal{H}$ satisfies*

$$\lim_{\delta\to 0}\left(\frac{\|\mathcal{H}(f,\delta)\|}{\|\delta\|}\right) = 0,$$

then N is Fréchet differentiable at f and N_f is the Fréchet derivative.

Theorem 2.4.7. *Let $(G, D(G))$ be the generator of the strongly continuous semigroup $(S_G(t))_{t\ge 0}$ on X, let N be a non-linear operator and let X be a Banach space. If N satisfies a local Lipschitz condition on X, then the semilinear ACP has a unique, local in time, mild solution.*

Proof. [25, Theorem 3.20, p. 119]. □

Theorem 2.4.8. *Let $(G, D(G))$ generate the strongly continuous semigroup $(S_G(t))_{t\ge 0}$ on X and let N satisfy the local Lipschitz condition*

$$\|N(f) - N(g)\| \le \kappa\|f - g\|$$

for all f, g in the closed ball $\overline{B}(u_0, r) \subseteq D = D(N)$. If

1. *N is Fréchet differentiable at any* $f \in B(u_0, r)$ *and the Fréchet derivative* N_f *is such that* $\|N_f g\| \leq \kappa_1 \|g\|$ *for all* $f \in B(u_0, r)$, $g \in X$ *where* κ_1 *is a positive constant independent of* f *and* g*,*

2. *the Fréchet derivative is continuous with respect to* $f \in B(u_0, r)$ *such that*

$$\|N_f g - N_{f_1} g\| \to 0 \quad \textit{as} \quad \|f - f_1\| \to 0 \quad \textit{where} \quad f, f_1 \in B(u_0, r),$$

   for any given $g \in X$*,*

3. $u_0 \in D(G)$*,*

then there exists $t_1 > 0$ *such that the continuous solution on* $[0, t_1)$ *of* (2.21) *is strongly differentiable on* $[0, t_1)$ *and satisfies the equation* (2.20).

Proof. [25, Theorems 3.30 and 3.32]. □

Chapter 3

Groups Fragmentation Process in a Moving Medium

3.1 Introduction

This chapter discusses the dynamics of groups in social grouping population. Existence of global solutions to continuous non-local convection-fragmentation equations is investigated in spaces of distributions with finite higher moments. Assuming that the velocity field is divergence free, use is made of the method of characteristics and Friedrichs lemma [56] to show that the transport operator generates a stochastic dynamical system. This allows for the use of substochastic methods and Kato-Voigt perturbation theorem [12] to ensure that the combined transport-fragmentation operator is the infinitesimal generator of a strongly continuous semigroup. In particular, it is shown that the solution represented by this semigroup is conservative.

3.2 Motivation

The world of today is full of interactions that range from simple to dynamic. Many, if not all, of the Earth's processes affect human life. The Earth's processes are greatly stochastic and seem chaotic to the naked eye [85]. Climate change, global warming, the spread of diseases and pollution have aroused general interest in the type of relationships that living organisms have with each other, with their natural settlement and in interactions between these organisms and the physical environment. Most of the fundamental elements of ecology, ranging from individual behaviour to species abundance, diversity and population dynamics exhibit spatial variation. The spatial variation influences the relationships between living organisms and their natural environment and has a deep impact on the ecology. The rate of evolution of a population in an (aquatic) (eco)system may affect its balance. For instance, phytoplankton is a key food item in both aquaculture and mariculture since both use phytoplankton as food for the animals being farmed. So phytoplankton clusterings acting on a slower time scale may be catastrophic for these two types of farming. This is one of the sources of the motivation and it is therefore necessary to study the behaviour of some populations in their midst. However, a plethora of patterns can be noticed and are brought forth by using population modeling as a tool. For instance, Ecological Population Modeling is concerned with changes in population size and age distribution within a population as a consequence of interactions of organisms with the physical environment, with individuals of their own species, and with organisms of other species [82].

Mathematically, a model of animal dynamics could be represented by differential equations or integro-differential equations for more complex models, which describe the system using mathematical concepts and language. Partial differential equation models provide a means of combining organism movement with population processes and have been used extensively to elucidate the effects of spatial variation on populations. They also allow a better understanding of how complex interactions and processes work.

Clusters in social grouping include swarms of locust, mosquitoes, flies or midges, flocks of sheep, herds of elephants, schools of fish, the marine zooplankton and phytoplankton swarming. A group size can change due to splitting (fission or fragmentation) into groups of smaller sizes or combining (aggregation, fusion or coagulation) to form groups of bigger sizes. The dynamics in population grouping is not limited only to fragmentation and aggregation, but also includes other processes, like advection, diffusion, direction changing, flow (transport). It is obvious that some clusterings and direction changes act on a faster time scale (school of fish) or a slower time scale (herd of elephants). There are short and long-term changes in the size and age composition of populations, and the biological and environmental processes influencing these changes. Population dynamics deals with the way populations are affected by birth and death rates, and also by immigration and emigration. A typical example of transport problem with fragmentation and aggregation is the dynamics of phytoplankton in a flowing water.

In phytoplankton dynamics, a system of particles called TEP (Transparent Exopolymer Particles) plays a major role. They are by-product of the growth of phytoplankton and their stickiness causes cells to remain together upon contact [32, 73]. On the other hand, the low level of concentration of TEP results in fragmentation of the aggregate due to external causes, like currents or turbulence on one hand, and internal unspecified forces of biotic nature on the other. The aggregate size can change due to splitting, death, growth or combining aggregates into bigger ones.

The global question of interest is how clustering of phytoplankton affects the evolution of the population throughout a seascape and what the consequences are for the ecology and other populations involved. Because population of phytoplankton includes numerous groups, even a group-level description is too numerically costly to use to model the entire population. Instead, one turns to the statistical description that estimates the frequency of groups of various sizes and characteristics, based on the rates at which such groups appear and disappear. It is then possible to use mathematical models of animal grouping (with fragmentation, aggregation, transport, direction changing processes) to establish a direct linkage between the behaviour of individuals and their consequences for ecological or environmental dynamics.

3.3 Description of the model

If we define a spatial dynamical system in which locally a group-size distribution can be estimated, but in which we also allow immigration and emigration from adjacent areas with different group-size distributions, we obtain the general model of the dynamics of phytoplankton as described above and which is a spatially explicit group-size distribution model as presented in [67]:

$$\frac{\partial}{\partial t}p_n + div(\omega p_n) = \frac{1}{2}\sum_{m=1}^{n-1} c(m, n-m)p_m p_{n-m} - \sum_{m=1}^{\infty} c(n,m)p_n p_m$$
$$-\frac{1}{2}\sum_{m=1}^{n-1} h(n,m)p_n + \sum_{m=n+1}^{\infty} h(m,n)p_m, \tag{3.1}$$

where the velocity $\omega = \omega(x, n)$ of the transport is supposed to be a known quantity, depending on the size n of aggregates and their position x. $p_n \equiv p(t, x, n)$ is the density of n-groups (i.e. groups of size n) at the position x, with the velocity ω at time t. Equation (3.1) is really complex: the second member on its left-hand side represents the flow process (the transport part), while on the right-hand side, the terms represent respectively the fusion to form groups of size n (that is, the gain part of the coagulation process), the fusion of groups of size n (the loss part of the coagulation), the fission of groups of size n (the loss due to the fragmentation) and the fission to form groups of size n (the gain due to the fragmentation). Then $c(n, m) \geq 0$ is the fusion rate, that is, the rate at which n-groups and m-groups joint to form $n + m$-groups and $h(n, m) \geq 0$ $(n > m)$ is the number of m-groups produced upon splitting of n-groups. The analysis of such a model required the researcher to proceed step by step as indicated in the following sections.

3.4 Well posedness of the transport problem with fragmentation

First, the study is interested in solving the problem (3.1) with the transport and fragmentation processes only. So the following Cauchy Problem (the model with an initial condition) is considered:

$$\frac{\partial}{\partial t}p(t, x, n) = -div(\omega(x, n)p(t, x, n)) - a_n p(t, x, n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t, x, m)$$

$$p(0, x, n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ... \tag{3.2}$$

where $a_n = \frac{1}{2}\sum_{m=1}^{n-1} h(n,m)$ is the average fragmentation rate, that is the average number at which clusters of size n undergo splitting, $b_{n,m} \geq 0$ is the average number of n-groups produced upon the splitting of m-groups and given by $h(m,n) = b_{n,m}a_m = \frac{1}{2}\sum_{k=1}^{m-1} b_{n,m}h(m,k)$. The coefficients a_n and $b_{n,m}$ give a randomly spatial distribution and are better to analyse than the previous ones c and h which describe a binary process. The space variable x is supposed to vary in the whole of $\mathbb{R}^3 = \Omega$. The function $\overset{o}{p}_n$ represents the density of n-groups at the beginning of observation ($t = 0$) and it is integrable with respect to x over the full space $\mathbb{R}^3$, this integral multiplied by n is summable so that the total initial population is finite, see the definition of $U(t)$ below . The necessary assumptions that will be useful in the analysis are introduced below.

3.4.1 Fragmentation equation

Since a group of size $m \leq n$ cannot split to form a group of size n, we require that $b_{n,m} = 0$ for all $m \leq n$ and

$$a_1 = 0, \quad \sum_{m=1}^{n-1} m b_{m,n} = n, \quad (n = 2, 3, ...), \tag{3.3}$$

meaning that a cluster of size one cannot split and the sum of all individuals obtained by fragmentation of an n-group is equal to n. To proceed, it is necessary to recall the following assumptions and spaces which are crucial for the analysis. Because the total number of individuals in a population is not modified by interactions among groups, the following conservation law is supposed to be satisfied:

$$\frac{d}{dt}U(t) = 0 \tag{3.4}$$

where $U(t) = \int_{\mathbb{R}^3}\sum_{n=1}^{\infty} np(t,x,n)dx = \sum_{n=1}^{\infty} n\int_{\mathbb{R}^3} p(t,x,n)dx$ is the total number of individuals in the space. Since $p_n = p(t,x,n)$ is the density of groups of size n at the position x and time t and that mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$\begin{aligned} \mathcal{X}_1 := \{\mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \\ \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n|g_n(x)|dx < \infty\}. \end{aligned} \tag{3.5}$$

Recall that work is done in this space because they have many desirable properties, like controlling the norm of their elements which, in the case of this study, represents the total mass (or total number of individuals) of the system and must be finite. Because uniqueness of solutions of the system (3.2) proved to be a more difficult problem [15], the analysis is restricted to a smaller class of functions, so the introduction of the following class of Banach spaces (of distributions with finite higher moments)

$$\begin{aligned} \mathcal{X}_r &:= \{\mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \\ \|\mathbf{g}\|_r &:= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r |g_n(x)| dx < \infty\}, \end{aligned} \tag{3.6}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$. We assume that for each $t \geq 0$, the function $(x, n) \longrightarrow p(t, x, n) = p_n(t, x)$ is such that $\mathbf{p} = (p_n(t, x))_{n=1}^{\infty}$ is from the space $\mathcal{X}_r$ with $r \geq 1$. In $\mathcal{X}_r$, (3.2) can be rewritten in more compact form,

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{p} &= \mathfrak{D}\mathbf{p} - \mathcal{A}\mathbf{p} + \mathfrak{B}\mathcal{A}\mathbf{p} := \mathfrak{D}\mathbf{p} + \mathcal{F}\mathbf{p}, \\ \mathbf{p}_{|t=0} &= \mathring{\mathbf{p}}. \end{aligned} \tag{3.7}$$

Here, $\mathbf{p}$ is the vector $(p(t,x,n))_{n\in\mathbb{N}}$, $\mathcal{A}$ is the diagonal matrix $(a_n)_{n\in\mathbb{N}}$, $\mathfrak{B} = (b_{n,m})_{1\leq n\leq m-1, m\geq 2}$, $\mathfrak{D}$ the transport expression defined as $(p(t,x,n))_{n\in\mathbb{N}} \longrightarrow (-div(\omega(x,n)p(t,x,n)))_{n=1}^{\infty}$, $\mathring{\mathbf{p}}$ the initial vector $(\mathring{p}_n(x))_{n\in\mathbb{N}}$ which belongs to $\mathcal{X}_r$ and $\mathcal{F}$ the fragmentation expression defined by

$$(\mathcal{F}\mathbf{p})_{n=1}^{\infty} := \left(-a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m)\right)_{n=1}^{\infty}.$$

In this study, for any subspace $S \subseteq \mathcal{X}_r$, S_+ will denote the subset of S defined as $S_+ = \{\mathbf{g} = (g_n)_{n=1}^{\infty} \in S; g_n(x) \geq 0, n \in \mathbb{N}, x \in \mathbb{R}^3\}$. Note that any $\mathbf{g} \in (\mathcal{X}_r)_+$ possesses moments

$$M_q(\mathbf{g}) := \sum_{n=1}^{\infty} n^q g_n$$

of all orders $q \in [0, r]$. Although some of the results, such as the existence of a strongly continuous semigroup of contractions associated with the transport problem and the fragmentation process hold in $\mathcal{X}_r$ for all $r \geq 1$, other important properties will require $r > 1$, especially when considering the coagulation process. In fact, by the substochastic semigroup theory developed in [12], one can look at (3.1) as a perturbation of the transport-fragmentation semigroup by the non-linear operator defining the coagulation process. Imposing $r > 1$ ensures that a significant amount of mass after fragmentation is concentrated in small particles. This has the physical interpretation that surface effects are reduced, i.e. it is unlikely that a large cluster will fragment into large groups, therefore making more clusters with small sizes and concentrated at the origin. However, the calculations are practically the same for both cases and, in fact for other spaces corresponding to other moments of the solution. In $\mathcal{X}_r$, operators $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A}\mathbf{g} := (a_n g_n)_{n=1}^{\infty}, \quad D(\mathbf{A}) := \{\mathbf{g} \in \mathcal{X}_r : \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r a_n |g_n(x)| dx < \infty\}; \tag{3.8}$$

$$\mathbf{Bg} := (B_n g_n)_{n=1}^{\infty} = \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m g_m \right)_{n=1}^{\infty}, \quad D(\mathbf{B}) := D(\mathbf{A}). \tag{3.9}$$

Throughout, we assume that the coefficients a_n and $b_{n,m}$ satisfy the mass conservation conditions (3.3). Now let us prove that $\mathbf{B}$ is well-defined on $D(\mathbf{A})$ as stated in (3.9). Following the same method as in [15] and using the condition (3.3), we obtain

$$n^r - \sum_{m=1}^{n-1} m^r b_{m,n} \geq n^r - (n-1)^{r-1} \sum_{m=1}^{n-1} m b_{m,n} = n^r - n(n-1)^{r-1} \geq 0.$$

Hence,

$$\sum_{m=1}^{n-1} m^r b_{m,n} \leq n^r \tag{3.10}$$

for $r \geq 1$, $n \geq 2$. Note that the equality holds for $r = 1$. For every $\mathbf{g} \in D(\mathbf{A})$, we have

$$\begin{aligned}
\|\mathbf{Bg}\|_r &= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m |g_m(x)| \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{\infty} n^r b_{n,m} \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{m-1} n^r b_{n,m} \right) dx \\
&\leq \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| m^r dx \\
&= \|\mathbf{Ag}\|_r \\
&< \infty,
\end{aligned}$$

where the inequality (3.10) has been used and the fact that $m \geq m-1$. Then $\|\mathbf{Bg}\|_r \leq \|\mathbf{Ag}\|_r$, for all $\mathbf{g} \in D(\mathbf{A})$, so that $D(\mathbf{B}) := D(\mathbf{A})$ can be taken and $(\mathbf{A}+\mathbf{B}, D(\mathbf{A}))$ is well-defined.

3.4.2 Cauchy problem for the transport operator in $\Lambda = \mathbb{R}^3 \times \mathbb{N}$

The primary objective in this section is to analyse the solvability of the Cauchy problem for the transport equation

$$\frac{\partial}{\partial t} p(t,x,n) = -div(\omega(x,n)\, p(t,x,n)), \tag{3.11}$$

$$p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ...$$

in the space $\mathcal{X}_r$.
Let us fix $n \in \mathbb{N}$. We consider the function $\omega_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\omega_n(x) = \omega(x, n)$ and $\tilde{\mathcal{D}}_n$ the expression appearing on the right-hand side of the equation (3.11). Then

$$\begin{aligned} \tilde{\mathcal{D}}_n[p(t,x,n)] &:= -div\,(\omega(x,n)\,p(t,x,n)) \\ &= (\nabla \cdot \omega(x,n))p(t,x,n) + \omega(x,n) \cdot (\nabla p(t,x,n)). \end{aligned} \tag{3.12}$$

We assume that ω_n is divergence free and globally Lipschitz continuous. Then $div\,\omega_n(x) := \nabla \cdot \omega(x,n) = 0$ and (3.12) becomes

$$\tilde{\mathcal{D}}_n[p(t,x,n)] := \omega(x,n) \cdot (\nabla p(t,x,n)). \tag{3.13}$$

For $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, the initial value problem

$$\begin{aligned} \frac{d\dagger_n}{ds} &= \omega_n(\dagger_n), \qquad s \in \mathbb{R} \\ \dagger_n(t) &= x, \end{aligned} \tag{3.14}$$

has one and only one solution $\dagger_n(s)$ taking values in $\mathbb{R}^3$. Thus, we can consider the function $\phi : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by the condition that for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$s \longrightarrow \phi(x,t,s), \qquad s \in \mathbb{R}$$

is the only solution of the Cauchy Problem (3.14). The integral curves given by the ϕ-parameter family $(\dagger_n)_\phi$ (with $\dagger_n(s) = \phi(x,t,s)$, $s \in \mathbb{R}$, the only solution of (3.14)) are called the characteristics of $\tilde{\mathcal{D}}_n$. The function ϕ possesses many desirable properties [45, 81, 83] that will be relevant for studying the transport operator in $\mathcal{X}_r$. Some of them are listed in [12, Proposition 10.1].

Mathematical setting

It is observed that the operators on the right-hand side of (3.7) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus, there is a need to work with parameter-dependent operators that can be "glued"together in such a way that the resulting operator inherits the properties of the individual components. Let us provide a framework for such a technique called the method of semigroups with a parameter [12]. We consider the space $\mathcal{X} := L_p(S, X)$ where $1 \leq p < \infty$, (S, dm) is a measure space and X a Banach space. Let us suppose that we are given a family of operators $\{(A_s, D(A_s))\}_{s\in S}$ in X and define the operator $(\mathbb{A}, D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae,

$$\mathcal{D}(\mathbb{A}) := \{g \in \mathcal{X}; g(s) \in D(A_s) \text{ for almost every } s \in S,\ \ \mathbb{A}g \in \mathcal{X}\}, \tag{3.15}$$

and, for $g \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}g)(s) := A_s g(s), \tag{3.16}$$

for every $s \in S$. The following proposition is obtained

Proposition 3.4.1. *(see [12], Proposition 3.28). If for almost any $s \in S$ the operator A_s is m-dissipative in X, and the function $s \longrightarrow R(\lambda, A_s)g(s)$ is measurable for any $\lambda > 0$ and $g \in \mathcal{X}$, then the operator $\mathbb{A}$ is an m-dissipative operator in $\mathcal{X}$. If $(G_s(t))_{t\geq 0}$ and $(\mathcal{G}(t))_{t\geq 0}$ are the semigroups generated by A_s and $\mathbb{A}$, respectively, then for almost every $s \in S$, $t \geq 0$, and $g \in \mathcal{X}$, we have*

$$[\mathcal{G}(t)g](s) := G_s(t)g(s). \tag{3.17}$$

Using the above ideas, we introduce relevant operators in the present applications. In the transport part of (3.7), the n variable is the parameters and x is the main variable. We set

$$X_x := L_1(\mathbb{R}^3, dx) := \{\psi : \|\psi\| = \int_{\mathbb{R}^3} |\psi(x)|dx < \infty\}$$

and define in X_x the operators $(\mathcal{D}_n, D(\mathcal{D}_n))$ as

$$\begin{aligned} &\mathcal{D}_n p_n = \tilde{\mathcal{D}}_n p_n, \quad \text{with } \tilde{\mathcal{D}}_n p_n \ \text{ represented by (3.13)} \\ &D(\mathcal{D}_n) := \{p_n \in X_x, \ \ \mathcal{D}_n p_n \in X_x\}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.18}$$

Then we introduce the operator $\mathbf{D}$ in $\mathcal{X}_r$ defined by

$$\begin{aligned} &[\mathbf{D}\mathbf{p}](x,n) = [\mathfrak{D}\mathbf{p}](x,n) \\ &D(\mathbf{D}) = \{\mathbf{p} = (p_n)_{n\in\mathbb{N}} \in \mathcal{X}_r,\, p_n \in D(\mathcal{D}_n) \,\text{for almost every}\, n \in \mathbb{N},\, \mathbf{D}\mathbf{p} \in \mathcal{X}_r\}. \end{aligned} \tag{3.19}$$

Now, the transport operator $\mathbf{D}$ can properly been studied. Using the above proposition in the application, we can take $\mathbb{A} = \mathbf{D}$, $\mathcal{X} = \mathcal{X}_r = L_1(\mathbb{N}, X_x) = L_1(\Lambda, d\mu dm_r) = L_1(\mathbb{R}^3 \times \mathbb{N}, d\mu dm_r)$, where $\mathbb{N}$ is equipped with the weighted counting measure dm_r with weight n^r and $d\mu = dx$ is the Lebesgue measure in $\mathbb{R}^3$. In the notation of the proposition, $(\mathbb{N}, dm_r) = (S, dm)$, $X_x = X$ and $A_s = \mathcal{D}_n$, therefore $(\mathcal{D}_n, D(\mathcal{D}_n))_{n\in\mathbb{N}}$ is a family of operators in X_x and using (3.16), we get

$$(\mathbf{D}\mathbf{p})_n := \mathcal{D}_n p_n. \tag{3.20}$$

Here, $\mathcal{D}_n p_n$ is understood in the sense of distribution. Precisely speaking, if we take $C_0^1(\mathbb{R}^3)$ as the set of test functions, $p_n \in D(\mathcal{D}_n)$ if and only if $p_n \in X_x$ and there exists $g_n \in X_x$ such that

$$\int_{\mathbb{R}^3} \xi g_n d\mu = \int_{\mathbb{R}^3} p_n \partial \cdot (\xi\omega_n) d\mu = \int_{\mathbb{R}^3} p_n \omega_n \cdot \partial\xi d\mu, \tag{3.21}$$

for all $\xi \in C_0^1(\mathbb{R}^3)$, where

$$\omega_n \cdot \partial\xi(x) := \sum_{j=1}^{3} \omega_{n,j} \partial_j \xi(x) \tag{3.22}$$

with $\omega_{n,j} = \omega_j(x,n)$, the j^{th} component of the velocity $\omega(x,n)$. The middle term in (3.21) exists as ω_n is globally Lipschitz continuous, and the last equality follows as ω_n is divergence-free. If this is the case, we define $(\mathcal{D}_n p_n)_{n\in\mathbb{N}} = \mathbf{D}\mathbf{p} = \mathbf{g} = (g_n)_{n\in\mathbb{N}}$.

Now we can show that the operator $\mathbf{D}$ is the generator of a stochastic semigroup on $\mathcal{X}_r$

Theorem 3.4.2. *If for each $n \in \mathbb{N}$ the function $\omega_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is globally Lipschitz continuous and divergence-free, then the operator $(D(\boldsymbol{D}), \boldsymbol{D})$ defined by (3.19) is the generator of a strongly continuous stochastic semigroup $(G_{\boldsymbol{D}}(t))_{t\geq 0}$, given by*

$$[G_{\boldsymbol{D}}(t)\boldsymbol{p}](x) = (p_n(\phi(x,t,0)))_{n\in\mathbb{N}} \tag{3.23}$$

for any $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in \mathcal{X}_r$ and $t > 0$.

Proof. According to the relation (3.17), it suffices to prove that:

$$[G_{\mathcal{D}_n}(t)p_n](x) = p_n(\phi(x,t,0)), \tag{3.24}$$

for each $p_n \in D(\mathcal{D}_n)$, where $(G_{\mathcal{D}_n}(t))_{t\geq 0}$ is a strongly continuous stochastic semigroup generated by $\mathcal{D}_n$.
Let $(Z_0(t))_{t\geq 0}$ be the family defined by the right-hand side of the relation (3.24). The proof of the theorem will follow three steps.

(i): First, It is shown that $(Z_0(t))_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators. We need some properties of ϕ as listed in [12, 81, 83] and given as follows: The function ϕ has the following properties.

- (p_1): $\phi(x,t,t) = x$ for all $x \in \mathbb{R}^3,\ t \in \mathbb{R}$;
- (p_2): $\phi(\phi(x,t,s),s,\tau) = \phi(x,t,\tau)$ for all $x \in \mathbb{R}^3,\ t,\ s,\ \tau \in \mathbb{R}$;
- (p_3): $\phi(x,t,s) = \phi(x,t-s,0) = \phi(x,0,s-t)$ for all $x \in \mathbb{R}^3,\ t,\ s \in \mathbb{R}$;
- (p_4): $|\phi(x,t,s) - \phi(y,t,s)| \leq e^{K|t-s|}|x-y|$ for all $x \in \mathbb{R}^3,\ t,\ s \in \mathbb{R}$;
- (p_5): Function $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \ni (x,t,s) \to \phi(x,t,s)$ is continuous;
- (p_6): The transformation $\mathcal{T}$ defined by $t = t,\ s = s,\ y = \phi(x,t,s)$ is a topological homeomorphism which is bimeasurable and its inverse $\mathcal{T}^{-1}$ is represented by $t = t,\ s = s,\ x = \phi(y,s,t)$;
- (p_7): For all $t,\ s \in \mathbb{R}$ the transformation of $\mathbb{R}^3$ onto itself defined by $y = \phi(x,t,s)$ is measure-preserving.

Then, by the properties (p_6) and (p_7), it is observed that for any p_n, the composition $(x,t) \to p_n(\phi(x,t,0))$, in (3.24) is a measurable function satisfying the equality

$$\|Z_0(t)p_n\| = \|p_n\|, \tag{3.25}$$

hence the family $(Z_0(t))_{t\geq 0}$ consists of bounded linear operators from $X_x \longrightarrow X_x$. Then, the following relations can easily been verified:

- (i_a): $Z_0(0) = I$
- (i_b): $Z_0(t+s) = Z_0(t)Z_0(s)$, for all $t,\ s \in \mathbb{R}$;
- (i_c): $\lim_{t\to 0^+} \|Z_0(t)p_n - p_n\| = 0$, for each $p_n \in X_x$.

In fact, (i_a) and (i_b) follow immediately from the properties (p_1) and (p_2). To prove (i_c), we can follow the argument of Example 3.10 in [12]. Thus, it is enough to show (i_c) for $p_n \in C_0^\infty(\mathbb{R}^3)$. For such p_{n_s}, we have $\lim_{t\to 0^+}(Z_0(t)p_n)(x) = p_n(x)$ for all $x \in \mathbb{R}^3$. Furthermore, if $|p_n(x)| \leq M$ for all $x \in \mathbb{R}^3$ then $|(Z_0(t)p_n)(x)| \leq M$ for all $x \in \mathbb{R}^3$ and, because the support of $Z_0(t)p_n$ is bounded, the Lebesgue dominated convergence theorem shows that (i_c) is satisfied. Thus, $(Z_0(t))_{t\geq 0}$ is a C_0-semigroup.

(ii): Secondly, we prove that the generator T_0 of $(Z_0(t))_{t\geq 0}$ is an extension of $\mathcal{D}_n$. Let $\mathcal{Y}$ be the set of real-valued functions defined on $\mathbb{R}^3$, are Lipschitz continuous, and compactly supported. Obviously, $\mathcal{Y} \subset D(\mathcal{D}_n)$ because if $p_n \in \mathcal{Y}$, then the first-order partial derivatives of p_n are measurable, bounded, and compactly supported and thus, multiplied by Lipschitz continuous functions of ω_n, belong to $L_1(\mathbb{R}^3, d\mu)$. For a fixed $p_n \in \mathcal{Y}$, we now denote by ϑ the real-valued function defined on $\mathbb{R}^3 \times R^+$ by

$$\vartheta(x,t) = (Z_0(t)p_n)(x).$$

From the previous considerations and properties (p_3)-(p_5), there exists a measurable subset E of $\mathbb{R}^3 \times R^+$, with $\mu(\mathbb{R}^3 \times R^+ \setminus E) = 0$, such that at each point $(x,t) \in E$, the function ϑ has measurable first-order partial derivatives. In particular,

$$\frac{\partial \vartheta}{\partial t}(x,t) = (Z_0(t)\mathcal{D}_n p_n)(x), \quad (x,t) \in E,$$

and therefore, if we let $\lambda_{p_n} := \text{ess sup}_{(x)\in\mathbb{R}^3}|\mathcal{D}_n p_n|$, then

$$|\partial_t \vartheta(x,t)| \leq \lambda_{p_n}$$

for any $(x,t) \in E$.

From this and from part (i) of the proof, it follows that

$$\|h^{-1}(Z_0(h)p_n - p_n) - \mathcal{D}_n p_n\| = \|h^{-1}\int_0^h (Z_0(s) - I)\mathcal{D}_n p_n ds\| \to 0$$

as $h \to 0^+$. This proves that $\mathcal{Y} \subset D(T_0)$ and that $T_0 p_n = \mathcal{D}_n p_n$, for all $p_n \in \mathcal{Y}$. Next we prove that $\mathcal{Y}$ is a core of $\mathcal{D}_n$, that is, that $(\mathcal{D}_n, D(\mathcal{D}_n))$ is the closure of $(\mathcal{D}_n, \mathcal{Y})$. Let ϖ_ε, $\varepsilon > 0$, be a mollifier and for p_n, let $\varpi_\varepsilon * p_n$ be the mollification of p_n. We use the Friedrichs lemma, [56, pp. 313-315], or [80, Lemma 1.2.5], which states that there is $C > 0$, independent of ε, such that for any L_r function p_n, $1 \leq r < \infty$, we have

$$\|\mathcal{D}_n(\varpi_\varepsilon * p_n) - \varpi_\varepsilon * \mathcal{D}_n p_n\| \leq C\|p_n\| \tag{3.26}$$

and

$$\lim_{\varepsilon\to 0^+}(\|\varpi_\varepsilon * p_n - p_n\| + \|\mathcal{D}_n(\varpi_\varepsilon * p_n) - \mathcal{D}_n p_n\|) = 0. \tag{3.27}$$

Estimates (2.9) in [12] and the relation (3.26) above imply

$$\|\mathcal{D}_n(\varpi_\varepsilon * p_n)\| \leq C\|p_n\| + \|\mathcal{D}_n p_n\|$$

which shows that the mollification $p_n \to \varpi_\varepsilon * p_n$ is a continuous operator in $D(\mathcal{D}_n)$ (equipped with the graph norm) uniformly bounded with respect to ε.

Next, it is observed that the subset of $D(\mathcal{D}_n)$ consisting of compactly supported functions is dense in $D(\mathcal{D}_n)$ with the graph norm. Indeed, let $p_n \in D(\mathcal{D}_n)$. Because both p_n, $\mathcal{D}_n p_n \in X_x$, the absolute continuity of the Lebesgue integral implies that for any given $\delta > 0$, there exists a compact subset Ω' of $\mathbb{R}^3$ such that

$$\int\limits_{\mathbb{R}^3\setminus\Omega'} (|p_n| + |\mathcal{D}_n p_n|)d\mu < \delta$$

For this Ω' we choose $\psi \in C_0^\infty(\mathbb{R}^3)$ satisfying $0 \leq \psi(x) \leq 1$ for all $(x) \in \mathbb{R}^3$, and $\psi(x) = 1$ for all $(x) \in \Omega'$. Now it is easy to see that $\psi p_n \in D(\mathcal{D}_n)$ and has a compact support. Moreover,

$$\|\psi p_n - p_n\| \leq 2 \int\limits_{\mathbb{R}^3\setminus\Omega'} |p_n| d\mu,$$

$$\|\mathcal{D}_n(\psi p_n) - \mathcal{D}_n p_n\| \leq 2 \int\limits_{\mathbb{R}^3\setminus\Omega'} |\mathcal{D}_n p_n| d\mu + L \int\limits_{\mathbb{R}^3\setminus\Omega'} |p_n| d\mu$$

where $L = sup|\mathcal{D}_n\psi|$ can be made independent of Ω' due to the fact that $\mathbb{R}^3$ is the whole space.

Let $p_n \in D(\mathcal{D}_n)$ be compactly supported. It is well known [12] that $\varpi_\varepsilon * p_n$ is infinitely differentiable and compactly supported and thus belongs to $\mathcal{Y}$. Equation (3.27) yields that $\varpi_\varepsilon * p_n \to p_n$ as $\varepsilon \to 0^+$ in the graph norm of $D(\mathcal{D}_n)$. Because it has been shown above that compactly supported functions from $D(\mathcal{D}_n)$ are dense in $D(\mathcal{D}_n)$, it is observed that $(\mathcal{D}_n, D(\mathcal{D}_n))$ is the closure of $(\mathcal{D}_n, \mathcal{Y})$ and, because T_0 is a closed extension of $(\mathcal{D}_n, \mathcal{Y})$ and $\mathcal{D}_n \subset T_0$ is obtained.

(iii): Let us conclude the proof by recognising that $D(T_0) \subset D(\mathcal{D}_n)$ so that the operators T_0 and $\mathcal{D}_n$ coincide and $(G_{\mathcal{D}_n}(t))_{t\geq 0} = (Z_0(t))_{t\geq 0}$. Suppose $p_n \in D(T_0)$. Then for any fixed $\lambda > 0$, there exists a unique $g_n \in X_x$ such that $p_n = (\lambda I - T_0)^{-1} g_n$. For any $\psi \in C_0^1(\mathbb{R}^3)$, we obtain, using (3.21),

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{D}_n p_n \psi d\mu \\
&= \int_{\mathbb{R}^3} p_n(x)(\omega_n \cdot \partial\psi)(x)dx \\
&= \int_{\mathbb{R}^3} \left(\int_0^\infty e^{-\lambda t} g_n(\phi(x,t,0))dt \right) (\omega_n \cdot \partial\psi)(x)dx \\
&= \int_0^\infty \left(\int_{\mathbb{R}^3} e^{-\lambda t} g_n(\phi(x,t,0))(\omega_n \cdot \partial\psi)(x)dx \right) dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^3} e^{-\lambda t} g_n(y)(\omega_n \cdot \partial\psi)(\phi(y,0,t))(x)dy \right) dt \\
&= \int_{\mathbb{R}^3} \left(\int_0^\infty e^{-\lambda t} \frac{d}{dt}\psi(\phi(y,0,t))dt \right) g_n(y)dy \\
&= \int_{\mathbb{R}^3} \left(e^{-\lambda t}\psi(\phi(y,0,t))|_0^\infty g_n(y)dy + \lambda \int_{\mathbb{R}^3} \left(\int_0^\infty e^{-\lambda t}\psi(\phi(y,0,t))dt \right) \right) g_n(y)dy \\
&= -\int_{\mathbb{R}^3} g_n(y)\psi(y)dy + \lambda \int_{\mathbb{R}^3} \left(\int_0^\infty e^{-\lambda t} g_n(\phi(x,0,t))dt \right) \psi(x)dx \\
&= -\int_{\mathbb{R}^3} (g_n - \lambda p_n)\psi d\mu.
\end{aligned}$$

This implies that $p_n \in D(\mathcal{D}_n)$. Hence $T_0 \subset \mathcal{D}_n$ and $\mathcal{D}_n p_n = T_0 p_n$.

□

Remark 1. (Conservativeness of the transport model)
Because the flow process does not modify the total number of individuals in the system, let us show that the model (3.11) is conservative in the space $\mathcal{X}_r$, that is, the law (3.4) is satisfied. We have proven that the semigroup generated by the operator $\mathbf{D}$ is stochastic, this implies

$$\begin{aligned}
0 &= \int_{\Lambda} \mathbf{D}\mathbf{p}\, d\mu dm_r, \\
&= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \mathcal{D}_n p(t,x,n)dx, \qquad (3.28)
\end{aligned}$$

for all $\mathbf{p} \in D(\mathbf{D})$ $t \geq 0$, $r \geq 1$. Thus, $\int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n\mathcal{D}_n p(t,x,n)dx = 0$ for all $t \geq 0$ which leads to

$$\begin{aligned} \frac{d}{dt}U(t) &= \frac{d}{dt}\left(\sum_{n=1}^{\infty} n \int_{\mathbb{R}^3} p(t,x,n)dx\right) \\ &= \sum_{n=1}^{\infty} n \int_{\mathbb{R}^3} \partial_t p(t,x,n)dx \\ &= \sum_{n=1}^{\infty} n \int_{\mathbb{R}^3} \mathcal{D}_n p(t,x,n)dx \\ &= 0 \end{aligned}$$

and therefore proving the conservativeness of the transport model in (3.19).

3.5 Perturbed transport-fragmentation problems

Attention is now shifted to the transport problem with the loss part of the fragmentation process. We assume that there are two constants $0 < \theta_1$ and θ_2 such that for every $x \in \mathbb{R}^3$,

$$\theta_1 \alpha_n \leq a_n(x) \leq \theta_2 \alpha_n, \tag{3.29}$$

with $\alpha_n \in \mathbb{R}_+$ and independent of the state variable x. Then a_n is bounded for each $n \in \mathbb{N}$ and the loss operator $(A_n, D(A_n))$ can be defined in X_x as $A_n(x) = a_n(x)$ with $D(A_n) = X_x = L_1(\mathbb{R}^3)$. The corresponding abstract Cauchy problem reads as

$$\begin{aligned} \partial_t p(t,x,n) &= \mathcal{D}_n p(t,x,n) - A_n p(t,x,n) = F_n p(t,x,n) \\ p(0,x,n) &= \overset{o}{p}_n(x), \qquad n = 1,2,3,... \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} F_n &= \mathcal{D}_n - A_n \\ D(F_n) &:= D(\mathcal{D}_n), \end{aligned} \tag{3.31}$$

(where we have made use of (3.29) to have $D(F_n) = D(\mathcal{D}_n) \cap D(A_n) = D(\mathcal{D}_n) \cap L_1(\mathbb{R}^3)$ and
$D(\mathcal{D}_n) \subseteq L_1(\mathbb{R}^3)$). The problem (3.30) can be rewritten in $\mathcal{X}_r$ in more compact form

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{p} &= \mathbf{D}\mathbf{p} - \mathbf{A}\mathbf{p} = \mathbf{F}\mathbf{p} \\ \mathbf{p}_{|t=0} &= \overset{o}{\mathbf{p}}. \end{aligned} \tag{3.32}$$

with $\mathbf{F} = \mathbf{D} - \mathbf{A}$, where $\mathbf{A}$ is defined by (3.8). It is necessary to characterise the domain of $\mathbf{F}$. According to the condition (3.29), for each $n \in \mathbb{N}$, the operator A_n is the generator of a C_0-semigroup of contractions, say $(G_{A_n}(t))_{t\geq 0}$. The following theorem holds.

Theorem 3.5.1. *Assume that (3.29) is satisfied for each $n \in \mathbb{N}$.*
The operator $(\boldsymbol{F}, D(\boldsymbol{F})) = (\boldsymbol{D} - \boldsymbol{A}, D(\boldsymbol{D}) \cap D(\boldsymbol{A}))$ is the generator of a substochastic semigroup $(G_{\boldsymbol{F}}(t))_{t\geq 0}$ given by

$$[G_{\boldsymbol{F}}(t)\boldsymbol{p}](x) = \left(\lim_{v \longrightarrow \infty} \left[G_{\mathcal{D}_n}\left(\frac{t}{v}\right) G_{A_n}\left(\frac{t}{v}\right)\right]^v p_n \right)_{n\in\mathbb{N}} \tag{3.33}$$

for $\boldsymbol{p} \in \mathcal{X}_r$ and $t > 0$, where $(G_{\mathcal{D}_n}(t))_{t\geq 0}$ is defined by (3.24).

Proof. First of all it should be proven that for a fix $n \in \mathbb{N}$, F_n is the generator of a substochastic semigroup $(G_{F_n}(t))_{t\geq 0}$ in X_x given by

$$[G_{F_n}(t)p_n](x) = \lim_{v \longrightarrow \infty} \left[G_{\mathcal{D}_n}\left(\frac{t}{v}\right) G_{A_n}\left(\frac{t}{v}\right)\right]^v p(t, x, n) \tag{3.34}$$

for each $p_n \in D(F_n) := D(\mathcal{D}_n)$.
We need to show that $\mathcal{D}_n$ and A_n satisfy the conditions of the Corollary 5.5 in [74].
(a) From Theorem 5.4.1 and assumption (3.29), it is observed that $\mathcal{D}_n$ and A_n are generator of positive semigroups of contractions, then

$$\|G_{\mathcal{D}_n}(t)\| \leq 1 = 1e^{0t}, \text{ and } \|G_{A_n}(t)\| \leq 1 = 1e^{0t} \text{ for all } t \geq 0.$$

Thus, $\mathcal{D}_n$, $A_n \in \mathcal{G}(1, 0)$ and $G_{\mathcal{D}_n}(t) \geq 0$, $\quad G_{A_n}(t) \geq 0$ for all $t \geq 0$.
(b) According to Hille-Yosida Theorem 2.2.4, $\mathcal{D}_n$ is closed and densely defined in X_x and because $L_1(\mathbb{R}^3) = D(A_n) \supset D(\mathcal{D}_n)$, we have $D(\mathcal{D}_n) \cap D(A_n) = D(\mathcal{D}_n)$ is dense in X_x.
(c) From the above condition (a), we can write

$$\begin{aligned} \| \left(G_{\mathcal{D}_n}(t)G_{A_n}(t)\right)^v \| &\leq \|G_{\mathcal{D}_n}(t)\|^v \|G_{A_n}(t)\|^v \\ &\leq 1 \\ &= 1e^{0vt}, \quad v = 1,\ 2,\ 3... \end{aligned} \tag{3.35}$$

(d) According to the Bounded perturbation theorem [12, Theorem 4.9], $\mathcal{D}_n - A_n$ is the generator of a positive semigroup of contractions since $\mathcal{D}_n$ generates a positive semigroup of contractions (Theorem 5.4.1) and A_n is bounded (assumption (3.29)).
We know that $\lambda I - (\mathcal{D}_n - A_n) : \ D(\mathcal{D}_n) \longrightarrow X_x$ and by Hille-Yosida Theorem 2.2.4, $\lambda I - (\mathcal{D}_n - A_n)$ must be invertible for some $\lambda > 0$ and $(\lambda I - (\mathcal{D}_n - A_n))^{-1} \in \mathcal{L}(X_x)$ (the space of bounded linear operators from X_x into X_x). Then the range of $\lambda I - (\mathcal{D}_n - A_n) = X_x$. Thus, $\lambda I - (\mathcal{D}_n - A_n)$ is densely defined in X_x.
All the conditions of the Corollary 5.5 in [74] are satisfied by $\mathcal{D}_n$ and A_n, hence $\overline{F_n} = \overline{\mathcal{D}_n - A_n} = \mathcal{D}_n - A_n = F_n$ is the generator of a semigroup $(G_{F_n}(t))_{t\geq 0}$ defined by

$$G_{F_n}(t)p_n = \lim_{v \longrightarrow \infty} \left[G_{\mathcal{D}_n}\left(\frac{t}{v}\right) G_{A_n}\left(\frac{t}{v}\right)\right]^v p_n, \quad \text{for } p_n \in X_x. \tag{3.36}$$

where we have used the fact that $\mathcal{D}_n - A_n$ is closed since it is the generator of a positive semigroup of contractions (Hille-Yosida Theorem 2.2.4).

Let us show that $(G_{F_n}(t))_{t\geq 0}$ is substochastic. According to (3.34) and the above condition (a), we have $G_{F_n}(t) \geq 0$ for all $t \geq 0$, since $G_{F_n}(t)p_n$ is the limit of a sequence of elements of the positive cone of X_x

$$(X_x)_+ = \{g_n \in X_x; g_n \geq 0\}$$

which is closed. Lastly, by (3.35) and (3.34), we obtain

$$\begin{aligned} \|G_{F_n}(t)\| &\leq \| \lim_{v \longrightarrow \infty} \|G_{\mathcal{D}_n}(t)\|^v \|G_{A_n}(t)\|^v \\ &\leq 1 \end{aligned}$$

for all $t \geq 0$. Thus, by the relation (3.17), the operator $\mathbf{F}$ with the domain $D(\mathbf{F})$ defined by (3.15) is the generator of a substochastic semigroup $(G_{\mathbf{F}}(t))_{t\geq 0}$ in $\mathcal{X}_r$ and given by (3.33). Now we provide a characterisation of the domain $D(\mathbf{F})$. Because $\mathcal{D}_n$ is conservative, integration of (3.30) over $\mathbb{R}^3$ gives $\frac{d}{dt}\|p_n\| = \frac{d}{dt}\int_{\mathbb{R}^3} p(t,x,n)dx = -\int_{\mathbb{R}^3} a_n(x)p_n(x)dx$. Hence (3.29) yields

$$-\int_{\mathbb{R}^3} \theta_2\alpha_n p_n(x)dx \leq -\int_{\mathbb{R}^3} a_n(x)p_n(x)dx \leq -\int_{\mathbb{R}^3} \theta_1\alpha_n p_n(x)dx$$

for all $p_n \in (X_x)_+$ and using Gronwall's inequality, we obtain

$$-\theta_2\alpha_n\|p_n\| \leq \frac{d}{dt}\|p_n\| \leq -\theta_1\alpha_n\|p_n\|,$$

then

$$e^{-\theta_2\alpha_n t}\|\mathring{p}_n\| \leq \|p_n\| \leq e^{-\theta_1\alpha_n t}\|\mathring{p}_n\|.$$

This inequality for $p_n = G_{F_n}(t)\mathring{p}_n$ yields

$$e^{-\theta_2\alpha_n t}\|\mathring{p}_n\| \leq \|G_{F_n}(t)\mathring{p}_n\| \leq e^{-\theta_1\alpha_n t}\|\mathring{p}_n\| \tag{3.37}$$

where $\mathring{p}_n \in (C_0^\infty(\mathbb{R}^3))_+ \subseteq D(F_n)_+$. If we take $0 \leq \mathring{p}_n \in L_1(\mathbb{R}^3)$, then it can always be mollified by construction of approximations to the identity (mollifiers) $\varpi_\varepsilon(x) = C_\varepsilon\varpi(x/\varepsilon)$ (as in [12, Example 2.1]) where ϖ is a $C_0^\infty(\mathbb{R}^3)$ function defined by

$$\varpi(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

and C_ε are constants chosen so that $\int\limits_{\mathbb{R}^3} \varpi_\varepsilon(x)dx = 1$. Using the mollification of $\mathring{p}_n$ by taking the convolution

$$\mathring{p}_{n,\varepsilon} := \int\limits_{\mathbb{R}^3} \mathring{p}_n(x-y)\varpi_\varepsilon(y)dy = \int\limits_{\mathbb{R}^3} \mathring{p}_n(y)\varpi_\varepsilon(x-y)dy, \tag{3.38}$$

we obtain $\overset{o}{p}_{n,\varepsilon}$ in $L_1(\mathbb{R}^3)$ (since $\overset{o}{p}_n \in L_1(\mathbb{R}^3)$) and

$$\overset{o}{p}_n = \lim_{\varepsilon \longrightarrow 0^+} \overset{o}{p}_{n,\varepsilon} \quad \text{in } L_1(\mathbb{R}^3).$$

Moreover, $\overset{o}{p}_{n,\varepsilon}$ are also non-negative by (3.38) since $0 \leq \overset{o}{p}_n$, and the family $(\overset{o}{p}_{n,\varepsilon})_\varepsilon \subseteq C_0^\infty(\mathbb{R}^3)$. This shows that any non-negative $\overset{o}{p}_n$ taken in $L_1(\mathbb{R}^3)$ can be approximated by a sequence of non-negative functions of $C_0^\infty(\mathbb{R}^3)$. The inequality (3.37) is therefore valid for any non-negative $\overset{o}{p}_n \in L_1(\mathbb{R}^3)$. Using the fact that any arbitrary element $\overset{o}{g}$ of $L_1(\mathbb{R}^3)$ (equipped with the pointwise order almost everywhere) can be written in the form $\overset{o}{g} = \overset{o}{g}_+ - \overset{o}{g}_-$, where $\overset{o}{g}_+,\ \overset{o}{g}_- \in L_1(\mathbb{R}^3)_+$, the positive element approach, [22, 88] or [12, Theorem 2.64], allows the extension of the right inequality of (3.37) to all $X_x = L_1(\mathbb{R}^3)$, in order to obtain

$$\|G_{F_n}(t)p_n\| \leq e^{-\theta_1 \alpha_n t}\|p_n\|. \tag{3.39}$$

Using the semigroup representation of the resolvent, [12, Theorem 3.34] then, for $\lambda > 0$,

$$\begin{aligned}
\|R(\lambda, F_n)p_n\| &\leq \int_0^\infty e^{-\lambda t}\|G_{F_n}(t)p_n\|dt \\
&\leq \int_0^\infty e^{-\lambda t}e^{-\theta_1 \alpha_n t}\|p_n\|dt \\
&\leq \frac{1}{\lambda + \theta_1 \alpha_n}\|p_n\|.
\end{aligned}$$

by the right inequality of (3.29),

$$\|A_n R(\lambda, F_n)p_n\| \leq \frac{\theta_2 \alpha_n}{\lambda + \theta_1 \alpha_n}\|p_n\| \leq \frac{\theta_2}{\theta_1}\|p_n\|$$

is obtained. Passing to the whole space $\mathcal{X}_r$ using the gluing technique in Proposition 3.4.1, we have

$$\begin{aligned}
\|\mathbf{A}R(\lambda, \mathbf{F})\mathbf{p}\|_r &= \sum_{n=1}^\infty n^r \|A_n R(\lambda, F_n)p_n\| \\
&\leq \frac{\theta_2}{\theta_1}\sum_{n=1}^\infty n^r\|p_n\| \\
&= \frac{\theta_2}{\theta_1}\|\mathbf{p}\|_r
\end{aligned}$$

This relation states that $D(\mathbf{A}) \supseteq D(\mathbf{F})$, (the domain of $\mathbf{A}$ is at least that of $\mathbf{F}$). Because $F_n = \mathcal{D}_n - A_n$ and A_n is bounded for each $n \in \mathbb{N}$, we exploit the following relation for resolvents in $X_x := L_1(\mathbb{R}^3)$:

$$\begin{aligned}
\lambda I - F_n &= \lambda I - \mathcal{D}_n + A_n R(\lambda, F_n)(\lambda I - F_n) \\
I &= (\lambda I - \mathcal{D}_n)R(\lambda, F_n) + A_n R(\lambda, F_n) \\
R(\lambda, \mathcal{D}_n) &= R(\lambda, F_n) + R(\lambda, \mathcal{D}_n)A_n R(\lambda, F_n) \\
R(\lambda, F_n) &= R(\lambda, \mathcal{D}_n)(I - A_n R(\lambda, F_n))
\end{aligned}$$

for every $n \in \mathbb{N}$. Extending to the whole space $\mathcal{X}_r$ yields

$$R(\lambda, \mathbf{F}) = R(\lambda, \mathbf{D})[\mathcal{I} - \mathbf{A}R(\lambda, \mathbf{F})]$$

leading to $D(\mathbf{D}) \supseteq D(\mathbf{F})$ and therefore $D(\mathbf{F}) \subseteq D(\mathbf{D}) \cap D(\mathbf{A})$.
On the other hand, if $\mathbf{p} \in D(\mathbf{D}) \cap D(\mathbf{A})$, then $\|\mathbf{Dp}\|_r < \infty$ and $\|\mathbf{Ap}\|_r < \infty$. Therefore,

$$\|\mathbf{Dp} - \mathbf{Ap}\|_r \leq \|\mathbf{Dp}\|_r + \|\mathbf{Ap}\|_r < \infty,$$

indicating that $\mathbf{p} \in D(\mathbf{F})$ and thus $D(\mathbf{D}) \cap D(\mathbf{A}) \subseteq D(\mathbf{F})$. Consequently, we obtain $D(\mathbf{F}) = D(\mathbf{D}) \cap D(\mathbf{A})$, which ends the proof. □

Now, let us take the gain part of the fragmentation process defined by (3.9) with the coefficients satisfying the conservation law (3.3) and consider the perturbed transport equation

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{p} &= \mathbf{Dp} - \mathbf{Ap} + \mathbf{Bp} \\ \mathbf{p}_{|t=0} &= \mathring{\mathbf{p}} \end{aligned} \tag{3.40}$$

Theorem 3.5.2. *If the assumptions of the Theorem 5.5.1 hold, then there is an extension* $(\mathcal{K}, D(\mathcal{K}))$ *of* $(\boldsymbol{D} - \boldsymbol{A} + \boldsymbol{B},\quad D(\boldsymbol{D}) \cap D(\boldsymbol{A}))$ *that generates the smallest substochastic semigroup on* $\mathcal{X}_r$*, denoted by* $(G_{\mathcal{K}}(t))_{t\geq 0}$.

Proof. This theorem is a direct continuation of Theorem 5.5.1 using Kato's Theorem in L_1, see Theorem 2.3.5. Because $D(\mathbf{B}) = D(\mathbf{A})$ (relation (3.9)), then $D(\mathbf{B}) \supset D(\mathbf{D}) \cap D(\mathbf{A})$. Thus, to apply Kato's Perturbation Theorem, there is a need to show that for all $\mathbf{p} = (p_n)_{n=1}^{\infty} \in D(\mathbf{D} - \mathbf{A})_+ = (D(\mathbf{D}) \cap D(\mathbf{A}))_+$,

$$\int_{\Lambda} (\mathbf{Dp} - \mathbf{Ap} + \mathbf{Bp})\, d\mu dm_r \leq 0$$

or, equivalently,

$$\int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\mathcal{D}_n p(t,x,n) - a_n(x)p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m}a_m(x)p(t,x,m) \right) dx \leq 0. \tag{3.41}$$

Since $\mathbf{p} \in D(\mathbf{D})$ then $\sum_{n=1}^{\infty} \int_{\mathbb{R}^3} n^r \mathcal{D}_n p(t,x,n)dx < \infty$.

On the other hand, $\|\mathbf{Ap}\|_r < \infty,\quad \|\mathbf{Bp}\|_r < \infty$ and (3.41) can be split in order to get its left hand-side equal to

$$\int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \mathcal{D}_n p(t,x,n)dx + \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(-a_n(x)p(t,x,n)\right.$$

$$\left. + \sum_{m=n+1}^{\infty} b_{n,m}a_m(x)p(t,x,m)\right)dx.$$

The first term vanishes by the stochasticity (3.28) of the operator **D**. For the other term, using the relation (3.10) in its explicit form yields

$$\begin{aligned} & \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(-a_n p_n + \sum_{m=n+1}^{\infty} b_{n,m} a_m p_m \right) dx \qquad (3.42) \\ = & \int_{\mathbb{R}^3} (-a_2 p_2 (2^r - b_{1,2}) - a_3 p_3 (3^r - b_{1,3} - 2^r b_{2,3}) \\ & -a_4 p_4 (4^r - b_{1,4} - 2^r b_{2,4} - 3^r b_{3,4}) - \cdots) \, dx \\ = & \int_{\mathbb{R}^3} \left[-\sum_{n=2}^{\infty} a_n p_n \left(n^r - \sum_{m=1}^{n-1} m^r b_{m,n} \right) \right] dx \\ = & -c_r(\mathbf{p}) \qquad (3.43) \\ \leq & \ 0 \end{aligned}$$

with $p_n = p(t, x, n)$ and where we have used the fact that $-\sum_{n=2}^{\infty} a_n p_n \Delta_n^{(r)} \leq 0$ is valid for every $x \in \mathbb{R}^3$ with

$$\Delta_n^{(r)} = n^r - \sum_{m=1}^{n-1} m^r b_{m,n}, \quad n \geq 2, \ r \geq 0, \qquad (3.44)$$

and where c_r is a non-negative (possibly zero) functional defined on

$$(D(\mathbf{D}) \cap D(\mathbf{A})),$$

which proves the theorem. □

3.6 A continuous model for non-local fragmentation dynamics in a moving medium

The corresponding continuous model for non-local fragmentation dynamics in a moving medium reads as

$$\frac{\partial}{\partial t} p(t,x,m) = -div(\omega(x,m)p(t,x,m)) - a(x,m)p(t,x,m) + \int_m^{\infty} b(x,s,m)a(x,s)p(t,x,s)ds$$

$$p(0,x,m) = \overset{o}{p}(x,m), \quad a.e. \ (x,m) \in \mathbb{R}^3 \times \mathbb{R}_+ \qquad (3.45)$$

where in terms of the mass size m and the position x, the state of the system is characterised at any moment t by the particle-mass-position distribution $p = p(t,x,m)$, with $p : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}_+$. ω has the same definition as in (3.1). $a(x,m)$ describes the ability of aggregates of size m and position x to break into smaller particles. Once an aggregate of mass s and position x breaks, the expected number of daughter particles of size m is the non-negative measurable function $b(x, s, m)$ defined on $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$. The treatment of the system (3.45) is very similar to the analysis performed for the model (3.1) above. In the continuous case, there is no need for the "gluing" Theorem 3.4.1 and the analysis is straightforward. The complete analysis for the continuous model is found in the published paper [70].

3.7 Concluding remarks

In this chapter, the theory of strongly continuous semigroups of operators was used to analyse the well-posedness of an integro-differential equation modeling convection-fragmentation processes. This study, with the inclusion of the spatial transportation kernel, is seen as a generalisation of the preceding studies which did not consider such a kernel before. The result obtained here is that the combined fragmentation-transportation operator is the infinitesimal generator of a strongly continuous stochastic semigroup, thereby addressing the problem of existence of solutions for this model. However, the full identification of the generator and characterisation of its domain remain an open problem.

Chapter 4

Non-autonomous Fragmentation Dynamics

Fragmentation equations (discrete or continuous) where different coefficients are time dependent has caught the attention of researchers in the past decade. This chapter deals with two related aspects to the model: The first is about the analysis of the system using various approximation techniques and the second is concerned with the equivalent norm approach for a non-autonomous fragmentation model. The latter approach makes a contribution that may lead to the full characterisation of the infinitesimal generator of a C_0 semigroup for non-autonomous fragmentation and coagulation dynamics which remain unresolved.

4.1 Global analysis of a discrete non-local and non-autonomous fragmentation dynamics occurring in a moving process

4.1.1 Introduction

In this section a double approximation technique is used to show existence result for a non-local and non-autonomous fragmentation dynamics occurring in a moving process. The case where sizes of clusters are discrete and fragmentation rate is time, position and size dependent is considered. The system involving transport and non-autonomous fragmentation processes, where in addition, new particles are spatially randomly distributed according to some probabilistic law is investigated by means of forward propagators associated to the evolution semigroup theory and the perturbation theory. The full generator is considered as a perturbation of the pure non-autonomous fragmentation operator. Use can therefore be made of the truncation technique [57], the resolvent approximation [88], Duhamel formula [39] and Dyson-Phillips series [76] to establish the existence of a solution for a discrete non-local and non-autonomous fragmentation process in a moving medium, hereby, making a contribution that may lead to the proof of uniqueness of strong solutions to this type of transport and non-autonomous fragmentation problem which remains unresolved.

4.1.2 Preliminaries and assumptions

Fragmentation models have attracted considerable attention of late as they can be found in many important areas of science and engineering. Examples range from the splitting of phytoplankton clusters, astrophysics, rock crushing, colloidal chemistry, polymer science to depolymerisation. The dynamical behaviour of a non autonomous system of phytoplankton clusters (for example) which are undergoing break up to produce smaller particles in a moving medium can be derived by balancing loss and gain of clusters of size n (with position x) over a short period of time and is given by the following differential equation as presented in [67]:

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) = -div(\omega(x,n)p(t,x,n)) \\ \qquad\qquad + \sum\limits_{m=n+1}^{\infty} a(t,x,m)b(t,x,n,m)p(t,x,m) - a(t,x,n)p(t,x,n), \\ p(\tau_0,x,n) = p_{\tau_0}(x,n), \quad 0 \leq \tau_0 < t \leq T, \quad n = 1,2,3\ldots, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

where $p(t,x,n)$ is the particle mass distribution function with respect to the mass n at the position $x \in \mathbb{R}^3$ and time t, ($p_{\tau_0}(x,n) \equiv p_{n,\tau_0}(x)$ is the mass distribution at some fixed time $\tau_0 \geq 0$), $b(t,x,n,m) \equiv b_{n,m}(t,x)$ is the distribution of particle masses n and position x, spawned by the fragmentation of a particle of mass m at time t, $T \in \mathbb{R}$ and $a(t,x,n) \equiv a_n(t,x)$ is the evolutionary time-dependent fragmentation rate, that is, the rate at which mass n particles at position x break up at a time t. The velocity $\omega = \omega(x,n)$ of the transport is supposed to be a known quantity, depending on the size n of aggregates and their position x.

The combination of fragmentation equations and transport mechanisms have been successfully utilised to model a wide range of natural processes. Examples in chemical engineering include polymer breakup and solid drugs degradation in fluids. In aquaculture, such models are used to describe phytoplankton dynamics under the kinetic constraints of the flow. The mathematical investigation of fragmentation models presents several challenges both from the theoretical and numerical point of view. In Chapter 3 and also in [70], the authors investigated the existence of global solutions to continuous non-local convection-fragmentation equations in spaces of distributions with finite higher moments. But till today, models with time dependent coefficients (non-autonomous) remain barely touched. Moreover, models of transport and non-autonomous fragmentation process have not yet been studied in the same work and there are still only few studies devoted to their analysis (well-posedness, conservativeness, honesty,$\cdots$,) separately or in the same model. In [57], the authors used techniques of truncation to prove existence, uniqueness and mass conservation for a pure non-autonomous fragmentation model under certain conditions on initial data and the associated truncated system. The authors in [6] used evolution semigroups approach which allowed them to build on the substochastic semigroup theory and to obtain an equivalent result as in [57]. In the analysis of the book by Tosio Kato [50] and later improved by Da Prado *et al.* [33], it is generally assumed that the generators $A(t)$ and $B(t)$ involved in the perturbation are of class $\mathcal{G}(1,0)$. This condition is modified in [65] where the authors used semigroup perturbation and renormalisation approach to show that the closure of the involved operators is an anti-generator. However, in many applications of forward propagator (evolution semigroup) theory to evolution equations, like transport equations used in this study or population equations [42, 63], perturbation method remains essential no matter which generator is the perturbed or the perturbing operator.

As in [71] and also in Chapter 7, focus is on the case where after clusters fragmentation, the centres of new generating groups are dispersed and spread according to a given random law $h(\cdot, n, m, y)$. This is the probability density that after a break up of an m-aggregate (with the centre at y), the new formed n-group will be located at x. Therefore,

$$\int_{\mathbb{R}^3} h(x,n,m,y)dx = 1, \tag{4.2}$$

and the system (4.1) becomes

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) = -div(\omega(x,n)p(t,x,n)) \\ \qquad + \sum\limits_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)p(t,y,m)dy - a(t,x,n)p(t,x,n), \\ p(\tau_0,x,n) = p_{\tau_0}(x,n), \quad 0 \le \tau_0 < t \le T, \quad n = 1,2,3\ldots, \quad x \in \mathbb{R}^3. \end{cases} \tag{4.3}$$

Since a group of size $m \le n$ cannot split to form a group of size n then, it is required that

$$b_{n,m}(t,x) = 0 \tag{4.4}$$

at any time t and position x for all $m \le n$. The following assumptions are also required:

$$a_1(t,x) = 0 \quad \text{and} \quad \sum_{m=1}^{n-1} m b_{m,n}(t,x) = n \quad (n = 2,3,\ldots), \tag{4.5}$$

meaning that a cluster of size one cannot split and the sum of all individuals obtained by fragmentation at a position x of an n-group is equal to n all the time t. The second term on the right-hand side of (4.3) describes the increase in the number of particles of size n due to fission of larger particles (the gain due to the fragmentation). The third term describes the reduction in the number of particles of size n due to the fission of groups of same size (the loss due to the fragmentation). The space variable x is supposed to vary in the whole of $\mathbb{R}^3$.

4.1.3 Approximation and analysis of the fragmentation operator

Since $p = p(t, x, n)$ is the density of n-groups at the position x and time t and that total mass

$$U(t) = \sum_{n=1}^{\infty} n \int_{\mathbb{R}^3} p(t, x, n)dx \tag{4.6}$$

is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$Y_1 := L_1(\mathbb{R}^3 \times \mathbb{N}, d\mu dm_1), \tag{4.7}$$

where $\mathbb{N}$ is equipped with the weighted counting measure dm_1 with weight n and $d\mu = dx$ is the Lebesgue measure in $\mathbb{R}^3$. Note that elements'norm of Y_1 represent the total mass (or total number of individuals) of the system. Now, we put

$$\mathcal{Y}_1 = L_1(\mathfrak{J}, Y_1), \quad \mathfrak{J} = [0, T],$$

and (4.3) is recast as the non-autonomous abstract Cauchy problem in $\mathcal{Y}_1$:

$$\begin{cases} \frac{\partial}{\partial t}p(t, x, n) = [\tilde{\mathcal{D}}p](t, x, n) + Q(t)p(t, x, n), \\ p(\tau_0, x, n) = p_{\tau_0}(x, n), \ 0 \leq \tau_0 < t \leq T, \quad n = 1, 2, 3 \dots, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.8}$$

or in the compact form

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{p}(t) = \mathbf{D}\mathbf{p}(t) + \mathcal{Q}(t)\mathbf{p}(t), \\ \mathbf{p}(t)|_{t=\tau_0} = \mathbf{p}_{\tau_0}, \end{cases} \tag{4.9}$$

where $\mathbf{p}(t)$ is the vector $\mathbf{p}(t) = (p(t, x, n))_{n=1}^{\infty}$, $\mathbf{p}_{\tau_0}$ the mass distribution vector $(p_n(\tau_0, x))_{n=1}^{\infty}$ at the fixed time $\tau_0 \geq 0$ and position x, $\mathcal{Q}(t)$ the non-autonomous fragmentation operator defined by

$$\begin{aligned} \mathcal{Q}(t)\mathbf{p}(t) &= ([Q(t)p(t)](t, x, n))_{n=1}^{\infty} \\ &:= \left(\sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t, y, m)b(t, y, n, m)h(x, n, m, y)p(t, y, m)dy - a(t, x, n)p(t, x, n) \right)_{n=1}^{\infty}, \end{aligned} \tag{4.10}$$

where $Q(t)$ is seen as the pointwise operation

$$\psi(t, x, n) \longmapsto \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t, y, m)b(t, y, n, m)h(x, n, m, y)\psi(t, y, n)dy - a(t, x, n)\psi(t, x, n) \tag{4.11}$$

defined on the set of measurable functions. $Q(t)$ is defined by $Q(t) = \mathfrak{Q}(t)$ and represents the realisation of $\mathfrak{Q}(t)$ on the domain

$$D(Q(t)) = \{p \in \mathcal{Y}_1; \ \mathfrak{Q}(t)p(t) \in \mathcal{Y}_1\}, \tag{4.12}$$

with $\mathfrak{Q}(t)p(t)$ given by (4.11). The transport matrix $\mathbf{D}$ is given by $\mathbf{Dp} := (\tilde{\mathcal{D}}p(t,x,n))_{n=1}^{\infty}$ with

$$[\tilde{\mathcal{D}}p](t,x,n) = -div\,(\omega(x,n)\,p(t,x,n))\,. \tag{4.13}$$

To investigate and analyse the abstract Cauchy Problem (4.8) and show existence for this system, a two parameter family is needed.

Definition 4.1.1. (Evolution system [74] or propagator [65])
A two parameter family of bounded linear operators $U(t,\tau)$, $0 \le \tau \le t \le T$, on a Banach space is called forward propagator or evolution system if the following conditions are respected:
(i) $U(\tau,\tau) = I$ *for* $0 \le \tau \le T$.
(ii) $U(t,r)U(r,\tau) = U(t,\tau)$ *for* $0 \le \tau \le r \le t \le T$.
(iii) $(t,\tau) \longrightarrow U(t,\tau)$ *is strongly continuous for* $0 \le \tau \le t \le T$.

Recall that ([65]) the forward propagator $U(t,\tau)$, $0 \le \tau \le t \le T$ can be associated to the so-called evolution semigroup $(G_t(s))_{s\ge 0}$ defined in $\mathcal{Y}_1$, i.e. if for any fixed $t \in \mathfrak{I} = [0,T]$, the operator $Q(t)$ generates a forward propagator $U(t,\tau)$, $0 \le \tau \le t \le T$, then this propagator defines a C_0 semigroup $(G_t(s))_{s\ge 0}$ given by the relation

$$[G_t(s)p](\sigma) = (\sigma - s)\chi_{\mathfrak{I}}\,U(\sigma,\sigma - s)\,p(\sigma - s), \tag{4.14}$$

where $\chi_{\mathfrak{I}}$ is the characteristic function of $\mathfrak{I} = [0,T]$, $p \in \mathcal{Y}_1$ and $\sigma \in \mathfrak{I}$. In the rest of the chapter, when we say $Q(t)$ is the generators of C_0-semigroups in $\mathcal{Y}_1$, it means $Q(t)$ generates a propagator which defines a C_0-semigroup in $\mathcal{Y}_1$ satisfying the relation (4.14).

We introduce, for any given $k \in \mathbb{N}$, the projection operator q_k defined for a function $g \in \mathcal{Y}_1$ as

$$[q_k g](x,n) = \begin{cases} g(x,n), & k > n > 0 \text{ and } x \in \mathbb{R}^3 \\ 0, & \text{otherwise.} \end{cases}$$

The space

$$X_k = \{g \in \mathcal{Y}_1 :\ g(x,n) \equiv 0 \text{ on } \mathbb{R}^3 \times (k,\infty)\}$$

is therefore a closed subspace of $\mathcal{Y}_1$ on which the projection operator q_k acts. Let us associate to the fragmentation model

$$\begin{cases} \frac{dp}{dt}(t) = Q(t)p(t), \\ p(\tau_0) = p_{\tau_0},\ 0 \le \tau_0 < t \le T, \end{cases}$$

the following truncated version

$$\begin{cases} \frac{dp}{dt}(t) = Q(t)q_k p(t), \\ p(\tau_0) = p_{\tau_0},\ 0 \le \tau_0 < t \le T, \end{cases} \tag{4.15}$$

where $Q(t)$ is represented by (4.11). If $Q_k(t) = Q(t)q_k$ is set, then $Q_k(t)$ can be decomposed as $Q_k(t) = A_k(t) + B_k(t)$ where the loss and the gain fragmentation operators $A_k(t)$ and $B_k(t)$ are given by

$$A_k(t)g(t,x,n) = A(t)q_k g(t,x,n) = -a(t,x,n)g(t,x,n)$$

and

$$B_k(t)g(t,x,n) = B(t)q_k g(t,x,n) = \sum_{m=n+1}^{k} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)g(t,y,m)dy,$$

where $A(t)$ and $B(t)$ are expressed as

$$A_k(t) = A(t)$$

and

$$B(t)g(t,x,n) = \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)g(t,y,m)dy. \tag{4.16}$$

Thus, for all $g \in \mathcal{Y}_1$, $k \in \mathbb{N}$ and $(t,x,n) \in \mathfrak{I} \times \mathbb{R}^3 \times \mathbb{N}$,

$$Q_k(t)g(t,x,n) = \begin{cases} \sum\limits_{m=n+1}^{k} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)g(t,y,m) \\ \qquad -a(t,x,n)g(t,x,n), \ k > n > 0 \\ 0, \qquad \text{otherwise}, \end{cases} \tag{4.17}$$

with

$$D(Q_k(t)) = \{g \in \mathcal{Y}_1;\ Q_k(t)g \in X_k\}.$$

We assume that for t fixed in $\mathbb{R}$, there are two constants $0 < \theta_1$ and θ_2 such that

$$\theta_1\alpha(n,t) \le a_t(x,n) = a(t,x,n) \le \theta_2\alpha(n,t) \ \text{ for almost all } (x,n) \in \mathbb{R}^3 \times \mathbb{N}, \tag{4.18}$$

where $\alpha(n,t) \in L_{\infty,loc}(\overline{\mathbb{R}_+{}^2})$ is a real-valued function which can depend on n and t but is independent of the state variable x. This obviously implies that for any $k \in \mathbb{N}$, there exists a positive $\Theta_{t,k}$ such that

$$\text{ess } \sup_{\mathbb{R}^3\times(0,k)} a_t(x,n) \le \Theta_{t,k}. \tag{4.19}$$

Moreover, the sequence $\{\Theta_{t,k}\}_{k\in\mathbb{N}}$ (t fixed in $\mathbb{R}$) is bounded in the following way:

$$\text{ess } \sup_{0\le t\le T} \Theta_{t,k} \le \bar{\Theta}_{T,k} < \infty. \tag{4.20}$$

Lemma 4.1.2. *For t fixed in $\mathfrak{I}$, and $k \in \mathbb{N}$, there is a positive uniformly continuous semigroup of contractions on $X_k \subset \mathcal{Y}_1$, say $(G_{t,k}(s))_{s\ge0}$ generated by the operator $Q_k(t)$ such that $(G_{t,k}(s))_{s\ge0}$ is conservative on $(X_k)_+$ and given by*

$$G_{t,k}(s) = I + \sum_{j=1}^{\infty} \left[\left(\frac{s^j(Q(t))^j}{j!}\right] q_k. \tag{4.21}$$

Moreover, for any $r \ge k$, $s \ge 0$,

$$q_k G_{t,r}(s) q_k = G_{t,r}(s). \tag{4.22}$$

Proof. Let us fix t in $\mathfrak{I}$. The operator $A_k(t)$ is bounded by (4.19). Changing the order of summation by the Tonelli's theorem, for every $g \in X_k$,

$$\begin{aligned}
\|B_k(t)g\|_1 &= \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{k} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)|g(t,y,m)|dy \right) dx \\
&= \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{k} a(t,y,m)b(t,y,n,m)|g(t,y,m)| \right) dy \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{k} a(t,y,m)|g(t,y,m)| \left(\sum_{n=1}^{\infty} nb(t,y,n,m) \right) dy \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{k} a(t,y,m)|g(t,y,m)| \left(\sum_{n=1}^{m-1} nb(t,y,n,m) \right) dy \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{k} ma(t,y,m)|g(t,y,m)|dy \\
&= \int_{\mathbb{R}^3} \sum_{m=1}^{k} ma(t,y,m)|g(t,y,m)|dy \\
&\leq \int_{\mathbb{R}^3} \sum_{m=1}^{\infty} ma(t,y,m)|g(t,y,m)|dy \\
&= \|A_k(t)g\|_1 \\
&= \Theta_{t,k}\|g\|_1 \\
&< \infty,
\end{aligned}$$

where we have used (4.2) and (4.5) respectively. Then $B_k(t)$ is also bounded. Hence $Q_k(t)$ generates a uniformly continuous semigroup. We denote this semigroup by $(G_{t,k}(s))_{s\geq 0}$. Clearly, $A_k(t)$ generates a positive semigroup of contractions and $B_k(t)$ is a positive operator. Moreover, the above calculations also imply that $D(B_k(t)) \supset D(A_k(t))$ and

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{\mathbb{R}^3} n(A_k(t)g(t,x,n) + B_k(t)g(t,x,n))dx \\
&= \sum_{n=1}^{k} \int_{\mathbb{R}^3} n(A_k(t)g(t,x,n) + B_k(t)g(t,x,n))dx \leq 0
\end{aligned}$$

for all $g \in (D(A_k(t)))_+$ with

$$D(A_k(t)) = \{g \in \mathcal{Y}_1;\ -a_t g \in X_k\} \quad \text{and} \quad D(B_k(t)) = \{g \in \mathcal{Y}_1;\ B_k(t)g \in X_k\}.$$

Thus, the assumptions of Kato's Theorem in L_1-space, Theorem 2.3.5 hold. It is essentially noted that, for each fixed t, the operator $Q_k(t)$ becomes independent of time [57, Lemma 2.1] and Kato's Theorem is immediately applicable. Therefore, there is an extension $\breve{Q}_k(t)$ of $Q_k(t)$ which generates a substochastic semigroup. Because $a_t(x,n)$ is bounded in $\mathbb{R}^3 \times (0,k)$, this substochastic semigroup is conservative, it follows that $\breve{Q}_k(t) = \overline{Q}_k(t)$, where $\overline{Q}_k(t)$ is the closure of $Q_k(t)$. Since $Q_k(t)$ generates a uniformly (and hence strongly) continuous semigroup, $Q_k(t)$ is a closed operator. This yields $\breve{Q}_k(t) = Q_k(t)$, consequently, the uniformly continuous semigroup $(G_{t,k}(s))_{s\geq 0}$ is a positive strongly continuous semigroup of contractions, furthermore, $(G_{t,k}(s))_{s\geq 0}$ is honest.

The proof of (4.21) is clear since the usual power series definition can be used to define $G_{t,k}(s) = \exp(sQ_k(t))$. By induction, $(Q_k(t))^J = (Q(t)q_k)^J = (Q(t))^J q_k$ for $J = 1, 2, \cdots$, from which the exponential formula yields (4.21).

To prove (4.22), we have that $B(t)q_k g = q_k B(t) q_k g$ on $\overline{B(t)}(0_{\mathbb{R}^3}, k) \times [0,k]$ since for $k \geq n \geq 0$, $B(t)q_k g(t,x,n)$ is given by (4.16) and $B(t)q_k g(t,x,n) = 0$ for $k < n$. Moreover, it is obvious that $A(t)q_k g = q_k A(t) q_k g$, hence we have also

$$(A(t) + B(t))q_k = A(t)q_k + B(t)q_k = q_k A(t) q_k + q_k B(t) q_k = A_k(t) + B_k(t) = Q_k(t).$$

Next, by $q_k q_r = q_r q_k = q_k$ we have

$$q_k Q_r(t) q_k = q_k q_r Q(t) q_r q_k = q_k Q(t) q_k = Q_k(t)$$

if we assume, by induction, that $q_k(Q_r(t))^{j-1} q_k = (Q_k(t))^{j-1}$, then

$$\begin{aligned} q_k(Q_r(t))^j q_k &= q_k(Q_r(t))^{j-1} Q_r(t) q_k \\ &= q_k(Q_r(t))^{j-1} q_r Q(t) q_r q_k \\ &= q_k(Q_r(t))^{j-1} q_r q_k Q(t) q_k \\ &= q_k(Q_r(t))^{j-1} q_k Q_k(t) \\ &= (Q_k(t))^j. \end{aligned}$$

Now, using (4.21) and the the fact that $Q_r(t)$ is a bounded operator, the semigroup generated by $Q_r(t)$ is expressed by

$$\begin{aligned} q_k G_{t,r}(s) q_k &= \sum_{n=0}^{\infty} \frac{s^j q_k (Q_r(t))^j q_k}{n!} \\ &= \sum_{n=0}^{\infty} \frac{s^j (Q_k(t))^j}{j!} \\ &= G_{t,r}(s), \end{aligned}$$

which concludes the lemma. □

Next, we assume that a(t,x,n) satisfies the Lipschitz condition

$$|a(t,x,n) - a(\sigma,x,n)| \leq |t - \sigma| \Lambda(x,n), \quad t, \sigma \in \mathfrak{I} \tag{4.23}$$

where $\Lambda(x,n) \geq 0$ together with $\Lambda(x,n) \leq \Theta_k$ for all $k \geq n > 0$. We state the following lemma

Lemma 4.1.3. *The function $t \longrightarrow Q_k(t)$ is continuous in the uniform operator topology for each k fixed in $\mathbb{N}$*

Proof. Using Fubini's theorem and assumption (4.23) yields

$$\|Q_k(t)g - Q_k(\sigma)g\|_1 \leq k\Theta_k|t-\sigma|\|g\|_1, \quad \text{for all } g \in \mathcal{Y}_1$$

and the result follows. □

Making use of (4.14) and the Lemma 4.2.1, there is a forward propagator, let us say $\{U_k(t,\tau)\}_{0\leq\tau\leq t\leq T}$ defined in $X_k \subset \mathcal{Y}_1$ which is associated to the evolution semigroup $(G_{t,k}(s))_{s\geq 0},\ t \in \mathfrak{I}$. The propagator $\{U_k(t,\tau)\}_{0\leq\tau\leq t\leq T}$ shares certain properties with the family of semigroups $(G_{t,k}(s))_{s\geq 0},\ t \in \mathfrak{I}$, as stated in the following theorem and proven in [57, Theorem 4.1].

Theorem 4.1.4. *[74] For each $k \in \mathbb{N}$, the forward propagator $\{U_k(t,\tau)\}_{0\leq\tau\leq t\leq T}$ generated by the family of infinitesimal generators $\{Q_k(t)\}_{t\in\mathfrak{I}}$ have the following properties:*

1. *$U_k(t,\tau)$ is positive;*
2. *$\|U_k(t,\tau)g\|_1 = \|g\|_1$, for all $g \in \mathcal{Y}_1$;*
3. *$\sum_{n=0}^{\infty} n[U_k(t,\tau)g](t,x,n) = \sum_{n=0}^{\infty} ng(t,x,n)$, for all $g \in \mathcal{Y}_1,\ x \in \mathbb{R}^3$;*
4. *$\frac{\partial}{\partial t}U_k(t,\tau) = Q_k(t)U_k(t,\tau), \qquad 0 \leq \tau \leq t \leq T$;*
5. *$\frac{\partial}{\partial \tau}U_k(t,\tau) = -U_kQ_k(t,\tau), \qquad 0 \leq \tau \leq t \leq T$.*

Theorem 4.1.5. *The truncated problem* (4.15) *has a unique, strongly continuously differentiable, positive, mass-conserving solution for all initial data $p(\tau_0) = p_{\tau_0} \in X_k$. The solution is given by $p(t) = U_k(t,\tau_0)p_{\tau_0}$ $(0 \leq \tau_0 \leq t \leq T)$.*

Proof. This theorem is an immediate consequence of Lemmas 4.2.1 and 4.2.2, Theorem 4.2.1 associated with [74, Theorem 5.1]. □

Remark 2. According to [74, Definition 2.1, p. 130 and Remark 3.2, p. 138], for each fixed $k > 0$, the family of infinitesimal generators $\{Q_k(t)\}_{t\in\mathfrak{I}}$ of C_0-semigroups on X_k is stable with stability constants 1 and 0. Moreover, from Theorem 4.2.3 we obtain

$$\|U_k(t,\tau)\|_1 \leq 1, \text{ for } 0 \leq \tau \leq t \leq T. \tag{4.24}$$

4.1.4 Cauchy problem for the transport model in $\mathbb{R}^3 \times \mathbb{N}$

We consider the Cauchy problem for the transport equation

$$\frac{\partial}{\partial t} p(t,x,n) = -div(\omega(x,n)\, p(t,x,n)), \tag{4.25}$$

$$p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1,2,3,...$$

in the space $\mathcal{Y}_1$.
The function $\omega : \mathbb{R}^3 \times \mathbb{N} \longrightarrow \mathbb{R}^3$ defined by $(x,n) \longrightarrow \omega(x,n)$ is considered and it should be recalled that $\tilde{\mathcal{D}}$ in (4.13) is the expression that appears on the right-hand side of the equation (4.25). Then,

$$\begin{aligned}\left[\tilde{\mathcal{D}}p\right](t,x,n) &:= -div\,(\omega(x,n)\,p(t,x,n)) \\ &= (\nabla\cdot\omega(x,n))p(t,x,n) + \omega(x,n)\cdot(\nabla p(t,x,n)).\end{aligned} \tag{4.26}$$

We assume that

(H1) ω is divergence free;

(H2) ω globally Lipschitz continuous,

then $div\,\omega(x,n) := \nabla\cdot\omega(x,n) = 0$ and (4.26) becomes

$$\left[\tilde{\mathcal{D}}p\right](t,x,n)) := \omega(x,n)\cdot(\nabla p(t,x,n)). \tag{4.27}$$

We define in $\mathcal{Y}_1$ the operators $(\mathcal{D}, D(\mathcal{D}))$ as

$$\begin{aligned}\left[\mathcal{D}p\right](t,x,n) &= [\tilde{\mathcal{D}}p](t,x,n), \quad \text{with } [\tilde{\mathcal{D}}p](t,x,n) \ \text{ represented by } (4.27) \\ D(\mathcal{D}) &:= \{p \in \mathcal{Y}_1,\ \ \omega\cdot(\nabla p) \in \mathcal{Y}_1\}, \quad n \in \mathbb{N}.\end{aligned} \tag{4.28}$$

Remark 3. With the assumptions (H1) - (H2), it is proven [70] that $\mathcal{D}$ is the generator of a strongly continuous stochastic semigroup, say $(G_{\mathcal{D}}(t))_{t\geq 0}$ and $\|G_{\mathcal{D}}(t)\| \leq 1 = 1e^{0t}$. Then $\mathcal{D} \in \mathcal{G}(1,0)$ and because of Hille-Yosida's characterisation (Theorem 2.2.4), $(\mathcal{D}, D(\mathcal{D}))$ is a closed and densely defined operator satisfying for the resolvent set $\rho(\mathcal{D}) \supset [0,\infty)$ and $\|\lambda R(\lambda,\mathcal{D})\|_1 \leq K$ for some constant $K > 0$ and all $\lambda > 0$. Furthermore,

(i): for any $g \in \mathcal{Y}_1$,

$$\lim_{\lambda\to\infty} \lambda R(\lambda,\mathcal{D})g = g \tag{4.29}$$

(ii): $\mathcal{D}R(\lambda,\mathcal{D})$ are bounded operators and for any $g \in D(\mathcal{D})$,

$$\lim_{\lambda\to\infty} \lambda\mathcal{D}R(\lambda,\mathcal{D})g = \mathcal{D}g. \tag{4.30}$$

4.1.5 Perturbed approximated problem

Let $\mathcal{D}_\lambda = \lambda\mathcal{D}R(\lambda, \mathcal{D})$, Yosida [88] was the first to use the bounded operators $\mathcal{D}_\lambda$ to approximate the unbounded operator $\mathcal{D}$, for which semigroups can be defined *via* the exponential formula

$$e^{t\mathcal{D}} = I + \frac{t\mathcal{D}}{1!} + \frac{t^2\mathcal{D}^2}{2!} + \cdots .$$

This idea is exploited to analyse the following approximated problem associated to (4.3):

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) = Q_k(t)p(t,x,n) + [\mathcal{D}_\lambda p](t,x,n), \\ p(\tau_0, x, n) = p_{\tau_0}(x,n), \quad 0 \le \tau_0 < t \le T, \quad n = 1,2,3\ldots, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.31}$$

Lemma 4.1.6. *Let each k fixed in $\mathbb{N}$ and any $\lambda > 0$. Under the assumptions (4.5), (4.19), (4.20), (4.23), (H1) and (H2), the operator $(T_{k,\lambda}(t) = Q_k(t)+\mathcal{D}_\lambda, D(Q_k(t)))$, $(0 \le t \le T)$ which appears in approximated Cauchy problem (4.31) is a stable generator, with the stability constants 1 and $\|\mathcal{D}_\lambda\|_1$, of a forward propagator $\{\mathbb{U}_{k,\lambda}(t,\tau)\}_{0\le\tau\le t\le T}$ associated to an evolution semigroup $(\mathbb{G}_{t,k,\lambda}(s))_{s\ge0}$, positive, conserving the norm $\|\cdot\|_1$ and given by*

$$\mathbb{G}_{t,k,\lambda}(s)g = G_{t,k}(s)g + \sum_{i=1}^{\infty} G^i_\lambda(s)g, \tag{4.32}$$

where

$$G^i_\lambda(s)g = \int_0^s G_{t,k}(s-v)g\mathcal{D}_\lambda G^{i-1}_\lambda(v)gdv \quad and \quad g \in X_k$$

with $(G_{t,k}(s))_{s\ge0}$ defined in Lemma 4.2.1.

Furthermore, the problem (4.31) has a unique, strongly continuously differentiable, positive, mass-conserving solution for all initial data $p(\tau_0) = p_{\tau_0} \in X_k$. The solution is given by $p(t) = \mathbb{U}_{k,\lambda}(t,\tau_0)p_{\tau_0}$ $(0 \le \tau_0 \le t \le T)$.

Proof. Let us fix $k \in \mathbb{N}$ and $\lambda > 0$. The fact that $T_{k,\lambda}(t)$ is the generator of a forward propagator comes from Lemma 4.2.1, the Bounded perturbation theorem and the remark (4.14). According to Remark 2 we have $\rho(Q_k(t)) \supset (0,\infty)$. If $\nu > \|\mathcal{D}_\lambda\|_1$, it is obvious that $\nu \in \rho(T_{k,\lambda}(t))$ (Bounded perturbation) and the resolvent satisfies

$$R(\nu, T_{k,\lambda}(t)) = \sum_{i=1}^{\infty} R(\nu, Q_k(t))\left[\mathcal{D}_\lambda R(\nu, Q_k(t))\right]^i .$$

Henceforth, for any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_r \le T$, $r = 1, 2, \cdots$, we have

$$\prod_{n=1}^{r} R(\nu, T_{k,\lambda}(t_n)) = \prod_{n=1}^{r}\left[\sum_{i=1}^{\infty} R(\nu, Q_k(t_n))\left[\mathcal{D}_\lambda R(\nu, Q_k(t_n))\right]^i\right]. \tag{4.33}$$

The development of the right-hand side of (4.33) yields a series with the general term in the form

$$R(\nu, Q_k(t_m))\left[\mathcal{D}_\lambda R(\nu, Q_k(t_m))\right]^{i_m} \cdots R(\nu, Q_k(t_1))\left[\mathcal{D}_\lambda R(\nu, Q_k(t_1))\right]^{i_1}, \tag{4.34}$$

where $i_n \geq 0$. If $\sum_{n=1}^{r} i_n = i$, then using the stability of the family $Q_k(t)_{0\leq\leq t\leq T}$, we estimate estimate (4.34) by $\|\mathcal{D}_\lambda\|_1\nu^{-i-r}$. Therefore,

$$\left\|\prod_{n=1}^{r} R(\nu, T_{k,\lambda}(t_n))\right\|_1 \leq \nu^{-r}\sum_{i=1}^{\infty}\binom{i}{r}\left(\|\mathcal{D}_\lambda\|_1\nu^{-1}\right)^{-i} = \left[(\nu - \|\mathcal{D}_\lambda\|_1)\right]^{-1},$$

where $\binom{i}{r}$ is the number of terms in which $\sum_{n=1}^{r} i_n = i$ in this series. To prove (4.32), we use the Duhamel equation [39]

$$\mathbb{G}_{t,k,\lambda}(s)g = G_{t,k}(s)g + \int_0^s G_{t,k}(s-v)g\mathcal{D}_\lambda\mathbb{G}_{t,k,\lambda}(v)gdv,\ g \in X_k \tag{4.35}$$

whose the iteration leads to the Dyson–Phillips series given by

$$\mathbb{G}_{t,k,\lambda}(s) = \sum_{i=0}^{\infty} G_\lambda^i(s), \qquad \text{with} \quad G_\lambda^0(s)g = G_{t,k}(s)g$$

and (4.32) follows.

The second part of the theorem follows from the Theorem 4.2.4 and Remark 3. We just need to show that the model with the transport process is conservative. It has been proven [70] that the semigroup generated by the operator $\mathcal{D}$ is strongly continuous and stochastic, this implies

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3\times\mathbb{N}} \mathcal{D}p\, d\mu dm_1, \\ &= \int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\mathcal{D}p(t,x,n)dx, \end{aligned} \tag{4.36}$$

for all $p \in D(\mathcal{D})$ $t \geq 0$, which leads to

$$\begin{aligned} \frac{d}{dt}U(t) &= \frac{d}{dt}\left(\sum_{n=1}^{\infty} n\int_{\mathbb{R}^3} p(t,x,n)dx\right) \\ &= \sum_{n=1}^{\infty} n\int_{\mathbb{R}^3}\partial_t p(t,x,n)dx \\ &= \sum_{n=1}^{\infty} n\int_{\mathbb{R}^3}\mathcal{D}p(t,x,n)dx \\ &= 0 \end{aligned}$$

where $U(t)$ is the total mass of the system defined in (4.6) and therefore proving the conservativeness of the transport process. □

It is known, see [57, Corollary 6.4] that, under the assumptions of the previous Lemma, the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) = \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)p(t,y,m)dy \\ \qquad -a(t,x,n)p(t,x,n), \\ p(\tau_0,x,n) = p_{\tau_0}(x,n), \quad 0 \le \tau_0 < t \le T, \quad n = 1,2,3\ldots, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.37}$$

has a solution p defined on the product set $[0,\infty) \times \mathbb{R} \times \mathbb{N}$ whenever $p_{\tau_0} \in \mathcal{Y}_1$ and given by

$$p(t,x,n) = [U(t,\tau_0)p_{\tau_0}](x,n), \quad n \in \mathbb{N},\ 0 \le \tau_0 < t, \tag{4.38}$$

where

$$U(t,\tau)g = \lim_{k\to\infty} U_k(t,\tau)g \tag{4.39}$$

for all $g \in \mathcal{Y}_1$ with $\{U_k(t,\tau)\}_{0\le\tau<t\le T}$ defined in the Theorem 4.2.3. In the same way, if $(G_t(s))_{s\ge0}$ be the evolution semi-group associated to $U(t,\tau)$, (see (4.14)) then

$$G_t(s)g = \lim_{k\to\infty} G_{t,k}(s)g \tag{4.40}$$

for all $g \in \mathcal{Y}_1$ with $\{G_{t,k}(s)\}_{s\ge0}$ defined in the Lemma 4.2.1. Because $\mathcal{D}_\lambda$ is bounded, it follows that the perturbed Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) = \mathcal{D}_\lambda p(t,x,n)) \\ \qquad + \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)p(t,y,m)dy - a(t,x,n)p(t,x,n), \\ p(\tau_0,x,n) = p_{\tau_0}(x,n), \quad 0 \le \tau_0 < t \le T, \quad n = 1,2,3\ldots, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.41}$$

also has a solution p defined on the product set $[0,\infty) \times \mathbb{R} \times \mathbb{N}$ whenever $p_{\tau_0} \in \mathcal{Y}_1$. The following can thus be stated:

Lemma 4.1.7. *Let fix $\lambda > 0$. Consider the families $(\mathbb{G}_{t,k,\lambda}(s))_{s\ge0}$ and $\{\mathbb{U}_{k,\lambda}(t,\tau)\}_{0\le\tau\le t\le T}$ defined in Lemma 4.1.6. The family $(\mathbb{G}_{t,\lambda}(s))_{s\ge0}$ defined by*

$$\mathbb{G}_{t,\lambda}(s)g = \lim_{k\to\infty} \mathbb{G}_{t,k,\lambda}(s)g, \qquad g \in \mathcal{Y}_1 \tag{4.42}$$

exists for all $0 \le t \le T$ and $(\mathbb{G}_{t,\lambda}(s))_{s\ge0}$ forms a positive, C_0-semigroup on $\mathcal{Y}_1$ conserving the norm $\|\cdot\|_1$.

In the same way, the family $\{\mathbb{U}_\lambda(t,\tau)\}_{0\le\tau\le t\le T}$ defined by

$$\mathbb{U}_\lambda(t,\tau)g = \lim_{k\to\infty} \mathbb{U}_{k,\lambda}(t,\tau)g, \qquad g \in \mathcal{Y}_1 \tag{4.43}$$

exists for all $0 \le \tau < t \le T$ and $\{\mathbb{U}_\lambda(t,\tau)\}_{0\le\tau\le t\le T}$ forms a forward propagator on $\mathcal{Y}_1$ conserving the norm $\|\cdot\|_1$.

Proof. Let $g \in \mathcal{Y}_1$, from (4.21) and (4.22) we have

$$\mathbb{G}_{t,k,\lambda}(s)q_r g = \mathbb{G}_{t,r,\lambda}(s)g + q_r g - g, \qquad \text{where } k \geq r.$$

Then

$$\begin{aligned}\|\mathbb{G}_{t,k,\lambda}(s)g - \mathbb{G}_{t,r,\lambda}(s)g\|_1 &\leq \|\mathbb{G}_{t,k,\lambda}(s)g - \mathbb{G}_{t,k,\lambda}(s)q_r g\|_1 + \|g - q_r g\|_1 \\ &= 2\|g - q_r g\|_1 \\ &\longrightarrow 0, \quad \text{as } k, r \longrightarrow \infty\end{aligned}$$

Hence, the family $(\mathbb{G}_{t,k,\lambda}(s))_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{Y}_1$ and therefore convergent and its limits, uniform in s, exists as $\mathbb{G}_{t,\lambda}(s) : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_1$. Furthermore,

$$\begin{aligned}\|\mathbb{G}_{t,\lambda}(s)g\|_1 &= \left\|\lim_{k\to\infty} \mathbb{G}_{t,k,\lambda}(s)g\right\|_1 \\ &= \lim_{k\to\infty} \|\mathbb{G}_{t,k,\lambda}(s)g\|_1 \\ &= \lim_{k\to\infty} \|g\|_1 \\ &= \|g\|_1,\end{aligned}$$

where we have used the continuity of $\|\cdot\|_1$ on $\mathcal{Y}_1$ and the fact that $\{\mathbb{G}_{t,k,\lambda}(s)\}$ conserves the norm $\|\cdot\|_1$. Thus, $\{\mathbb{G}_{t,\lambda}(s)\}_{s\geq 0}$ conserves the norm $\|\cdot\|_1$.
$(\mathbb{G}_{t,k,\lambda}(s))_{s\geq 0}$ is a semigroup, in fact
(a)

$$\begin{aligned}\mathbb{G}_{t,\lambda}(s+v)g &= \lim_{k\to\infty} \mathbb{G}_{t,k,\lambda}(s+v)g \\ &= \lim_{k\to\infty} \mathbb{G}_{t,k,\lambda}(s) \lim_{k\to\infty} \mathbb{G}_{t,k,\lambda}(v)g.\end{aligned}$$

In the same way as previously, it is shown, using the definition of $(\mathbb{G}_{t,\lambda}(s))_{s\geq 0}$, that

$$\|\mathbb{G}_{t,k,\lambda}(s)\mathbb{G}_{t,k,\lambda}(v)g - \mathbb{G}_{t,\lambda}(s)\mathbb{G}_{t,\lambda}(v)g\|_1 \longrightarrow 0, \qquad \text{as } k, r \longrightarrow \infty$$

Hence

$$\mathbb{G}_{t,\lambda}(s)\mathbb{G}_{t,\lambda}(v)g \quad \longleftarrow \mathbb{G}_{t,k,\lambda}(s)\mathbb{G}_{t,k,\lambda}(v)g = \mathbb{G}_{t,k,\lambda}(s+v)g \longrightarrow \quad \mathbb{G}_{t,\lambda}(s+v)g, \text{ as}$$

$k \longrightarrow \infty$. This convergence, being strong in $\mathcal{Y}_1$ finally yields $\mathbb{G}_{t,\lambda}(s)\mathbb{G}_{t,\lambda}(v) = \mathbb{G}_{t,\lambda}(s+v)$
(b)$\mathbb{G}_{t,\lambda}(0) = I$ since

$$\mathbb{G}_{t,\lambda}(0)g = \lim_{k\to\infty} \mathbb{G}_{t,k,\lambda}(0)g = \lim_{k\to\infty} g = g$$

for all $g \in \mathcal{Y}_1$. (c) In a similar way, we show that $\lim_{s\to 0^+} \mathbb{G}_{t,k,\lambda} g = g$ since the limits (4.42) exists uniformly in s.
(d) $\{\mathbb{G}_{t,\lambda}(s)\}$ is positive since from the definition of $\mathbb{G}_{t,\lambda}(s)$ and [75, Corollary 5.11], there exists a subsequence $\{\mathbb{G}_{t,k_\delta,\lambda}(s)g\}$ $(\delta = 1, 2, \cdots)$ so that

$$[\mathbb{G}_{t,k_\delta,\lambda}(s)g](x,n) \longrightarrow [\mathbb{G}_{t,\lambda}(s)g](x,n), \quad \text{as } \delta \longrightarrow \infty \text{ for a.e. } (x.n) \in \mathbb{R}^3 \times \mathbb{N}.$$

Because $\{\mathbb{G}_{t,k_\delta,\lambda}(s)g\}$ is positive, then if $g > 0$, this pointwise limit gives

$$[\mathbb{G}_{t,\lambda}(s)g](x,n) \geq 0 \text{ for a.e. } (x.n) \in \mathbb{R}^3 \times \mathbb{N}$$

and the results follow. The proof of the second part of this lemma is very similar to the proof of the first part with the additional note that the limit (4.43) is uniform in t and τ, which concludes the proof of the lemma. □

The above discussions allow the stating of the following existence result for an approximated discrete and non-autonomous fragmentation model in a moving medium.

Theorem 4.1.8. *Let $\lambda > 0$. Under the assumptions of the Lemma 4.1.6, the families defined in (4.42) and (4.43) are respectively an evolution semigroup and a forward propagator generated by the operator $(T_\lambda(t) = Q(t) + \mathcal{D}_\lambda, D(Q(t)))$, $(0 \leq t \leq T)$ defining the Cauchy problem (4.41). Furthermore, the solution given, for all initial data $p(\tau_0) = p_{\tau_0} \in \mathcal{Y}_1$, by*

$$p(t,x,n) = [\mathbb{U}_\lambda(t,\tau_0))p_{\tau_0}](x,n), \quad n \in \mathbb{N},\ 0 \leq \tau_0 < t, \tag{4.44}$$

satisfies the perturbed Cauchy problem (4.41).

4.1.6 Existence results: Discussion

Now, what happens if we tend $\lambda \longrightarrow \infty$! It is known, *via* (4.30), that the perturbed Cauchy problem (4.41) tends to the discrete and non-autonomous fragmentation model in a moving medium (4.3) as $\lambda \longrightarrow \infty$. For this reason, the solution obtained in Theorem 4.1.8 can serve to approximate a solution for the Cauchy problem (4.3). The results obtained here, where we had to deal with a two parameter family of bounded linear operators, improve the preceding ones [70, 57] where the two processes involved in the system, namely, transport and non-autonomous fragmentation, were treated separately. However, the problem of characterising the full generator is still an open problem for this type of perturbed non-autonomous and transport model.

4.2 Analysis by approximation technique

4.2.1 Introduction

In this section, forward propagators are used to investigate the fragmentation part of the system (4.1). A part of the technique here is similar to the one used in the Section 4.1. Our analysis consists of approximating the solution of that fragmentation model which is discrete, non-local and non-autonomous by a sequence of solutions of cut-off problems of a similar form. Then, the classical argument of Dini [50, Lemma 4], and Duhamel formula can be exploited to show existence of strong solutions of the model in the class of Banach spaces (of functions with finite higher moments) $\mathcal{X}_r := L_1(\mathfrak{I}, X_r)$, where

$$X_r = \{g : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g(x,n),\ \|g\|_r := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r |g(x,n)| dx < \infty\}.$$

This result is a great contribution to the proof of uniqueness of strong solutions to the discrete, non-local and non-autonomous fragmentation model which is still ongoing. Taking into account assumption (4.2), the fragmentation part of the system (4.1) yields

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) &= \sum\limits_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)p(t,y,m)dy \\ & -a(t,x,n)p(t,x,n), \\ p(\tau_0,x,n) &= p_{\tau_0}(x,n), \quad 0 \le \tau_0 < t \le T, \quad n = 1,2,3\ldots, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.45}$$

with the coefficients a and b satisfying the conditions (4.4)-(4.5). As in the previous section, the total mass $\sum_{n=1}^{\infty} n \int_{\mathbb{R}^3} p(t,x,n)dx$ is expected to remain unchanged, then the suitable Banach space to work in reads as

$$\{g : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g(x,n),\ \|g\|_1 := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n|g(x,n)| dx < \infty\} \tag{4.46}$$

We restrict the analysis, as in the previous chapter, to the class of Banach spaces of functions with finite higher moments:

$$X_r := \{g : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g(x,n),\ \|g\|_r := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r |g(x,n)| dx < \infty\}, \tag{4.47}$$

$r \ge 1$, which coincides with (4.46) for $r = 1$.

Let us put

$$\mathcal{X}_r = L_1(\mathfrak{I}, X_r), \quad \mathfrak{I} = [0,T].$$

Let us recall that until now, models with time dependent coefficients (non-autonomous) remain barely touched and there are still only few studies devoted to their analysis (well-posedness, conservativeness, honesty.) In [57], the authors used techniques of truncation to prove existence, uniqueness and mass conservation for a model of type (4.45) under certain conditions on initial data and associated truncated system. The authors in [6] used the evolution semigroups approach which allowed them to build on the substochastic semigroup theory and obtain an equivalent result as in [57]. In the analysis of the book by Tosio Kato [50], and later improved by Da Prado *et al.* [33], it is generally assumed that the generators $A(t)$ and $B(t)$ involved in the perturbation are of class $\mathcal{G}(1,0)$. This condition is modified in [65] where the authors used the semigroup perturbation and renormalisation approach to show that the closure of the involved operators is an anti-generator.

Let us rewrite (4.45) as non-autonomous abstract Cauchy problem in $\mathcal{X}_r$:

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,n) = Q(t)p(t,x,n), \\ p(\tau_0,x,n) = p_{\tau_0}(x,n), \;\; 0 \le \tau_0 < t \le T, \;\; n = 1,2,3\ldots, \;\; x \in \mathbb{R}^3, \end{cases} \tag{4.48}$$

or in the compact form

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{p}(t) = \mathcal{Q}(t)\mathbf{p}(t) \\ \mathbf{p}(t)|_{t=\tau_0} = \mathbf{p}_{\tau_0}, \end{cases} \tag{4.49}$$

where $\mathbf{p}(t)$ is the vector $\mathbf{p}(t) = (p(t,x,n))_{n=1}^{\infty}$, $\mathbf{P}_{\tau_0}$ the mass distribution vector $(p_n(\tau_0,x))_{n\in\mathbb{N}}$ at the fixed time $\tau_0 \ge 0$ and position x and $\mathcal{Q}(t)$ the non-autonomous fragmentation operator defined by

$$\begin{aligned} \mathcal{Q}(t)\mathbf{p}(t) &= ([Q(t)p(t)](t,x,n))_{n=1}^{\infty} \\ &:= \left(\sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)p(t,y,m)dy - a(t,x,n)p(t,x,n) \right)_{n=1}^{\infty}, \end{aligned} \tag{4.50}$$

here, $Q(t)$ is seen as the pointwise operation

$$\psi(t,x,n) \longmapsto \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)\psi(t,y,n)dy - a(t,x,n)\psi(t,x,n) \tag{4.51}$$

defined on the set of measurable functions. $Q(t)$ is defined by $Q(t) = \mathfrak{Q}(t)$ and represents the realisation of $\mathfrak{Q}(t)$ in the domain

$$D(Q(t)) = \{p \in \mathcal{X}_r;\;\; \mathfrak{Q}(t)p(t) \in \mathcal{X}_r\}, \tag{4.52}$$

with $\mathfrak{Q}(t)p(t)$ given by (4.51). To investigate and analyse the abstract Cauchy Problem (4.48) and show existence for this system, we will also need the two parameter family defined previously as propagator or evolution system (see Definition 4.1.1) together with the relation (4.14) .

4.2.2 Mathematical setting and analysis in $\mathcal{X}_r$

We have set

$$X_r := L_1(\mathbb{R}^3 \times \mathbb{N}, d\mu dm_r)$$

to get $\mathcal{X}_r = L_1(\mathfrak{I}, X_r)$, where $\mathbb{N}$ is equipped with the weighted counting measure dm_r with weight n^r and $d\mu = dx$ is the Lebesgue measure in $\mathbb{R}^3$ and $\mathfrak{I} = [0,T]$. The operator $(Q(t), D(Q(t)))$ represented by (4.51)-(4.52) can also be defined in X_r. For any given $K \in \mathbb{N}$, we introduce the projection operator q_K defined for a function $g \in \mathcal{X}_r$ defined as

$$[q_K g](x,n) = \begin{cases} g(x,n), & K > n > 0 \text{ and } x \in \mathbb{R}^3 \\ 0, & \text{otherwise.} \end{cases} \tag{4.53}$$

The space

$$\mathcal{X}_{r,K} = \{g \in \mathcal{X}_r : \ g(x,n) \equiv 0 \ \text{ on } \ \mathbb{R}^3 \times (K,\infty)\} \tag{4.54}$$

is therefore a closed subspace of $\mathcal{X}_r$ on which the projection operator q_K acts. We associate to (4.48) the following shortened version

$$\begin{cases} \frac{dp}{dt}(t) = Q(t)q_K p(t), \\ p(\tau_0) = p_{\tau_0}, \ \ 0 \le \tau_0 < t \le T, \end{cases} , \tag{4.55}$$

where $Q(t)$ is represented by (4.51). Let us set $Q_K(t) = Q(t)q_K$, then $Q_K(t)$ can be decomposed as $Q_K(t) = A_K(t) + B_K(t)$ where the loss and the gain fragmentation operators $A_K(t)$ and $B_K(t)$ are given by

$$A_K(t)g(t,x,n) = A(t)q_K g(t,x,n) = -a(t,x,n)g(t,x,n) \tag{4.56}$$

and

$$\begin{aligned} B_K(t)g(t,x,n) &= B(t)q_K g(t,x,n) \\ &= \sum_{m=n+1}^{K} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)g(t,y,m)dy, \end{aligned} \tag{4.57}$$

where $A(t)$ and $B(t)$ are expressed as

$$A_K(t) = A(t) \tag{4.58}$$

and

$$B(t)g(t,x,n) = \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)g(t,y,m)dy. \tag{4.59}$$

Thus, for all $g \in \mathcal{X}_r$, $K \in \mathbb{N}$ and $(t,x,n) \in \mathfrak{J} \times \mathbb{R}^3 \times \mathbb{N}$,

$$Q_K(t)g(t,x,n) = \begin{cases} \sum_{m=n+1}^{K} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)g(t,y,m) \\ \qquad -a(t,x,n)g(t,x,n), \qquad K > n > 0 \\ 0, \qquad \text{otherwise}. \end{cases} \tag{4.60}$$

We assume that for t fixed in $\mathbb{R}$, there are two constants $0 < \theta_1$ and θ_2 such that

$$\theta_1\alpha(n,t) \le a_t(x,n) = a(t,x,n) \le \theta_2\alpha(n,t) \ \text{ for almost all } \ (x,n) \in \mathbb{R}^3 \times \mathbb{N}, \tag{4.61}$$

where $\alpha(n,t) \in L_{\infty,loc}(\overline{\mathbb{R}_+{}^2})$ is a real-valued function which can depend on n and t but is independent of the state variable x. This obviously implies that for any $K \in \mathbb{N}$, there exists a positive $\Theta_{t,K}$ such that

$$\text{ess } \sup_{\mathbb{R}^3 \times (0,K)} a_t(x,n) \le \Theta_{t,K}. \tag{4.62}$$

The sequence $\{\Theta_{t,K}\}_{K\in\mathbb{N}}$ (t fixed in $\mathbb{R}$) is not necessarily bounded. Using the conditions (4.5), we can prove that

$$\sum_{m=1}^{n-1} m^r b_{m,n}(t,x) \leq n^r \tag{4.63}$$

for any $r \geq 1$, $n \geq 2$, $x \in \mathbb{R}^3$ at any time t. In fact,

$$n^r - \sum_{m=1}^{n-1} m^r b_{m,n}(t,x) \geq n^r - (n-1)^{r-1} \sum_{m=1}^{n-1} m b_{m,n}(t,x) = n^r - n(n-1)^{r-1} \geq 0.$$

Note that the equality holds for $r = 1$.

Lemma 4.2.1. *For t fixed in $\mathfrak{J}$, and $K \in \mathbb{N}$, there is a positive uniformly continuous semigroup of contractions on $\mathcal{X}_{r,K} \subset \mathcal{X}_r$, say $(G_{t,K}(s))_{s\geq 0}$ generated by the operator $Q_K(t)$ such that $(G_{t,K}(s))_{s\geq 0}$ is conservative on $(\mathcal{X}_{r,K})_+$ and given by*

$$G_{t,K}(s) = I + \sum_{j=1}^{\infty}[(\frac{s^j(Q(t))^j}{j!}]q_K. \tag{4.64}$$

Moreover, for any $N \geq K$, $s \geq 0$,

$$q_K G_{t,N}(s) q_K = G_{t,N}(s) \tag{4.65}$$

Proof. Let us fix t in $\mathfrak{J}$. The operator $A_K(t)$ is bounded by (4.62). Changing the order of summation by the Fubini theorem, for every $g \in \mathcal{X}_{r,K}$,

$$\begin{aligned}
\|B_K(t)g\|_r &= \int_{\mathbb{R}^3}\left(\sum_{n=1}^{\infty} n^r \sum_{m=n+1}^{K} \int_{\mathbb{R}^3} a(t,y,m)b(t,y,n,m)h(x,n,m,y)|g(t,y,m)|dy\right) dx \\
&= \int_{\mathbb{R}^3}\left(\sum_{n=1}^{\infty} n^r \sum_{m=n+1}^{K} a(t,y,m)b(t,y,n,m)|g(t,y,m)|\right) dy \\
&= \int_{\mathbb{R}^3}\sum_{m=2}^{K} a(t,y,m)|g(t,y,m)|\left(\sum_{n=1}^{\infty} n^r b(t,y,n,m)\right) dy \\
&= \int_{\mathbb{R}^3}\sum_{m=2}^{K} a(t,y,m)|g(t,y,m)|\left(\sum_{n=1}^{m-1} n^r b(t,y,n,m)\right) dy \\
&\leq \int_{\mathbb{R}^3}\sum_{m=2}^{K} m^r a(t,y,m)|g(t,y,m)|dy \\
&= \int_{\mathbb{R}^3}\sum_{m=1}^{K} m^r a(t,y,m)|g(t,y,m)|dy
\end{aligned}$$

$$\begin{aligned}
&\le \int_{\mathbb{R}^3} \sum_{m=1}^{\infty} m^r a(t,y,m)|g(t,y,m)|dy \\
&= \|A_K(t)g\|_r \\
&= \Theta_{t,K}\|g\|_r \\
&< \infty,
\end{aligned}$$

where (4.2), (4.5) and (4.63) have been used respectively. Then $B_K(t)$ is also bounded. Hence, $Q_K(t)$ generates a uniformly continuous semigroup. This semigroup is denoted by $(G_{t,K}(s))_{s\geq 0}$. Clearly, $A_K(t)$ generates a positive semigroup of contractions and $B_K(t)$ is a positive operator. Moreover, the above calculations also imply that $D(B_K(t)) \supset D(A_K(t))$ and

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{\mathbb{R}^3} n^r (A_K(t)g(t,x,n) + B_K(t)g(t,x,n))dx \\
&= \sum_{n=1}^{K} \int_{\mathbb{R}^3} n^r (A_K(t)g(t,x,n) + B_K(t)g(t,x,n))dx \le 0
\end{aligned}$$

for all $g \in (D(A_K(t)))_+$ with

$$D(A_K(t)) = \{g \in \mathcal{X}_r;\ -a_t g \in \mathcal{X}_{r,K}\} \quad \text{and} \quad D(B_K(t)) = \{g \in \mathcal{X}_r;\ B_K(t)g \in \mathcal{X}_{r,K}\}.$$

Thus, the assumptions of Kato's Theorem 2.3.5 hold. We essentially note that for each fixed t the operator $Q_K(t)$ becomes independent of time [57, Lemma 2.1] and Kato's Theorem is immediately applicable. Therefore, there is an extension $\breve{Q}_K(t)$ of $Q_K(t)$ which generates a substochastic semigroup. Because $a_t(x,n)$ is bounded in $\mathbb{R}^3 \times (0,K)$, this substochastic semigroup is conservative, it follows that $\breve{Q}_K(t) = \overline{Q}_K(t)$, where $\overline{Q}_K(t)$ is the closure of $Q_K(t)$. Since $Q_K(t)$ generates a uniformly (and hence strongly) continuous semigroup, $Q_K(t)$ is a closed operator. Therefore, we have that $\breve{Q}_K(t) = Q_K(t)$, consequently, the uniformly continuous semigroup $(G_{t,K}(s))_{s\geq 0}$ is a positive strongly continuous semigroup of contractions, furthermore, $(G_{t,K}(s))_{s\geq 0}$ is honest.

The proof of (4.64) is clear since the usual power series definition can be used to define $G_{t,K}(s) = \exp(sQ_K(t))$. By induction, $(Q_K(t))^J = (Q(t)q_K)^J = (Q(t))^J q_K$ for $J = 1, 2, \cdots$, from which the exponential formula yields (4.64).

To prove (4.65), it is stated that $B(t)q_K g = q_K B(t) q_K g$ on $\overline{B(t)}(0_{\mathbb{R}^3}, K) \times [0, K]$ since for $K \geq n \geq 0$, $B(t)q_K g(t,x,n)$ is given by (4.59) and $B(t)q_K g(t,x,n) = 0$ for $K < n$. Moreover, it is obvious that $A(t)q_K g = q_K A(t) q_K g$, hence, we have also

$$(A(t) + B(t))q_K = A(t)q_K + B(t)q_K = q_K A(t)q_K + q_K B(t)q_K = A_K(t) + B_K(t) = Q_K(t).$$

Next, by $q_K q_N = q_N q_K = q_K$ we have

$$q_K Q_N(t) q_K = q_K q_N Q(t) q_N q_K = q_K Q(t) q_K = Q_K(t)$$

if we assume, by induction, that $q_K(Q_N(t))^{j-1}q_K = (Q_K(t))^{j-1}$, then

$$\begin{aligned} q_K(Q_N(t))^j q_K &= q_K(Q_N(t))^{j-1}Q_N(t)q_K \\ &= q_K(Q_N(t))^{j-1}q_N Q(t) q_N q_K \\ &= q_K(Q_N(t))^{j-1}q_N q_K Q(t) q_K \\ &= q_K(Q_N(t))^{j-1}q_K Q_K(t) \\ &= (Q_K(t))^j. \end{aligned}$$

Now using (4.64) and the the fact that $Q_N(t)$ is a bounded operator, the semigroup generated by $Q_N(t)$ is expressed by

$$\begin{aligned} q_K G_{t,N}(s) q_K &= \sum_{n=0}^{\infty} \frac{s^j q_K(Q_N(t))^j q_K}{n!} \\ &= \sum_{n=0}^{\infty} \frac{s^j (Q_K(t))^j}{j!} \\ &= G_{t,N}(s), \end{aligned}$$

which concludes the lemma. □

Next, we assume that a(t,x,n) satisfies the Lipschitz condition

$$|a(t,x,n) - a(\sigma,x,n)| \leq |t-\sigma|\Lambda(x,n), \quad t,\sigma \in \mathfrak{I} \tag{4.66}$$

where $\Lambda(x,n) \geq 0$ together with $\Lambda(x,n) \leq \Theta_K$ for all $K \geq n > 0$, we state the following lemma

Lemma 4.2.2. *The function $t \longrightarrow Q_K(t)$ is continuous in the uniform operator topology for each K fixed in $\mathbb{N}$*

Proof. Using Fubini's theorem and assumption (4.66) yields

$$\|Q_K(t)g - Q_K(\sigma)g\|_r \leq K\Theta_K|t-\sigma|\|g\|_r, \quad \text{for all } g \in \mathcal{X}_r$$

and the result follows. □

Using (4.14) and the Lemma 4.2.1, there is a forward propagator, say $\{U_K(t,\tau)\}_{0\leq\tau\leq t\leq T}$ defined in $\mathcal{X}_{r,K} \subset \mathcal{X}_r$ which is associated to the evolution semigroup $(G_{t,K}(s))_{s\geq 0}$, $t \in \mathfrak{I}$. The propagator $\{U_K(t,\tau)\}_{0\leq\tau\leq t\leq T}$ shares certain properties with the family of semigroups $(G_{t,K}(s))_{s\geq 0}$, $t \in \mathfrak{I}$, as stated in the following theorem and proven in [57, Theorem 4.1].

Theorem 4.2.3. *For each $K \in \mathbb{N}$, the forward propagator $\{U_K(t,\tau)\}_{0\leq\tau\leq t\leq T}$ generated by the family of infinitesimal generators $\{Q_K(t)\}_{t\in\mathfrak{I}}$ have the following properties:*

1. $U_K(t,\tau)$ *is positive;*

2. $\|U_K(t,\tau)g\|_r = \|\|g\|_r$, *for all* $g \in \mathcal{X}_r$;

3. $\sum_{n=0}^{\infty} n^r[U_K(t,\tau)g](t,x,n) = \sum_{n=0}^{\infty} n^r g(t,x,n)$, *for all* $g \in \mathcal{X}_r$, $x \in \mathbb{R}^3$;

4. $\frac{\partial}{\partial t}U_K(t,\tau) = Q_K(t)U_K(t,\tau), \qquad 0 \le \tau \le t \le T;$

5. $\frac{\partial}{\partial \tau}U_K(t,\tau) = -U_K Q_K(t,\tau), \qquad 0 \le \tau \le t \le T;$

Theorem 4.2.4. *The truncated problem* (4.55) *has a unique, strongly continuously differentiable, positive, mass-conserving solution for all initial data* $p(\tau_0) = p_{\tau_0} \in \mathcal{X}_{r,K}$. *The solution is given by* $p(t) = U_K(t,\tau_0)p_{\tau_0}$ $(0 \le \tau_0 \le t \le T)$.

Proof. This theorem is an immediate consequence of Lemmas 4.2.1 and 4.2.2, Theorem 4.2.1 associated with [74, Theorem 5.1]. □

Now we consider the strong limit as $K \longrightarrow \infty$ of the forward propagator $\{U_K(t,\tau)\}_{0\le\tau\le t\le T}$. From Theorem 4.2.3, $U_K(t,\tau)$ is not two-sided differentiable with respect to t at $t = \tau$, but we have that $\frac{\partial}{\partial t}U_K(t,\tau)p$ is continuously differentiable [74] and

$$\frac{\partial}{\partial t}U_K(t,\tau)p = Q_K(t)U_K(t,\tau)p, \qquad 0 \le \tau \le t \le T, \qquad p \in \mathcal{X}_{r,K}.$$

The family $\{U_K(t,\tau)\}_{0\le\tau\le t\le T}$ defined in Theorem 4.2.3 can be extended to the uniformly continuous family of operators defined on $\mathcal{X}_r$ by

$$\tilde{U}_K(t,\tau) = q_K U_K(t,\tau) q_K.$$

It should be noted that $\tilde{U}_K(\tau,\tau) \ne I_{\mathcal{X}_r}$ for $0 \le \tau \le T$, therefore $\tilde{U}_K(t,\tau)$ is no longer a propagator. On the other hand, the operator $Q_K(t)$, as a bounded operator on $\mathcal{X}_r$, generates a uniformly continuous forward propagator, denoted by $\{\mathcal{U}_K(t,\tau)\}_{0\le\tau\le t\le T}$. As the restriction of $Q_K(t)$ to the complement of $\mathcal{X}_{r,K}$ is the zero operator, it generates there a constant propagator and we have

$$\mathcal{U}_K(t,\tau) = q_K U_K(t,\tau) q_K + (I_X - q_K), \tag{4.67}$$

where $I_{\mathcal{X}_r}$ is the identity on $\mathcal{X}_r$. Thus,

$$\mathcal{U}_K(t,\tau) q_K p = \tilde{U}_K(t,\tau)p.$$

Proposition 4.2.5. *For a fixed* t *in* $\mathfrak{I}$, *the families* $\{\mathcal{U}_K(t,\tau)\}_{0\le\tau\le t\le T}$ *and* $(\tilde{U}_K(t,\tau))_{s\ge 0}$ *have the following properties:*

1. *For any fixed* τ *in* $[0,t)$, *the family* $(\tilde{U}_K(t,\tau))_{0\le\tau\le t\le T}$ *is increasing with* K;

2. *There is a positive, strongly continuous family of forward propagators, let us say* $(U(t,\tau))_{0\le\tau\le t\le T}$*, such that for* $p \in \mathcal{X}_r$*, and* $0 \le \tau \le t \le T$

$$U(t,\tau)p = \lim_{K\to\infty} \tilde{U}_K(t,\tau)p = \lim_{K\to\infty} \mathcal{U}_K(t,\tau)p \quad \text{in } \mathcal{X}_r; \tag{4.68}$$

3. *Both limits in* (4.68) *are uniform in* t *and* τ *on bounded intervals. In particular, for* $p_{\tau_0} \in \mathcal{X}_{r,K}$*,*

$$U(t,\tau)p_{\tau_0} = q_N U_N(t,\tau) q_N p_{\tau_0} \quad \text{for any } N \ge K. \tag{4.69}$$

Proof. (*1*) Let $p \ge 0$ and define

$$p_K(t) = q_K U_K(t,\tau) q_K p = \tilde{U}_K(t,\tau)p \ge 0, \quad \tau \le t.$$

By the monotonicity of the projection operators we have

$$(q_{K+1} - q_K)p_{K+1}(t) \ge 0.$$

On the other hand, because

$$\frac{d}{dt}p_{K+1} = Q_{K+1}(t)p_{K+1},$$

then,

$$\frac{d}{dt}q_K p_{K+1} = q_K Q_{K+1}(t) q_K p_{K+1} + q_K Q_{K+1}(t)(q_{K+1} - q_K)p_{K+1}.$$

However, $q_K Q_{K+1}(t) q_K = Q_K(t)$ and $q_K A_{K+1} = q_K A_K$ so that

$$\begin{aligned} q_K Q_{K+1}(t)(q_{K+1} - q_K)p_{K+1} &= q_K A_K (q_{K+1} - q_K)p_{K+1} + q_K B_{K+1}(q_{K+1} - q_K)p_{K+1} \\ &= q_K B_{K+1}(q_{K+1} - q_K)p_{K+1} \ge 0, \end{aligned}$$

and

$$q_K p_{K+1}(0) = q_K p = p_K(0).$$

Thus, by the Duhamel formula in $\mathcal{X}_{r,K}$,

$$\begin{aligned} q_K p_{K+1}(t) &= U_K(t,\tau) q_K p + \int_0^t U_K(t-\sigma,\tau) q_K B_{K+1}(q_{K+1} - q_K)p_{K+1}(\sigma)d\sigma \\ &\ge U_K(t,\tau) q_K p \end{aligned}$$

and

$$q_K p_{K+1}(t) = q_K q_K p_{K+1}(t) \ge q_K U_K(t,\tau) q_K p = \tilde{U}_K(t,\tau)p.$$

Combining the estimates, we obtain

$$\tilde{U}_{K+1}(t,\tau)p = p_{K+1}(t) = q_{K+1}p_{K+1}(t) \ge q_K p_{K+1}(t) \ge \tilde{U}_K(t,\tau)p,$$

and the result follows.
(2) The family $\{\mathcal{U}_K(t,\tau)\}_{0\le\tau\le t\le T}$ is not increasing with K; we have, however, $\mathcal{U}_K \ge \tilde{U}_K$. Because the space $\mathcal{X}_r$ is a KB-space and the sequence $(\tilde{U}_K(t,\tau))_{0\le\tau\le t\le T}$ is increasing with

$$\|\tilde{U}_K(t,\tau)p\|_r = \|U_K(t,\tau)p\|_{\mathcal{X}_{r,K}} = \|q_K p\|_{\mathcal{X}_{r,K}} \le \|p\|_r$$

provided $p \ge 0$, we can define

$$U(t,\tau)p = \lim_{K\to\infty} \tilde{U}_K(t,\tau)p, \quad 0 \le \tau \le t \le T \ \ p \ge 0,$$

and by linearity this definition can be extended to arbitrary $p \in \mathcal{X}_r$. Moreover, from (4.67), we get

$$\mathcal{U}_K(t) - \tilde{U}_K(t,\tau) = (I_{\mathcal{X}_r} - q_K)$$

and because

$$\lim_{K\to\infty}(I_{\mathcal{X}_r} - q_K)p = 0$$

for any fixed p then,

$$U(t,\tau)p = \lim_{K\to\infty} \mathcal{U}_K(t)p, \quad 0 \le \tau \le t \le T \quad \text{for any } p \in \mathcal{X}_r.$$

Therefore, $(U(t,\tau))_{0\le\tau\le t\le T}$ is the strong limit of a sequence of uniformly bounded positive propagators. We need to show that $(U(t,\tau))_{0\le\tau\le t\le T}$ is a positive strongly continuous propagators. The relation $U(t,r)U(r,\tau) = U(t,\tau)$ for $0 \le \tau \le r \le t \le T$. is just the limit relation for $\{\mathcal{U}_K(t,\tau)\}_{0\le\tau\le t\le T}$. For any $p \in \mathcal{X}_r$, we set $p_K = q_K p$ for some fixed K; Then for $N > K$ we have

$$\mathcal{U}_N(t,\tau)p = \tilde{U}_N(t,\tau)p$$

and for such N, as $t \to \tau^+$,

$$\begin{aligned}
\|U(t,\tau)p_K - p_K\|_r &\le \|U(t,\tau)p_K - \tilde{U}_N(t,\tau)p_K\|_r + \|\tilde{U}_N(t,\tau)p_K - p_K\|_r \\
&= \|U(t,\tau)p_K\|_r - \|\tilde{U}_N(t,\tau)p_K\|_r + \|\tilde{U}_N(t,\tau)p_K - p_K\|_r \\
&\le \|p_K\|_r - \|\mathcal{U}_N(t,\tau)p_K\|_r + \|\mathcal{U}_N(t,\tau)p_K - p_K\|_r \to 0.
\end{aligned}$$

For arbitrary p, the density of compactly supported functions in $\mathcal{X}_r$ and the boundedness of $(U(t,\tau))_{0\le\tau\le t\le T}$ are used to conclude the assertion.
(3) The uniform convergence of $(\tilde{U}_K(t,\tau))_{0\le\tau\le t\le T}$ follows from the classical argument of Dini, as in ([50], Lemma 4). To prove this statement for $\{\mathcal{U}_K(t,\tau)\}_{0\le\tau\le t\le T}$, it is enough to note that the difference between $\{\mathcal{U}_K(t,\tau)\}_{0\le\tau\le t\le T}$ and $(\tilde{U}_K(t,\tau))_{0\le\tau\le t\le T}$ is independent of the parameter t. Equation (4.69) follows directly from relation (4.14) and the last statement of Lemma 4.2.1. □

4.2.3 The existence of solutions to the discrete, non-local and non-autonomous fragmentation model: Discussions

At this stage of the analysis, there is a temptation to use time-dependent analogue of the Hille-Yosida Theorem 2.2.4, see also [88], to find the infinitesimal generator of the limit forward propagator $(U(t,\tau))_{0\le\tau\le t\le T}$ and to verify that this generator coincides with the operator $Q(t)$ expressed in (4.48). However, the challenge we face is that there is no such a time-dependent analogue, and there is no way of finding a family of generators that uniquely characterises the forward propagator $(U(t,\tau))_{0\le\tau\le t\le T}$ or at least not yet (the question remains an open problem). We have proven (Theorem 4.2.4) that the discrete, non-local and non-autonomous fragmentation model (4.45) has a unique solution whenever the initial mass distribution $p_{\tau_0}(x,n) \equiv p_{n,\tau_0}(x)$ at some fixed time $\tau_0 \ge 0$ is in $\bigcup_K \mathcal{X}_{r,K}$. However, the existence of solutions to such a model with $p_{\tau_0}(x,n) \in \mathcal{X}_r$ is guaranteed by imposing the additional condition (4.62) and investigating the pointwise limit

$$p(t,x,n) = \left[\lim_{K\to\infty} U(t,\tau_0)p_{\tau_0}\right](x,n) \quad x \in \mathbb{R}^3,\ n \in \mathbb{N},\ 0 \le \tau_0 \le t \le T,$$

where $(U(t,\tau))_{0\le\tau\le t\le T}$ is the limit forward propagator defined in (4.68). This analysis, which differs from the one used in [57], is what was done in Lemma 4.2.1 and Proposition 4.2.5 above, by approximating the solution of (4.45) by a sequence of solutions of cut-off problems of a similar form and showing mass conservativeness in the closed subspace $\mathcal{X}_{r,K}$. Analysis that tend to prove uniqueness of strong solutions to the discrete, non-local and non-autonomous fragmentation model considered in this study is still in progress around the world and there is hope that this study will make a significant contribution in improving this situation.

4.3 An equivalent norm approach and conservativeness for a non-autonomous fragmentation model

4.3.1 Introduction

To conclude this chapter, the equivalent norm approach and semigroup perturbation theory are used to state and set conditions for a non-autonomous fragmentation system to be conservative. It is commonly assumed that the generators are of class $\mathcal{G}(1,0)$, see [33, 50], but this condition is modified in this section. Instead, we assume that the renormalisable generators involved in the perturbation process are in the class of quasi-contractive semigroups. We can henceforth exploit Hille-Yosida's condition (Theorem 2.2.4), the uniform boundedness [50], Hermitian conjugate [74] and the admissibility with respect to the involved operators to show that the sum of these operators is closable, its closure generates a propagator (evolution system) and, therefore, a C_0 semigroup, leading to the existence result and conservativeness of the model.

The dynamical non-autonomous system under investigation in this section is given by the integro-differential system:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = -a(t,x)u(t,x) + \int_x^\infty a(t,y)b(t,x,y)u(t,y)dy \\ u(\tau,x) = u_\tau(x), \qquad\qquad\qquad\qquad 0 \le \tau < t \le T,\ x > 0, \end{cases} \tag{4.70}$$

where u is the particle mass distribution function ($u(\tau, x) = u_\tau(x)$ is the mass distribution at some fixed time $\tau \geq 0$) with respect to the mass x, $b(t, x, y)$ is the distribution of particle masses x spawned by the fragmentation of a particle of mass y, at the time $t \leq T \in \mathbb{R}$ and $a(t, x)$ is the evolutionary time-dependent fragmentation rate, that is, the rate at which mass x particles break up at a time t. In a practical point of view, the first term on the right-hand side of (4.70) represents the reduction in the number of particles in the mass range $(x; x + dx)$ due to the fragmentation of particles in the same range. The second term describes the increase in the number of particles in the range due to fragmentation of larger particles.
The idea here is to analyse the equation (4.70) in the Banach space $L_1(\mathfrak{I}, X_1)$ where $\mathfrak{I} = [0, T]$ and

$$X_1 = L_1([0, \infty), xdx) = \left\{ \psi : \quad \|\psi\|_{X_1} := \int_0^\infty x|\psi(x)|\, dx < \infty \right\},$$

using the theory of evolution semigroup.

Throughout, the following regularity assumptions will be considered:

$(t, x) \to a(t, x) \in L_1\left([0, T'], L_\infty([k, l])\right)$, for any $0 < k < l < \infty$ and $T' \in (0, T)$
$b(t, x, y)$ is a positive measurable function with

$$b(t, x, y) = 0 \text{ for all } x \geq y \text{ and } 0 \leq t \leq T, \tag{4.71}$$

and the local conservative law

$$\int_0^y xb(t, x, y)dx = y \tag{4.72}$$

for all $y \geq 0$ *and* $0 \leq t \leq T$.

Until now, existence results and honesty have been proven for a number of fragmentation (autonomous or non-local) models, see for e.g. [35, 70, 71, 12, 36], where the authors used various methods including the substochastic semigroup theory. As said earlier, Da Prado *et al.* [33] assumed that the generators $A(t)$ and $B(t)$ involved in the perturbation are of class $\mathcal{G}(1, 0)$, but this condition is modified in this study as we will see later in the analysis.

We recast (4.70) as the non-autonomous abstract Cauchy problem in X_1:

$$\begin{cases} \frac{d}{dt}u(t) = Q(t)u(t) & 0 \leq \tau < t \leq T \\ u(\tau) = u_\tau, \end{cases} \tag{4.73}$$

where $Q(t)$ is defined by $Q(t) = \mathcal{Q}(t)$ and represents the realisation of $\mathcal{Q}(t)$ on the domain $D(Q(t)) = \{u \in X_1;\ \mathcal{Q}(t)u(t) \in X_1\}$, with $(\mathcal{Q}u)$ defined as

$$(\mathcal{Q}u)(t, x) = (\mathcal{Q}u)(t)(x) = -a(t, x)u(t, x) + \int_x^\infty a(t, y)b(t, x, y)u(t, y)dy,$$

$\mathcal{Q}(t)$ is seen as the pointwise operation

$$\psi(t,x) \longmapsto -a(t,x)\psi(t,x) + \int_x^\infty a(t,y)b(t,x,y)\psi(t,x)dy$$

defined on the set of measurable functions. $\mathcal{Q}(t)$ indeed defines various operators. The aim here is to analyse the problem by rephrasing it in abstract form (abstract Cauchy Problem (ACP)) as an ordinary differential equation.

Let us start with something simple and come back to the abstract Cauchy Problem (4.73); From Definition 4.1.1, it is clear that for $0 \leq t \leq T$, $Q(t)$ is a bounded linear operator in X_1 and that $t \longrightarrow Q(t)$ is continuous in the uniform operator topology. Next, we will find the propagator $U(t,\tau)$, see Definition 4.1.1, associated with (4.73) such that $u(t) = U(t,\tau)u_\tau$ is in some sense, a solution of (4.73) satisfying the initial condition $u(\tau) = u_\tau$.

Lemma 4.3.1. *Let $Q(t)$ be a bounded linear operator in X_1 for $0 \leq t \leq T$. If the function $t \longrightarrow Q(t)$ is continuous in the uniform operator topology, then for every $u_\tau \in X_1$, the abstract Cauchy Problem (4.73) has a unique classical solution u given by the relation:*

$$u(t) = u_\tau + \int_\tau^t Q(\varsigma)u(\varsigma)\, d\varsigma. \tag{4.74}$$

Proof. see [74, Theorem 5.1, Chapter 5], the proof is done in a Banach space X which is also true in X_1. □

Theorem 4.3.2. *There is a propagator $U(t,\tau)$ associated with the initial value problem (4.73) such that $U(t,\tau)u_\tau$ is its solution satisfying the initial condition $u(\tau) = u_\tau$.*

Proof. From the Lemma (4.3.1), the existence and uniqueness of the solution can already be noticed. Let $u(t)$ be this solution. The so-called solution operator of (4.73) is defined by

$$U(t,\tau)u_\tau = u(t) \qquad \text{for } 0 \leq \tau \leq t \leq T. \tag{4.75}$$

- For every $u_\tau \in X_1$, $U(\tau,\tau)u_\tau = u(\tau) = u_\tau$ then $U(\tau,\tau) = I$ (condition (i)).
- For every $u_\tau \in X_1$, we have $U(t,\tau)u_\tau = u(t)$ and $U(t,r)U(r,\tau)u_\tau = U(t,r)u(r) = u(t)$, then condition (ii) follows from the uniqueness of the solution of (4.73).
- It is obvious that $U(t,\tau)$ is a linear operator defined in all X_1 since (4.73) is linear. The relation (4.74) implies $\|u(t)\| \leq \|u_\tau\| + \int_\tau^t \|Q(\varsigma)\| \, \|u(\varsigma)\| \, d\varsigma$ and from Gronwall's inequality, we also have $\|u(t)\| \leq \|u_\tau\| \exp\left(\int_\tau^t \|Q(\varsigma)\| \, d\varsigma\right)$. Then, (4.75) yields $\|U(t,\tau)u_\tau\| = \|u_\tau\| \exp\left(\int_\tau^t \|Q(\varsigma)\| \, d\varsigma\right)$,

leading to

$$\|U(t,\tau)\| = \exp\left(\int_\tau^t \|Q(\varsigma)\| \, d\varsigma\right). \tag{4.76}$$

Hence, $U(t,\tau)$ is bounded and, therefore, strongly continuous. This concludes the proof. □

The fact that $Q(t)$ is bounded actually makes this existence result easier to obtain. Unfortunately, $Q(t)$ is not always bounded and then, in the following subsection, a different approach is used to obtain an equivalent result.

4.3.2 Equivalent norm approach

Let us come back to the equation (4.73) and split it to have (4.70) written in the abstract form:

$$\begin{cases} \frac{d}{dt}u(t) = A(t)u(t) + B(t)u(t) & 0 \leq \tau < t \leq T \\ u(\tau) = u_\tau, \end{cases} \tag{4.77}$$

where $A(t)$ is defined as $A(t) = \mathcal{A}(t)$ and represents the realisation of $\mathcal{A}(t)$ on the domain $D(A(t)) = \{u \in X_1;\ \mathcal{A}(t)u \in X_1\}$, with

$$[\mathcal{A}(t)u_\tau](x) = -a(t,x)u_\tau(x) \tag{4.78}$$

and $B(t)$ is defined as $B(t) = \mathcal{B}(t)$ and represents the realisation of $\mathcal{B}(t)$ on the domain $D(B(t))$ with

$$[\mathcal{B}(t)u_\tau](x) = \int_x^\infty a(t,y)b(t,x,y)u_\tau(x)dy. \tag{4.79}$$

Making use of the assumptions (4.71) and (4.72), it is easy to show that (see [70] or [71]) for any $u \in X_1$, $B(t)u \in X_1$, so we can take $D(B(t)) = D(A(t))$ and $(A(t) + B(t), D(A(t)))$ is a well-posed operator. Let us put

$$\begin{aligned} \mathcal{X}_1 &= L_1(\mathfrak{I}, X_1) \\ &:= \left\{\psi : [0,T] \times \mathbb{R} \ni (\sigma, x) \to u(\sigma, x), \quad \|\psi\|_1 := \int_0^T \int_0^\infty x|\psi(\sigma,x)|d\sigma dx < \infty\right\}, \end{aligned}$$

in the following sections the subscript t in A_t means the operator A depends on time t but is defined in $\mathcal{X}_1$ instead of X_1. The aim here is to set some conditions in $\mathcal{X}_1$ under which the operator sum K_t:

$$K_t\psi = A_t\psi + B_t\psi, \text{ on } D(A_t) \cap D(B_t) = D(A_t) \tag{4.80}$$

is closable, its closure generates a propagator and therefore a C_0 semigroup. We rely on the following theorem which was originally proven by Tosio Kato [50] and later improved by Da Prado *et al.* [33].

Theorem 4.3.3. *Consider in $\mathcal{X}_1$ the operators A_t and B_t be generators both belonging to the class $\mathcal{G}(1,0)$. If $D(A_t) \cap D(B_t)$ is dense in $\mathcal{X}_1$ and $ran(A_t + B_t + \xi)$ is dense in $\mathcal{X}$ for some $\xi < 0$, then K_t is closable and its closure $\bar{K}_t$ is a generator from the class $\mathcal{G}(1,0)$.*

The condition A_t and $B_t \in \mathcal{G}(1,0)$ is dropped in this study to provide stronger results. Let the integral operator in (4.77) be treated as a perturbation of the much easier operator of multiplication by a on X_1

$$[A(t)u_\tau](x) = -a(t,x)u_\tau(x). \tag{4.81}$$

Recall that (Theorem 4.3.2 and [6]) $A(t)_{t\in\mathfrak{I}}$ ($\mathfrak{I} = [0,T]$) is a family of generators of C_0 semigroups in X_1, then, for any fixed $t \in \mathfrak{I} = [0,T]$, $A(t)$ generates a propagator $U(t,\tau)$, $0 \leq \tau < t \leq T$ and this propagator defines a C_0 semigroup $(S_{A_t}(s))_{s\geq 0}$ in $\mathcal{X}_1$ by the relation (4.14) given here as:

$$\begin{aligned}[S_{A_t}(s)u_\tau](\sigma) &= (\sigma - s)\chi_{\mathfrak{I}}\, U(\sigma,\sigma - s)\, u_\tau(\sigma - s)\\ &= (\sigma - s)\chi_{\mathfrak{I}} \exp\left(-\int_{\sigma-s}^{\sigma} a(\xi,.)d\xi\right) u_\tau(\sigma - s), \qquad u_\tau \in \mathcal{X}_1,\end{aligned} \tag{4.82}$$

Then, as mentioned earlier, A is said to be the generators of C_0-semigroups in $\mathcal{X}_1$ means A generates a propagator which defines a C_0-semigroups in $\mathcal{X}_1$ satisfying the relation (4.82). In the following definition, we assume that Y is a subspace of $\mathcal{X}_1$ which is closed with respect to the norm $\|.\|_Y$, not necessary in the norm $\|.\|_1$ (hence Y is itself a Banach space).

Definition 4.3.4. *Let $S_{A_t}(s)_{s\geq 0}$ be a C_0-semigroup and A_t its infinitesimal generator. A subspace Y of $\mathcal{X}_1$ is called A_t-admissible if it is an invariant subspace of $S_{A_t}(s)$, $s \geq 0$ i.e. $S_{A_t}(s)Y \subseteq Y$, and the restriction of $S_{A_t}(s)$ to Y (i.e. $S_{\check{A}_t}(s) := S_{A_t}(s)_{|Y}$, $s \geq 0$) is a C_0-semigroup in Y (i.e. it is strongly continuous in the norm $\|.\|_Y$).*
If $T : Y \longrightarrow \mathcal{X}_1$ is the embedding operator of Y into $\mathcal{X}_1$, we have

$$S_{A_t}(\alpha)Tu = TS_{\check{A}_t}(\alpha)u,\ u \in Y,$$

which gives

$$A_tTu = T\check{A}_tu,\ u \in D(\check{A}_t),$$

with

$$D(\check{A}_t) = \{u \in D(A_t) \cap Y : A_tu \in Y\}. \tag{4.83}$$

It should be recalled that the adjoint A_t^* of A_t is a linear operator from $D(A_t^*) \subset \mathcal{X}_1^*$ into $\mathcal{X}_1^*$ (the dual of $\mathcal{X}_1$) and is defined as follows: $D(A_t^*)$ is the set of all elements $x^* \in \mathcal{X}_1^*$ for which there is a $y^* \in \mathcal{X}_1^*$ such that

$$\langle x^*, A_tx\rangle = \langle y^*, x\rangle \quad \text{for all } x \in D(A_t) \tag{4.84}$$

and if $x^* \in D(A_t^*)$ then $y^* = A_t^*x^*$ where y^* is the element of $\mathcal{X}_1^*$ satisfying (4.84). With the the assumptions (4.71) and (4.72) in mind, the following lemma can be stated:

Lemma 4.3.5. *Let $\check{A}_t$ and $\check{B}_t$ two operators defined by (4.83) and satisfying, for all $\lambda \in (0,\infty) \subset \rho(\check{A}_t)$ and $\kappa \in (0,\infty) \subset \rho(\check{B}_t)$,*

$$\|(\lambda I - \check{A}_t)^{-1}\|_Y \leq \frac{1}{\lambda}, \tag{4.85}$$

$$\|(\kappa I - \check{B}_t)^{-1}\|_Y \leq \frac{1}{\kappa}, \tag{4.86}$$

in the Banach space Y. If either $\check{A}_t{}^*$ or $\check{B}_t{}^*$ are densely defined in Y^*, then for any $\eta < 0$, we have the inequality:

$$|\eta|\|v\|_{Y^*} \leq \|\check{A}_t{}^* v + \check{B}_t{}^* v + \eta v\|_{Y^*}, \ v \in D(\check{A}_t{}^*) \cap D(\check{B}_t{}^*). \tag{4.87}$$

Proof. Suppose that $D(\check{B}_t{}^*)$ is dense in Y^* and defines the sum

$$\check{K}_{t,\varepsilon} := \check{A}_t u + \check{B}_t(I + \varepsilon \check{B}_t)^{-1} u, \ \ u \in D(\check{K}) = D(\check{A}_t), \ \ \varepsilon < 0.$$

It is obvious that $\check{K}_{t,\varepsilon}$ also satisfies the relations (4.85) or (4.86) since $\check{A}_t$ and $\check{B}_t$ do. Then the relation (4.85) yields

$$\begin{aligned} \|(\lambda I - \check{K}_{t,\varepsilon})^{-1} u\|_Y &\leq \|(\lambda I - \check{K}_{t,\varepsilon})^{-1}\|_Y \|u\|_Y \\ &\leq \frac{1}{\lambda}\|u\|_Y, \ \ u \in Y, \ \lambda > 0, \ \varepsilon < 0, \end{aligned}$$

leading to

$$\begin{aligned} \|u\|_Y &\leq \frac{1}{\lambda}\|(\lambda I - \check{K}_{t,\varepsilon})u\|_Y, \ \ u \in Y, \ \lambda > 0, \ \varepsilon < 0 \\ &\leq \frac{1}{\lambda}\|(\check{K}_{t,\varepsilon} - \lambda I)u\|_Y, \ \ u \in Y, \ \lambda > 0, \ \varepsilon < 0 \\ &\leq \frac{1}{|\eta|}\|(\check{K}_{t,\varepsilon} + \eta I)u\|_Y, \ \ u \in Y, \ \varepsilon < 0, \quad \text{where we have set } -\lambda = \eta < 0 \end{aligned}$$

or

$$\|(\check{K}_{t,\varepsilon} + \eta I)u\|_Y \geq |\eta|\|u\|_Y, \ \ u \in D(\check{K}_{t,\varepsilon}) = D(\check{A}_t), \ \varepsilon < 0, \ \eta < 0.$$

Immediate properties of Hermitian conjugate give

$$\|(\check{K}^*{}_{t,\varepsilon} + \eta I)v\|_{Y^*} \geq |\eta|\|v\|_{Y^*}, \ \ v \in D(\check{K}^*{}_{t,\varepsilon}) = D(\check{A}_t^*), \ \varepsilon < 0, \ \eta < 0, \tag{4.88}$$

and

$$\check{K}^*{}_{t,\varepsilon} v = \check{A}_t^* v + \check{B}_t^*(I + \varepsilon \check{B}_t^*)^{-1} v, \ \ v \in D(\check{K}^*{}_{t,\varepsilon}) = D(\check{A}_t^*), \ \varepsilon < 0. \tag{4.89}$$

Since $\check{B}_t{}^*$ is densely defined in Y^*, then

$$(I + \varepsilon \check{B}_t^*)^{-1} \longrightarrow I \quad as \quad \varepsilon \nearrow 0$$

and then,

$$\check{K}^*{}_{t,\varepsilon} v \longrightarrow \check{A}_t^* v + \check{B}_t^* v \quad as \quad \varepsilon \nearrow 0.$$

Substituting the latter relation in (4.88) yields (4.87).
The approach is the same if we consider that it is rather $\check{A}_t{}^*$ which is densely defined in Y^*. □

Corollary 4.3.6. *Let A_t and B_t two closed and densely defined operators satisfying, for all $\lambda \in (0,\infty) \subset \rho(A_t)$ and $\kappa \in (0,\infty) \subset \rho(B_t)$,*

$$\|(\lambda I - A_t)^{-1}\|_1 \leq \frac{1}{\lambda}, \tag{4.90}$$

$$\|(\kappa I - B_t)^{-1}\|_1 \leq \frac{1}{\kappa} \tag{4.91}$$

on $\mathcal{X}_1$, let $Y \hookrightarrow \mathcal{X}_1$ be admissible with respect to A_t and B_t and let the operator B_t verify

$$Y \subseteq D(B_t). \tag{4.92}$$

We assume that the induced generators $\check{A}_t$ and $\check{B}_t$, given by (4.83), are closed, densely defined and satisfy the relations (4.85) and (4.86) respectively. If $D(B_t^*)$ is dense in $\mathcal{X}_1^*$, then

$$|\eta|\|v\|_{Y^*} \leq \|\check{A}_t{}^* v + \check{B}_t{}^* v + \eta v\|_{Y^*}, \quad v \in D(\check{A}_t{}^*) \cap T^*\mathcal{X}_1^*, \ \eta < 0\,, \tag{4.93}$$

where $T : Y \longrightarrow \mathcal{X}_1$ is the embedding operator.

Proof. Let $v \in D(\check{A}_t{}^*) \cap T^*\mathcal{X}_1^*$, then there is $w \in \mathcal{X}_1^*$ such that $v = T^*w$. We also have $T^*\mathcal{X}_1^* \subset D(\check{B}_t{}^*)$ thanks to the condition (4.92). Therefore, the relation (4.89) of the previous lemma applies to $v = T^*w$ as:

$$\check{K^*}_{t,\varepsilon} T^* w = \check{A}_t^* T^* w + \check{B}_t^* (I + \varepsilon \check{B}_t^*)^{-1} T^* w, \quad \varepsilon < 0.$$

Since T is the embedding operator of Y into $\mathcal{X}_1$, we have

$$\check{K^*}_{t,\varepsilon} T^* w = \check{A}_t^* T^* w + \check{B}_t^* T^* (I + \varepsilon B_t^*)^{-1} w, \quad \varepsilon < 0,$$

which is well posed since the operator $B_t T : Y \to \mathcal{X}_1$ is bounded thanks to (4.92). Since B_t^* is densely defined in $\mathcal{X}_1^*$, we have

$$(I + \varepsilon B_t^*)^{-1} \longrightarrow I \quad as \quad \varepsilon \nearrow 0$$

and then,

$$\check{K^*}_{t,\varepsilon} T^* w = \check{A}_t^* T^* w + \check{B}_t^* T^* (I + \varepsilon B_t^*)^{-1} w \longrightarrow \check{A}_t^* T^* w + \check{B}_t^* T^* w \quad as \quad \varepsilon \nearrow 0.$$

Substituting the latter relation in (4.88) with $v = T^*w$ yields (4.93). □

Remark 4. It is in general possible to find in the Banach Space $\mathcal{X}_1$ a new norm $\|.\|$, which is equivalent to its natural norm

$$\|u\|_1 := \int_0^T \int_0^\infty x|u(\sigma, x)|\, d\sigma dx,$$

such that the operator A_t becomes a generator of the contraction semigroups on $\mathcal{X}_1$.

Indeed, since A_t is the generator of a C_0 semigroup, say $(S_{A_t}(s))_{s\geq 0}$, there is $M > 0$ and ω such that $\forall s \geq 0$, $\|(S_{A_t}(s)\|_1 \leq M\mathrm{e}^{\omega s}$.
We have

$$\begin{aligned} \|(S_{A_t}(s)u\|_1 &\leq M\mathrm{e}^{\omega s}\|u\|_1, \ \forall u \in \mathcal{X}_1 \\ &\leq M_{A_t}\mathrm{e}^{\omega s}. \end{aligned} \tag{4.94}$$

Let the following be set

$$\|u\| = (MM_{A_t})^{-1} \sup_{s\geq 0} \mathrm{e}^{-\omega s} \int_0^T \int_0^\infty x|S_{A_t}(s)u(\sigma, x)|\, d\sigma dx$$

Simple calculations show that

$$\int_0^T \int_0^\infty x|u(\sigma, x)|\, d\sigma dx = \|u\|_1 \leq MM_{A_t}\|u\| \leq M^2M_{A_t}\|u\|_1, \ \forall u \in \mathcal{X}_1 \tag{4.95}$$

which proves that the norm $\|.\|$ is equivalent to $\|u\|_1$.
On the other hand we have

$$\|S_{A_t}(\varsigma)u\| = (MM_{A_t})^{-1}\mathrm{e}^{\omega\varsigma} \sup_{s\geq 0} \mathrm{e}^{-\omega(s+\varsigma)} \int_0^T \int_0^\infty x|(S_{A_t}(s)S_{A_t}(\varsigma)u(\sigma, x)|\, d\sigma dx$$

$$\|S_{A_t}(\varsigma)u\| \leq (MM_{A_t})^{-1}\mathrm{e}^{\omega\varsigma} \sup_{s\geq 0} \mathrm{e}^{-\omega(s+\varsigma)} MM_{A_t}\mathrm{e}^{\omega\varsigma}\mathrm{e}^{\omega s}$$

which gives

$$\|S_{A_t}(\varsigma)u\| \leq \mathrm{e}^{\omega\varsigma}$$

This proves that the semigroup $S_{A_t}(s)_{s\geq 0}$ is in the class $\mathcal{G}(1, \omega)$ of quasi-contractive semigroups in the Banach space $\mathcal{X}_1$ equipped with the norm $\|.\|$. Next, we extend this result and characterise the existence of an equivalent norm in $\mathcal{X}_1$ for the pair of generators $\{A_t, B_t\}$.

Definition 4.3.7. *Let A_t and B_t be the generators of C_0-semigroups $S_{A_t}(s)_{s\geq 0}$ and $S_{B_t}(s)_{s\geq 0}$ in $\mathcal{X}_1$. The pair $\{A_t, B_t\}$ is called renormalisable with constants ω and ν if for any sequences $\{\alpha_k\}_{k=1}^N$, $\alpha_k \geq 0$ and $\{\delta_k\}_{k=1}^N$, $\delta_k \geq 0$, $n \in \mathbb{N}$, one has*

$$\sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N}}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)u\|_1 < \infty \tag{4.96}$$

for each $u \in \mathcal{X}_1$.

Lemma 4.3.8. *Let A_t and B_t be two generators of C_0-semigroups in $\mathcal{X}_1$ equipped with its natural norm*

$$\|u\|_1 := \int_0^T \int_0^\infty x|u(\sigma, x)|\, d\sigma dx.$$

The pair $\{A_t,\, B_t\}$ is renormalisable with constants ω and ν if and only if there is an equivalent norm $\|.\|$ in $\mathcal{X}_1$ such that A_t and B_t are closed, densely defined and we have $(\omega, \infty) \subset \rho(A_t)$ and $(\nu, \infty) \subset \rho(B_t)$, so that for all $\lambda > \omega$, $\kappa > \nu$,

$$\|(\lambda I - A_t)^{-1}\| \leq \frac{1}{\lambda - \omega}, \tag{4.97}$$

$$\|(\kappa I - B_t)^{-1}\| \leq \frac{1}{\kappa - \nu}, \tag{4.98}$$

with

$$\rho(A_t) = \{\lambda \in \mathbb{C}, \lambda I - A_t : D(A_t) \to \mathcal{X}_1 \text{ invertible}\}$$

and

$$\rho(B_t) = \{\lambda \in \mathbb{C}, \lambda I - B_t : D(A_t) \to \mathcal{X}_1 \text{ invertible}\}$$

the resolvent sets of A_t and B_t respectively.

Proof. Let us suppose that there is an equivalent norm $\|.\|$ in $\mathcal{X}_1$ such that A_t and B_t are closed, densely defined and satisfy the relations (4.97) and (4.98), then using the Theorem 2.2.4 (Hille-Yosida's condition), there are ω and ν such that $\forall \alpha,\, \delta \geq 0$,

$$\|S_{A_t}(\alpha)u\| \leq \|u\| \mathrm{e}^{\omega\alpha},\ \ \|S_{B_t}(\delta)u\| \leq \|u\|\mathrm{e}^{\nu\delta},\ \ for\, all\, u \in \mathcal{X}_1.$$

Since $\|.\|$ and $\|.\|_1$ are equivalent, there are $M \geq 0$ and $N \geq 0$ such that

$$\|S_{A_t}(\alpha)u\|_1 \leq M\|S_{A_t}(\alpha)u\| \leq M\mathrm{e}^{\omega\alpha}$$

and

$$\|S_{B_t}(\delta)u\|_1 \leq N\|S_{B_t}(\delta)u\| \leq N\mathrm{e}^{\nu\delta},$$

leading to

$$\mathrm{e}^{-\omega\alpha}\|S_{A_t}(\alpha)u\|_1 \leq M_{A_t} < \infty,\, \forall \alpha \geq 0$$

and

$$\mathrm{e}^{-\nu\delta}\|S_{B_t}(\delta)u\|_1 \leq N_{B_t} < \infty,\, \forall \delta \geq 0 \,\mathrm{and}\, u \in \mathcal{X}_1.$$

It is observed that for any sequences $\{\alpha_k\}_{k=1}^N$, $\alpha_k \geq 0$, and $\{\delta_k\}_{k=1}^N$, $\delta_k \geq 0$, $n \in \mathbb{N}$, one has

$$\sup_{\substack{\alpha_1 \geq 0, \dots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \dots, \delta_n \geq 0 \\ n \in \mathbb{N}}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \|S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n) u\|_1 < \infty$$

and the pair $\{A_t,\, B_t\}$ is renormalisable with constants ω and ν. Conversely, we consider the pair $\{A_t,\, B_t\}$ renormalisable with constants ω and ν. Then,

$$M := \sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N},\, \|u\|_1\leq 1}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\int_0^T\int_0^\infty x|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)u(\sigma,x)|\,d\sigma dx < \infty. \tag{4.99}$$

At this stage, we use the uniform boundedness principle shown in [50] and defined in $\mathcal{X}_1$ the norm:

$$\|u\| := M^{-2}\sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N}}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\int_0^T\int_0^\infty x|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)u(\sigma,x)|\,d\sigma dx.$$

Acknowledging the fact that

$$\int_0^T\int_0^\infty x|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)u(\sigma,x)|\,d\sigma dx \leq M\mathrm{e}^{\omega\Sigma\alpha_k}\mathrm{e}^{\nu\Sigma\delta_k}\int_0^T\int_0^\infty x|u(\sigma,x)|\,d\sigma dx, \tag{4.100}$$

it is clear that

$$\|u\| \leq M^{-2}M\int_0^T\int_0^\infty x|u(\sigma,x)|\,d\sigma dx \; and \; \int_0^T\int_0^\infty x|u(\sigma,x)|\,d\sigma dx \leq M^2\|u\| \quad \text{for}\, u\in\mathcal{X}_1. \tag{4.101}$$

Then,

$$\|u\| \leq M^{-1}\|u\|_1 \; and \; \|u\|_1 \leq M^2\|u\| \quad \text{for}\, u\in\mathcal{X}_1. \tag{4.102}$$

Hence, the norms $\|.\|$ and $\|.\|_1$ are equivalent. Moreover, (4.100), (4.102) and the fact that $A_t \in \mathcal{G}(M,\omega)$ also yield

$$\begin{aligned}\|S_{A_t}(\varsigma)u\| \;\leq\; & M^{-2}\sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N}}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\times\\ & \int_0^T\int_0^\infty x|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)S_{A_t}(\varsigma)u(\sigma,x)|\,d\sigma dx\end{aligned}$$

$$\begin{aligned}
&\leq M^{-2} \sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N}}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)S_{A_t}(\varsigma)u\|_1\\
&\leq M^{-2} \sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N}}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)\|_1\|S_{A_t}(\varsigma)u\|_1\\
&\leq M^{-2} \sup_{\substack{\alpha_1\geq 0,\ldots,\alpha_n\geq 0\\ \delta_1\geq 0,\ldots,\delta_n\geq 0\\ n\in\mathbb{N}}} \mathrm{e}^{-\omega\Sigma\alpha_k}\mathrm{e}^{-\nu\Sigma\delta_k}\|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)\ldots S_{A_t}(\alpha_n)S_{B_t}(\delta_n)\|_1 M\mathrm{e}^{\omega\varsigma}\\
&\leq M^{-2}\mathrm{e}^{\omega\varsigma}\|u\|_1\\
&\leq \mathrm{e}^{\omega\varsigma}\|u\|
\end{aligned}$$

We have

$$\|S_{A_t}(\varsigma)u\| \leq \mathrm{e}^{\omega\varsigma}\|u\|, \ u\in\mathcal{X}_1.$$

In the same way, we obtain

$$\|S_{B_t}(\varsigma)u\| \leq \mathrm{e}^{\nu\varsigma}\|u\|, \ u\in\mathcal{X}_1.$$

This means that the generators $A_t \in \mathcal{G}(1,\omega)$ and $B_t \in \mathcal{G}(1,\nu)$ in the Banach space $\mathcal{X}_1$ endowed with the norm $\|.\|$. Using the Theorem 2.2.4, A_t and B_t are closed, densely defined and satisfy the relations (4.97) and (4.98) in $(\mathcal{X}_1, \|.\|)$. □

Actually, we are in possession of all the essential elements that allow us to state the following perturbation theorem:

Theorem 4.3.9. *Let A_t and B_t be a renormalisable pair of generators of C_0-semigroups on $\mathcal{X}_1$ and the induced generators $\check{A}_t$ and $\check{B}_t$ be closed, densely defined and satisfy the relations (4.85) and (4.86) respectively. Further, let the Banach space $Y \hookrightarrow \mathcal{X}_1$ be admissible with respect to operators A_t and B_t so that $Y \subseteq D(B_t)$. If either $\check{A_t}^*$ or $\check{B_t}^*$ is densely defined in Y^*, or only B_t^* is densely defined in $\mathcal{X}_1^*$, then the closure $\overline{K_t}$ of the operator sum K_t:*

$$K_t\psi = A_t\psi + B_t\psi, \ \text{on} \ D(A_t)\cap D(B_t) = D(A_t) \tag{4.103}$$

exists and $\overline{K_t}$ is the generator of a C_0-semigroup.

Proof. We just need to prove that the range of $(K_t+\eta)$ for some $\eta<0$ ie dense in $\mathcal{X}_1$ and apply the Theorem 4.3.3. Let T be the embedding operator of Corollary 4.3.6, we have, from the Definition 4.3.4, that

$$A_tTu = T\check{A}_tu, \quad for \ u\in D(\check{A}_t)$$

and

$$B_tTu = T\check{B}_tu, \quad for \ u\in D(\check{B}_t).$$

We also have $D(\check{B}_t{}^*) \supseteq T^*\mathcal{X}_1^*$ since $D(B_t) \supseteq Y$. Therefore, $D(K_t)$ is dense in $\mathcal{X}_1$ and we obtain $D(K_t) \supseteq TD(\check{A}_t)$ since $\check{A}_t$ is closed, densely defined in Y which is itself densely embedded in $\mathcal{X}_1$.
Now, let $v \in D(K_t^*) \subseteq \mathcal{X}_1^*$, then, we obtain

$$\langle K_t Tu, v\rangle = \langle A_t Tu, v\rangle + \langle B_t Tu, v\rangle$$

or

$$\langle u, K_t^* T^* v\rangle = \langle T\check{A}_t u, v\rangle + \langle u, B_t^* T^* v\rangle.$$

Then,

$$\begin{aligned} \langle \check{A}_t u, T^* v\rangle &= \langle u, K_t^* T^* v\rangle - \langle u, B_t^* T^* v\rangle \\ &= \langle u, A_t^* T^* v\rangle, \quad u \in D(\check{A}_t), \end{aligned}$$

which means $T^*v \in D(\check{A}_t{}^*)$ and, then, $T^*D(K_t^*) \subseteq D(\check{A}_t{}^*)$. Since $D(\check{B}_t{}^*) \supseteq T^*\mathcal{X}_1^*$, we have

$$T^*D(K_t^*) \subseteq D(\check{A}_t{}^*) \cap D(\check{B}_t{}^*). \tag{4.104}$$

Assuming now by contradiction that $\mathrm{ran}(K_t+\eta)$ is not dense in $\mathcal{X}_1$ for some $\eta < 0$, then there is $v \in \mathcal{X}_1^*$ such that

$$\langle (K_t+\eta)u, v\rangle = 0, \quad u \in D(K_t),$$

which means

$$v \in D(K_t^*) \text{ and } (K_t^*+\eta)v = 0.$$

Hence,

$$T^*v \in D(\check{A}_t{}^*) \cap D(\check{B}_t{}^*), \text{ since } T^*D(K_t^*) \subseteq D(\check{A}_t{}^*) \cap D(\check{B}_t{}^*).$$

If B_t^* is densely defined in $\mathcal{X}_1^*$, then we apply the corollary (4.3.6) and find that $T^*v = 0$. If either $\check{A}_t{}^*$ *or* $\check{B}_t{}^*$ is densely defined in Y^*, then we apply the lemma (4.3.5) to also find that $T^*v = 0$. Therefore, we obtain $v = 0$, which is impossible. Hence, $\mathrm{ran}(K_t + \eta)$ is dense in $\mathcal{X}_1$ for all $\eta < 0$. Because A_t and B_t are a renormalisable pair of generators of C_0-semigroups on $\mathcal{X}_1$, we can use the Lemma (4.3.8) and Hille-Yosida Theorem 2.2.4 to say that A_t and B_t are of class $\mathcal{G}(1,0)$. Therefore, the operator $K_t = A_t + B_t$ is closable and the relation

$$|\eta|\|u\|_1 \leq \|K_t u + \eta u\|_1, \quad u \in D(K_t), \ \eta < 0,$$

yields the existence of the closure $\overline{K_t}$ of K_t. The theorem (4.3.3) completes the proof. □

Corollary 4.3.10. *Let the operators $A(t) = A$ and $B(t) = B$, independent of t and satisfying the conditions of Theorem 4.3.9, then, the closure $\overline{K(t)} = \overline{K}$ given as*

$$\overline{K}\psi = \overline{A\psi + B\psi}, \text{ on } D(A) \cap D(B) = D(A) \tag{4.105}$$

exists and is the generator of a C_0-semigroup.

Proof. In concrete applications, $A(t),\ t \in \mathfrak{I}$ is often a measurable family of generators or generators belonging uniformly to the class $\mathcal{G}(M, \omega)$, for some constants M and ω, and, since we are in one dimensional case, one can easily verify, as shown in [74], that, in this case the induced multiplication operator A is an anti-generator or generator in $Lp(\mathfrak{I}, X_1)$, for some $p \in [1, \infty)$ with $\mathfrak{I} \subseteq \mathbb{R}_+$. This reduces the problem to finding certain conditions for the operator sum

$$K\psi = A\psi + B\psi, \text{ on } \ D(A) \cap D(B) = D(A) \tag{4.106}$$

to be closable and its closure generates a C_0 semigroup and Theorem 4.3.9 ends the proof. □

Remark 5. From the relation (4.82), it follows that the closure of $A(t) + B(t)$ generates a propagator.

This allows the following conservativeness result to be stated:

Theorem 4.3.11. *(a) The C_0 semigroup $(S_{\overline{K_t}}(s))_{s\geq 0}$ generated by $\overline{K}_t = \overline{A_t + B_t}$ is conservative if and only if the associated propagator $U(t, \tau)$, $0 \leq \tau < t \leq T$, is conservative.* *(b) If the operators A_t and B_t, satisfy the conditions of Theorem 4.3.9, then the model (4.77) is conservative.*

Proof. (*a*) We make use of the relation (4.82) and properties of U given in Definition 4.1.1. The rest of the proof follows from [6, proposition 5.1]. (*b*) The second part of the proof follows from (*a*) and is based on [12, Theorem 6.13]. □

4.3.3 Discussion

We have set conditions on the generators involved in the semigroup perturbation and used the renormalisation method (which is different from the preceding ones [6, 57]), to analyse (4.77). We dropped the class $\mathcal{G}(1, 0)$ for the class $\mathcal{G}(1, \nu)$ of quasi-contractive semigroups in $\mathcal{X}_1 = L_1([0, T] \times [0, \infty),\ xd\sigma dx)$, and showed existence results and conservativeness for the non-autonomous fragmentation model (4.77), therefore, giving a stronger result than [33, 50], where the model was autonomous with coefficients independent of time. The result obtained here can lead to the full characterisation of the infinitesimal generator for the non-autonomous fragmentation model (4.77) and later for non-autonomous fragmentation-coagulation or non-autonomous transport-fragmentation-coagulation models, which remain open problems.

Chapter 5

Special Coagulation Process in a Moving Medium

5.1 Introduction

Existence of a global solution to continuous, non-common and non-linear convection-coagulation equations is investigated in space $L_1(\mathbb{R}^3 \times \mathbb{R}_+, mdmdx)$. As discussed in Chapter 3 (also see [70]), the method of characteristics and Friedrichs lemma are applied here to show that the transport operator generates a stochastic dynamical system, with the assumption that the velocity field is globally Lipschitz continuous and divergence free. We then proceed by using substochastic methods and Kato-Voigt perturbation theorem to ensure that the linear operator (transport-coagulation) is the infinitesimal generator of a strongly continuous semigroup. Once the existence for the linear problem has been established, the solution of the full non-linear problem follows by showing that the coagulation term is globally Lipschitz. In particular, we are able to prove that the solution of the full coagulation-transport model is unique.

5.2 Motivation

The process of clusters undergoing coagulation (coalescence), also referred to as inverse of the fragmentation dynamics, can be seen in many branches of natural sciences like biology, ecology, physics, chemistry, engineering and numerous domains of applied sciences. Among concrete examples one can count agglutination and splitting of blood cells, formation and splitting of aerosol droplets, evolution of phytoplankton aggregates, depolymerisation, rock fractures and breakage of droplets. The coagulation kernel can be size and position dependent and new particles resulting from the coagulation can be spatially distributed across the space. Coagulation equations, combined with transport terms (sometimes combined with fragmentation process), have been used to describe a wide range of phenomena. For instance, in ecology or aquaculture, there are phytoplankton population evolving in flowing water. Various types of coagulation equations have been comprehensively analysed in numerous works: The authors in [16, 69] only considered growth processes modeled by a first order partial differential operator and showed existence result for fragmentation-coagulation model with coagulation kernel taking into account that not all particles in an aggregate have the same ability to combine with particles of other aggregates which result in a "damped" coagulation process. In [5], the authors used similar kernels to model the evolution of phytoplankton. The author in [59] exploited the contraction mapping principle to prove existence and uniqueness results for the non-autonomous coagulation and multiple-fragmentation equation. As already known for transport and fragmentation processes, transport and coagulation dynamics combined in the same model are still not widely exploited in the domain of mathematical and abstract analysis. A special and non-common type of transport model was analysed in [70] where the authors proved the existence of the smallest substochastic semigroup generated by the linear part, consisting of the transport operator combined to fragmentation terms. Kinetic-type Models with diffusion were globally investigated in [12] and later extended in [23], where the author showed that the diffusive part does not affect the breach of the conservation laws and very recently, in [20], the author investigated equations describing fragmentation and coagulation processes with growth or decay and proved an analogous result.

In this chapter, the model we analyse is presented as follows: In social grouping population where we have defined a spatial dynamical system in which locally group-size distribution can be estimated, but in which immigration and emigration are also allowed from adjacent areas with different distributions, we obtain the general model consisting of transport, direction changing, fragmentation and coagulation processes describing the dynamics a population of, for example, phytoplankton which is a spatially explicit group-size distribution model as presented in [67]. We analyse the model consisting of transport and coagulation processes with the coagulation part different from the classical one where the kernel $k(m, n)$ is defined as the rate at which particles of mass m coalesce with particles of mass n and is derived by assuming that the average number of coalescences between particles having mass in $(m; m + dm)$ and those having mass in $(n; n + dn)$ is $k(m, n)p(t, m)p(t, n)dmdndt$ during the time interval $(t; t + dt)$. In the current model, we assume that any individual in the populations is viewed as a collection of joined cells.

5.3 Well posedness of the transport problem with coagulation

The model of coagulation dynamics occurring in a moving process [67, 70, 16], as described above, is given by

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,m) = -div(\omega(x,m)p(t,x,m)) - d(x,m)p(t,x,m) \\ \qquad +\chi_{U_{\mathbb{R}}}(m,x)\dfrac{\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx}{m\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}\eta d(x,\eta)p(t,x,\eta)d\eta dx} \\ p(0,x,m) = \overset{\circ}{p}(x,m), \quad a.e.\ (x,m)\in\mathbb{R}^3\times\mathbb{R}_+ \end{cases} \tag{5.1}$$

where in terms of the mass size m and the position x, the state of the system is characterised at any moment t by the particle-mass-position distribution $p = p(t,x,m)$, (p is also called the *density* or *concentration* of particles), with $p : \mathbb{R}_+ \times \mathbb{R}^3 \times (m_0,\infty) \to \mathbb{R}_+$. The space variable x is supposed to vary in the whole of $\mathbb{R}^3$. The function $\overset{\circ}{p}(x,m)$ represents the density of groups of size m at position x at the beginning ($t = 0$). In the model (5.1), we assume that the quasi non-local coagulation process at a position x occurs in the following way: Two clusters in a neighbourhood of x coalesce to form a third group which becomes located at x. The other terms and elements are defined in the following subsection.

5.3.1 The coagulation equation

Because the space variable x varies in the whole of $\mathbb{R}^3$ (unbounded) and since the total number of individuals in a population is not modified by interactions among groups, the following conservation law is supposed to be satisfied:

$$\frac{d}{dt}\mathcal{I}(t) = 0, \tag{5.2}$$

where $\mathcal{I}(t) = \int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty} p(t,x,m)mdmdx$, is the total number of individuals in the space (or total mass of the ensemble) with the assumption that $m_0 > 0$ is the smallest mass/size a monomer can have in the system. Henceforth we assume that for each $t > 0$, the density of groups of size m at the position x and time t is the function $(x,m) \longrightarrow p(t,x,m)$ taken from the Banach space

$$\mathcal{X}_1 := L_1(\mathbb{R}^3 \times \mathbb{R}_+, mdmdx)$$

and $\overset{\circ}{p} \in \mathcal{X}_1$. When any subspace $S \subseteq \mathcal{X}_1$, we will denote by S_+ the subset of S defined as $S_+ = \{g \in S; g(x,m) \geq 0, m \in \mathbb{R}_+, x \in \mathbb{R}^3\}$.

In $\mathcal{X}_1$, we define from the right-hand side of (5.1), the coagulation expression $\mathcal{N}$ given by

$$[\mathcal{N}p](x,m) := [\mathcal{C}p - \mathcal{D}p](x,m) \tag{5.3}$$

where

$$[\mathcal{C}p](x,m) = \chi_{U_{\mathbb{R}}}(m,x)\frac{\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx}{m\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}\eta d(x,\eta)p(t,x,\eta)d\eta dx}, \tag{5.4}$$

for any $p \in \mathcal{X}_{1+} \setminus \{0\}$,

$$\mathcal{C}(0) = 0,$$

and

$$[\mathcal{D}p](x,m) = d(x,m)p(x,m). \tag{5.5}$$

We assume that no particle of mass $m < 2m_0$ can emerge as a result of coagulation, then $\chi_{U_{\mathbb{R}}}$ is the characteristic function of the set $U_{\mathbb{R}} = \mathbb{R}^3 \times U = \mathbb{R}^3 \times [2m_0, \infty)$. Following [5], we assume that only a part of the aggregates has the competence to join. This could for example be due to the fact that only cells of some species have the necessary devices to glue or to attach to others. The coefficient of competence is a function $d(x,m)$ depending also on the position of the cluster. We assume that d is a positive and bounded function in the sense that there are two constants $0 < \theta_1$ and θ_2 such that for every $x \in \mathbb{R}^3$,

$$\theta_1\alpha_m \leq d(x,m) \leq \theta_2\alpha_m, \tag{5.6}$$

with $\alpha_m \in \mathbb{R}_+$ and independent of x.

Proposition 5.3.1. *The coagulation model described by* (5.3) *is formally conservative.*

Proof. The aim is to show that (5.2) is satisfied, that is,

$$\frac{d}{dt}\mathcal{I}(t) = \frac{d}{dt}\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty} p(t,x,m)mdmdx = \int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty} m\frac{\partial}{\partial t}p(t,x,m)dmdx = 0.$$

According to assumption (5.6), we just need to prove that

$$\int\limits_{\mathbb{R}^3}\int_{m_0}^{\infty}\left(\chi_{U_{\mathbb{R}}}(m,x)\int\limits_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx\right)$$

$$dmdx = \int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty} md(x,m)p(t,x,m)dmdx \cdot \int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}\eta d(x,\eta)p(t,x,\eta)d\eta dx. \tag{5.7}$$

Making use of the Fubini integration theorem, we obtain

$$\int_{\mathbb{R}^3}\int_{m_0}^{\infty}\left[\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx\right]dmdx$$
$$=\int_{\mathbb{R}^3}\int_{2m_0}^{\infty}\left[\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx\right]dmdx$$
$$=\int_{\mathbb{R}^3}\int_{m_0}^{\infty} nd(x,n)p(t,x,n)\left[\int_{\mathbb{R}^3}\int_{n+m_0}^{\infty}(m-n)d(x,m-n)p(t,x,m-n)dmdx\right]dndx$$
$$=\int_{\mathbb{R}^3}\int_{m_0}^{\infty} nd(x,n)p(t,x,n)\left[\int_{\mathbb{R}^3}\int_{m_0}^{\infty}(\eta)d(x,\eta)p(t,x,\eta)d\eta dx\right]dndx$$
$$=\int_{\mathbb{R}^3}\int_{m_0}^{\infty} nd(x,n)p(t,x,n)dndx\times\int_{\mathbb{R}^3}\int_{m_0}^{\infty}(\eta)d(x,\eta)p(t,x,\eta)d\eta dx,$$

which ends the proof. □

The total number of cells in all aggregates that, at time t, are implicated in the coagulation process is given by:

$$M(t):=\int_{\mathbb{R}^3}\int_{m_0}^{\infty}\eta d(x,\eta)p(t,x,\eta)\,d\eta dx,$$

and

$$f(t,x,m):=\frac{md(x,m)p(t,x,m)}{M(t)}$$

is the fraction of cells in size-m aggregates and position x competent for the coagulation process with respect to the total population of cells in aggregates prone to join. In terms of the quantities introduced so far, we can express the time rate of cells forming aggregates of size m and position x:

$$M(t,x)\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} f(t,x,m-n)f(t,x,n)\,dndx,$$

If coagulation were the only process, the equation would read:

$$\frac{\partial}{\partial t}mp(t,x,m)=M(t)\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} f(t,x,m-n)f(t,x,n)dndx-md(x,m)p(t,x,m),$$

which, after basic algebra, leads to:

$$\frac{\partial}{\partial t}p(t,x,m)=\left[\mathcal{C}p-\mathcal{D}p\right](t,x,m) \tag{5.8}$$

with $\mathcal{C}$ and $\mathcal{D}$ given by (5.4) and (5.5) respectively.

5.4 Cauchy problem for the transport operator in $\Lambda = \mathbb{R}^3 \times \mathbb{R}_+$

Λ is endowed with the Lebesgue measure $d\mu = d\mu_{m,x} = dmdx$. The primary objective in this section is to analyse the solvability of the transport problem

$$\frac{\partial}{\partial t}p(t,x,m) = -div(\omega(x,m)\, p(t,x,m)), \tag{5.9}$$

$$p(0,x,m) = \overset{\circ}{p}(x,m), \qquad m \in \mathbb{R}_+,\ x \in \mathbb{R}^3$$

in the space $\mathcal{X}_1$.

Furthermore, to simplify the notation, we put $\mathbf{v} = (x,m) \in \Lambda$. Assuming that ω is divergence free and globally Lipschitz continuous, then $div\,\omega(\mathbf{v}) := \nabla \cdot \omega(\mathbf{v}) = 0$. To properly study the transport operator, we consider the function $\omega : \Lambda \longrightarrow \mathbb{R}^3$ and denote by $\tilde{\mathcal{T}}$ the expression appearing on the right-hand side of the equation (5.9). Then

$$\begin{aligned}\tilde{\mathcal{T}}[p(t,\mathbf{v})] &:= -div\,(\omega(\mathbf{v})\, p(t,\mathbf{v}))\\ &= (\nabla \cdot \omega(\mathbf{v}))p(t,\mathbf{v}) + \omega(\mathbf{v}) \cdot (\nabla p(t,\mathbf{v})),\end{aligned} \tag{5.10}$$

which becomes

$$\tilde{\mathcal{T}}[p(t,\mathbf{v})] := \omega(\mathbf{v}) \cdot (\nabla p(t,\mathbf{v})). \tag{5.11}$$

For $\mathbf{v} \in \Lambda$ and $t \in \mathbb{R}$, the Cauchy problem

$$\begin{aligned}\frac{d\mathfrak{r}}{ds} &= \omega(\mathfrak{r}), \qquad s \in \mathbb{R}\\ \mathfrak{r}(t) &= \mathbf{v},\end{aligned} \tag{5.12}$$

has a unique solution $\mathfrak{r}(s)$ with values in Λ. Let the function $\phi : \Lambda \times \mathbb{R}^2 \longrightarrow \Lambda$ be defined by the condition that for $(\mathbf{v},t) \in \Lambda \times \mathbb{R}$,

$$s \longrightarrow \phi(\mathbf{v},t,s), \qquad s \in \mathbb{R}$$

is the unique solution of the Cauchy problem (5.12). We obtain the characteristics of $\tilde{\mathcal{T}}$ given by the integral curves ϕ-parameterized family $(\mathfrak{r})_\phi$ (with $\mathfrak{r}(s) = \phi(\mathbf{v},t,s)$, $s \in \mathbb{R}$, the only solution of (5.12)). Recall that the function ϕ possesses many desirable properties [45, 81, 83] that will be relevant for studying the transport operator in $\mathcal{X}_1$. We can take

$$\begin{aligned}\mathcal{T}p = \tilde{\mathcal{T}}p, &\quad \text{with } \tilde{\mathcal{T}}p \ \text{ represented by (5.11)}\\ D(\mathcal{T}) := \{p \in \mathcal{X}_1,&\ \ \mathcal{T}p \in \mathcal{X}_1\},\end{aligned} \tag{5.13}$$

Note that $\mathcal{T}p$ is understood in the sense of distribution. Precisely speaking, if we take $C_0^1(\Lambda)$ as the set of the test functions, each $p \in D(\mathcal{T})$ if and only if $p \in L_1(\Lambda)$ and there exists $g \in \mathcal{X}_1$ such that

$$\int_\Lambda \xi g d\mu = \int_\Lambda p\,\partial \cdot (\xi\omega) d\mu = \int_\Lambda p\,\omega \cdot \partial\,\xi d\mu, \tag{5.14}$$

for all $\xi \in C_0^1(\Lambda)$, where

$$\omega \cdot \partial\xi(\mathbf{v}) := \sum_{j=1}^{3} \omega_j \partial_j \xi(\mathbf{v}) \tag{5.15}$$

with $\omega_j = \omega_j(\mathbf{v})$, the j^{th} component of the velocity $\omega(\mathbf{v})$. The middle term in (5.14) exists as ω is globally Lipschitz continuous, and the last equality follows as ω is divergence-free. If this is the case, we can define $\mathcal{T}p = g$. Now, we prove that the operator $\mathcal{T}$ is the generator of a stochastic semigroup on $\mathcal{X}_1$.

Theorem 5.4.1. *If the function ω is globally Lipschitz continuous and divergence-free, then the operator $(D(\mathcal{T}), \mathcal{T})$ defined by (5.13) is the generator of a strongly continuous stochastic semigroup $(G_{\mathcal{T}}(t))_{t\geq 0}$, given by*

$$\left[G_{\mathcal{T}}(t)p\right](\boldsymbol{v}) = p(\phi(\boldsymbol{v}, t, 0)) \tag{5.16}$$

for any $p \in \mathcal{X}_1$ and $t > 0$.

Proof. The proof of this theorem is fully given in detail in [70, Theorem 2]. □

Remark 6. The previous theorem allows us to show that the model (5.9) is conservative in the space $\mathcal{X}_1$, that is, the law (5.2) is satisfied. In fact, the semigroup generated by the operator $\mathcal{T}$ is stochastic, then

$$\begin{aligned} 0 &= \int_{\Lambda} \mathcal{T}p \, d\mu, \quad \text{for all } p \in D(\mathcal{T}), \text{ then} \qquad (5.17) \\ 0 &= \int_{\mathbb{R}^3} \int_0^\infty m\mathcal{T}p(t,x,m)dmdx, \quad \text{for all } t \geq 0, \end{aligned}$$

which leads to

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) &= \frac{d}{dt}\left(\int_{\mathbb{R}^3}\int_0^\infty mp(t,x,m)dmdx\right) \\ &= \int_{\mathbb{R}^3}\int_0^\infty m\partial_t p(t,x,m)dmdx \\ &= \int_{\mathbb{R}^3}\int_0^\infty m\mathcal{T}p(t,x,m)dmdx \\ &= 0. \end{aligned}$$

This is an expected result since the flow process alone does not modify the total number of individuals in the system.

5.5 Coagulation competency in the moving Process

We consider the coalescence competence operator $(\mathcal{D}, D(\mathcal{D}))$ defined in (5.5), as a perturbation of the transport system (5.9). We obtain the abstract Cauchy problem

$$\begin{aligned} \partial_t p(t,\mathbf{v}) &= \mathcal{T}p(t,\mathbf{v}) - \mathcal{D}p(t,\mathbf{v}) = Fp(t,\mathbf{v}) \\ p(0,\mathbf{v}) &= \overset{o}{p}(\mathbf{v}), \qquad \mathbf{v} \in \Lambda, \end{aligned} \tag{5.18}$$

where

$$F = \mathcal{T} - D. \tag{5.19}$$

Remark 7. $(F, D(F)) = (\mathcal{T} - \mathcal{D}, D(\mathcal{T}))$ is a well-defined operator.

To show this assertion, it is first noted that $D(\mathcal{T}) \cap D(\mathcal{D}) = D(\mathcal{T})$ since $D(\mathcal{D}) = \mathcal{X}_1$. Because $\mathcal{T}$ is conservative, integration of (5.18) over Λ gives

$$\frac{d}{dt}\|p\|_1 = \frac{d}{dt}\int_{\mathbb{R}^3}\int_0^\infty mp(t,x,m)dmdx = -\int_{\mathbb{R}^3}\int_0^\infty d(x,m)mp(x,m)dmdx.$$

Hence, (5.6) leads to

$$\begin{aligned} -\int_{\mathbb{R}^3}\int_0^\infty \theta_2\alpha_m mp(x,m)dmdx &\leq -\int_{\mathbb{R}^3}\int_0^\infty d(x,m)mp(x,m)dmdx \\ &\leq -\int_{\mathbb{R}^3}\int_0^\infty \theta_1\alpha_m mp(x,m)dmdx \end{aligned}$$

for all $p \in (\mathcal{X}_1)_+$ and using Gronwall's inequality, we obtain

$$-\theta_2\alpha_m\|p\|_1 \leq \frac{d}{dt}\|p\|_1 \leq -\theta_1\alpha_m\|p\|_1,$$

then,

$$e^{-\theta_2\alpha_m t}\|\overset{o}{p}\|_1 \leq \|p\|_1 \leq e^{-\theta_1\alpha_m t}\|\overset{o}{p}\|_1.$$

This inequality for $p = G_F(t)\overset{o}{p}$ yields

$$e^{-\theta_2\alpha_m t}\|\overset{o}{p}\|_1 \leq \|G_F(t)\overset{o}{p}\|_1 \leq e^{-\theta_1\alpha_m t}\|\overset{o}{p}\|_1 \tag{5.20}$$

where $\overset{o}{p} \in (C_0^\infty(\Lambda))_+ \subseteq D(F)_+$. If we take $0 \leq \overset{o}{p} \in \mathcal{X}_1$ then, we can use approximations to the identity (mollifiers) $\varpi_\varepsilon(\mathbf{v}) = C_\varepsilon\varpi(\mathbf{v}/\varepsilon)$ where ϖ is a $C_0^\infty(\Lambda)$ function defined by

$$\varpi(\mathbf{v}) = \begin{cases} \exp\left(\frac{1}{|\mathbf{v}|^2-1}\right) & \text{for } |\mathbf{v}| < 1 \\ 0 & \text{for } |\mathbf{v}| \geq 1 \end{cases}$$

and C_ε are constants chosen so that $\int\limits_\Lambda \varpi_\varepsilon(\mathbf{v})dx = 1$. Using the mollification of $\overset{o}{p}$ by taking the convolution

$$\overset{o}{p}_\varepsilon := \int\limits_\Lambda \overset{o}{p}(\mathbf{v}-\mathbf{y})\varpi_\varepsilon(\mathbf{y})d\mu_y = \int\limits_\Lambda \overset{o}{p}(\mathbf{y})\varpi_\varepsilon(\mathbf{v}-\mathbf{y})d\mu_y, \tag{5.21}$$

we obtain $\overset{o}{p}_\varepsilon$ in $\mathcal{X}_1$ (since $\overset{o}{p} \in \mathcal{X}_1$) and $\overset{o}{p} = \lim\limits_{\varepsilon \longrightarrow 0^+} \overset{o}{p}_\varepsilon$ in $\mathcal{X}_1$. Moreover, $\overset{o}{p}_\varepsilon$ are also non-negative by (5.21) since $0 \le \overset{o}{p}$, and the family $(\overset{o}{p}_\varepsilon)_\varepsilon \subseteq C_0^\infty(\Lambda)$. This shows that any non-negative $\overset{o}{p}$ taken in $\mathcal{X}_1$ can be approximated by a sequence of non-negative functions of $C_0^\infty(\Lambda)$. The inequality (5.20) is therefore valid for any non-negative $\overset{o}{p} \in \mathcal{X}_1$. Using the fact that any arbitrary element $\overset{o}{g}$ of $\mathcal{X}_1$ (equipped with the pointwise order almost everywhere) can be written in the form $\overset{o}{g} = \overset{o}{g}_+ - \overset{o}{g}_-$, where $\overset{o}{g}_+,\ \overset{o}{g}_- \in (\mathcal{X}_1)_+$, the positive element approach, [22, 88] allows us to extend the right inequality of (5.20) to all $\mathcal{X}_1$ in order to have

$$\|G_F(t)p\|_1 \le e^{-\theta_1\alpha_m t}\|p\|_1. \tag{5.22}$$

Using the semigroup representation of the resolvent, we obtain for $\lambda > 0$

$$\begin{aligned}\|R(\lambda, F)p\|_1 &\le \int\limits_0^\infty e^{-\lambda t}\|G_F(t)p\|_1 dt\\ &\le \int\limits_0^\infty e^{-\lambda t}e^{-\theta_1\alpha_m t}\|p\|_1 dt\\ &\le \frac{1}{\lambda+\theta_1\alpha_m}\|p\|_1.\end{aligned}$$

According to the right inequality of (5.6), we obtain that

$$\|\mathcal{D}R(\lambda, F)p\|_1 \le \frac{\theta_2\alpha_m}{\lambda+\theta_1\alpha_m}\|p\|_1 \le \frac{\theta_2}{\theta_1}\|p\|_1.$$

This relation states that $D(\mathcal{D}) \supseteq D(F)$, (the domain of $\mathcal{D}$ is at least that of F). Because $F = \mathcal{T} - \mathcal{D}$ and $\mathcal{D}$ is bounded, we exploit the following relation for resolvent in $\mathcal{X}_1$:

$$\begin{aligned}\lambda I - F &= \lambda I - \mathcal{T} + \mathcal{D}R(\lambda, F)(\lambda I - F)\\ I &= (\lambda I - \mathcal{T})R(\lambda, F) + \mathcal{D}R(\lambda, F)\\ R(\lambda, \mathcal{T}) &= R(\lambda, F) + R(\lambda, \mathcal{T})\mathcal{D}R(\lambda, F)\\ R(\lambda, F) &= R(\lambda, \mathcal{T})(I - \mathcal{D}R(\lambda, F))\end{aligned}$$

for every $m \in \mathbb{R}_+$. This leads to $D(\mathcal{T}) \supseteq D(F)$ and therefore, $D(F) \subseteq D(\mathcal{T}) \cap D(\mathcal{D})$. On the other hand, if $p \in D(\mathcal{T}) \cap D(\mathcal{D})$ then $\|\mathcal{T}p\|_1 < \infty$ and $\|\mathcal{D}p\|_1 < \infty$. Therefore,

$$\|\mathcal{T}p - \mathcal{D}p\|_1 \le \|\mathcal{T}p\|_1 + \|\mathcal{D}p\|_1 < \infty,$$

meaning that $p \in D(F)$ and thus $D(\mathcal{T}) \cap D(\mathcal{D}) \subseteq D(F)$ and the remark is completed.

Assumption (5.6) implies that the operator $\mathcal{D}$ generates a C_0-semigroup of contractions $(G_\mathcal{D}(t))_{t\ge 0}$, which yields the following theorem [70, Theorem 5].

Theorem 5.5.1. *The operator $(F, D(F))$ is the infinitesimal generator of a substochastic semigroup $(G_F(t))_{t\geq 0}$ defined by*

$$[G_F(t)p](v) = \left[\lim_{v\longrightarrow\infty}\left[G_{\mathcal{T}}\left(\frac{t}{v}\right)G_{\mathcal{D}}\left(\frac{t}{v}\right)\right]^v p\right](v) \qquad (5.23)$$

for $p \in \mathcal{X}_1$ and $t > 0$, where $(G_{\mathcal{T}}(t))_{t\geq 0}$ is defined by (5.16).

In the next section, the non-linear perturbation is used to analyse the full model (5.1).

5.6 Global solution for the full model

The coagulation process appearing in a moving medium mathematically reads as:

$$\begin{aligned} \partial_t p(t,x,m) &= \mathcal{T}p(t,x,m) - \mathcal{D}p(t,x,m) + \mathcal{C}p(t,x,m) \\ p(0,x,m) &= \overset{o}{p}(x,m), \quad a.e.\ (x,m) \in \mathbb{R}^3 \times \mathbb{R}_+ \end{aligned}$$

where $\mathcal{C}$, given by (5.4), is defined on the set $\mathcal{X}_{1+} = \{g \in \mathcal{X}_1 : g \geq 0\}$. We recall that $\mathcal{C}(0) = 0$. We need the following lemma:

Lemma 5.6.1. *The operator $\mathcal{C}$ satisfies a global Lipschitz condition on the set $\mathcal{X}_{1+}$.*

Proof. We set:

$$\Psi h(x,m) = md(x,m)h(x,m) \quad \text{and} \quad \alpha(h) = \int_{\mathbb{R}^3}\int_{m_0}^{\infty} \Psi h(x,m)\, dmdx.$$

Thanks to (5.6), we also set $\vartheta = \operatorname{ess\,sup}_{\mathbb{R}^3\times(m_0,\infty)} d(x,n) < \infty$.

Remark 8. For every $h \in \mathcal{X}_{1+}\setminus\{0\} \subset \mathcal{X}_{1+} = D(\mathcal{D})$, the operator α satisfies

$$\alpha(h) = \int_{\mathbb{R}^3}\int_{m_0}^{\infty} \Psi h(x,m)dmdx = \int_{\mathbb{R}^3}\int_{m_0}^{\infty} md(x,m)h(x,m)\, dmdx = \|\mathcal{D}h\|_1 \leq \vartheta\|h\|_1 < \infty.$$

In terms Ψ and α the operator $\mathcal{C}$ takes the expression

$$\mathcal{C}h(x,m) = \chi_{U_{\mathbb{R}}}(m,x)\frac{(\Psi h * \Psi h)(m)}{m\alpha(h)},$$

where $h \in \mathcal{X}_{1+}\setminus\{0\}$ and

$$(\Psi h * \Psi h)(m) := \int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} \Psi h(x,n)\Psi h(x,m-n)\, dndx.$$

Let g_0 be a function fixed in $\mathcal{X}_{1+} \setminus \{0\}$. We set $\kappa := \alpha(g_0)\vartheta^{-1}$. Let g be any function from $\mathcal{X}_{1+} \setminus \{0\}$ such that $\|g - g_0\|_1 \leq \kappa$. Then

$$\alpha(g) = \alpha(g_0) + \alpha(g - g_0) \leq 2\alpha(g_0). \tag{5.24}$$

Making use of the linearity of α and properties of the convolution $*$ we have the following:

$$\begin{aligned}
\mathcal{C}g(x,m) - \mathcal{C}g_0(x,m) &= \chi_{U_{\mathbb{R}}}(m,x)\frac{[(\Psi g * \Psi g)(m)]\alpha(g_0 - g)}{m\alpha(g_0)\alpha(g)} \\
+ &\ \chi_{U_{\mathbb{R}}}(m,x)\frac{(\Psi g * \Psi g)(m)}{m\alpha(g_0)} + \chi_{U_{\mathbb{R}}}(m,x)\frac{(\Psi g_0 * \Psi g_0)(m)}{m\alpha(g_0)} \\
= &\ \chi_{U_{\mathbb{R}}}(m,x)\frac{[(\Psi g * \Psi g)(m)]\alpha(g_0 - g)}{m\alpha(g_0)\alpha(g)} + \chi_{U_{\mathbb{R}}}(m,x)\frac{[\Psi(g + g_0) * \Psi(g - g_0)](m)}{m\alpha(g_0)}.
\end{aligned}$$

It follows that

$$\|\mathcal{C}g - \mathcal{C}g_0\|_1 \leq \frac{\alpha(|g_0 - g|)\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}(\Psi g * \Psi g)(m)\,dmdx}{\alpha(g_0)\alpha(g)} + \frac{\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}[\Psi(g + g_0) * |\Psi(g - g_0)|\,](m)\,dmdx}{\alpha(g_0)}. \tag{5.25}$$

According to the Remark 8 we have

$$\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}(\Psi g * \Psi g)(m)\,dmdx = \left[\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}(\Psi g)(m)\,dmdx\right]^2 = (\alpha(g))^2 \leq \vartheta\|h\|_1 < \infty,$$

and

$$\int\limits_{\mathbb{R}^3}\int\limits_{m_0}^{\infty}[\Psi(g + g_0) * |\Psi(g - g_0)|](m)dmdx = \alpha(|g - g_0|)\alpha(g_0 + g).$$

Therefore, using again the linearity of α and applying (5.24) yield

$$\begin{aligned}
\|\mathcal{C}g - \mathcal{C}g_0\|_1 &\leq \frac{\alpha(g)\alpha(|g_0 - g|)}{\alpha(g_0)} + \frac{\alpha(g_0 + g)\alpha(|g - g_0|)}{\alpha(g_0)} \\
&\leq 5\alpha(|g_0 - g|) \\
&\leq 5\vartheta\|g - g_0\|_1.
\end{aligned} \tag{5.26}$$

Next, we prove that the later inequality is valid for all $h, g \in \mathcal{X}_{1+} \setminus \{0\}$. Let us fix h, g in $\mathcal{X}_{1+}\setminus\{0\}$ and let $h_t = (1-t)h+tg$ for $t \in [0,1]$. Since the function $t \mapsto \alpha(h_t)$ is continuous and $\alpha(h_t) > 0$ for each $t \in [0,1]$ we have $\inf_t \alpha(h_t) > 0$. Let $\overline{\kappa} = \vartheta^{-1} \inf_t \alpha(h_t)$. Then (5.26) yields

$$\|\mathcal{C}h_s - \mathcal{C}h_t\|_1 \leq 5\vartheta \|h_s - h_t\|_1 \quad \text{provided that} \quad \|h_s - h_t\|_1 \leq \overline{\kappa}.$$

Let $\mathbf{n}$ be an integer such that $\mathbf{n} \geq \|h-g\|_1/\overline{\kappa}$ and let $t_i = i/\mathbf{n}$ for $i = 0, 1, ..., \mathbf{n}$. Then, $\|h_{t_i} - h_{t_{i-1}}\|_1 \leq \overline{\kappa}$ and then:

$$\begin{aligned} \|\mathcal{C}h - \mathcal{C}g\|_1 &\leq \sum_{i=1}^{\mathbf{n}} \|\mathcal{C}h_{t_i} - \mathcal{C}h_{t_{i-1}}\|_1 \\ &\leq 5\vartheta \sum_{i=1}^{\mathbf{n}} \|h_{t_i} - h_{t_{i-1}}\|_1 \\ &= 5\vartheta \|h - g\|_1, \end{aligned} \tag{5.27}$$

where we used the fact that $h_{t_i} - h_{t_{i-1}} = \dfrac{g-h}{\mathbf{n}}$ for any $i = 0, 1, ..., \mathbf{n}$. Furthermore, from (5.7) and Remark 8, $\|\mathcal{C}g - \mathcal{C}0\|_1 = \|\mathcal{C}g\|_1 \leq \int_{\mathbb{R}^3} \int_{m_0}^{\infty} m d(x,m) g(x,m)\, dm dx \leq \vartheta \|g\|_1$ for any $g \in \mathcal{X}_{1+}$. This concludes that the operator $\mathcal{C}$ is continuous at 0. Therefore, inequality (5.27) passes to the limit at $h = 0$ or $g = 0$, which concludes the proof. □

Theorem 5.6.2. *Let $\overset{o}{p} \in D(F) \cap \mathcal{X}_{1+}$, the Cauchy problem*

$$\begin{aligned} \frac{du}{dt}(t) &= F[p(t)] + \mathcal{C}[p(t)] \\ p_{|t=0} &= \overset{o}{p}, \end{aligned} \tag{5.28}$$

has a global unique solution.

Proof. First, we recall that the solution p of (5.28) is the unique solution of the integral equation

$$p(t) = G_F(t)p_0 + \int_0^t G_F(t-s)\mathcal{C}[p(s)]\, ds, \quad t \geq 0, \tag{5.29}$$

where $(G_F(t))_{t \geq 0}$ is the semigroup generated by the operator F given in (5.23). We consider

$$\mathcal{Y} := C([0, t_1], \mathcal{X}_{1+})$$

and its norm

$$\|g\|_{\mathcal{Y}} := \max\{\|g(t)\|_1 : 0 \leq t \leq t_1\}.$$

Furthermore, we let

$$\Xi := \{g \in \mathcal{Y} : \ g(t) \in \overline{B}(p_0, r_1) \cap \mathcal{X}_{1+} \ \forall t \in [0, t_1]\},$$

with $r_1 \in \mathbb{R}_+$. Now, we define $\mathcal{M}$ on Ξ as the mapping

$$(\mathcal{M}g)(t) \ := \ G_F(t)f + \int_0^t G_F(t-s)\mathcal{C}[g(s)]\, ds, \ 0 \leq t \leq t_1.$$

Then, $\mathcal{M}(\Xi) \subset \mathcal{Y}$ and $(\mathcal{M}g)(t) \in \mathcal{X}_{1+}$ for all $t \in [0, t_1]$. The proof of the existence of a unique solution $p \in \Xi$ to the equation $p = \mathcal{M}p$ follows in the standard way [74, Theorem 6.1.2] since $\mathcal{X}_{1+}$ is a complete metric space as a closed subspace of a Banach space. Consequently, the integral equation (5.29) has a unique solution $p \in C([0, t_1], \mathcal{X}_{1+})$. The existence of a global strong solution to problem (5.28) immediately follows from the fact that $\mathcal{C}$ is globally Lipschitz, as shown in the Lemma 5.6.1. □

5.7 Concluding remarks

In this chapter, the theory of strongly continuous semigroups of operators [74] was used to analyse the well-posedness and show existence result of an integro-differential equation modelling convection-coagulation processes. This study is an innovation in the domain of applied analysis thanks to the inclusion of the spatial transportation kernel which was not considered in previous studies. We proved that the full model with combined coagulation-transport operator has global unique solution, thereby addressing the problem of existence of solutions for this model. This may help to analyse in the same way, a model with combined coagulation-fragmentation-transport-direction changing whose full identification of the generator and characterisation of the domain remain open.

Chapter 6

Some Applications for Fragmentation Models

6.1 Introduction

In this chapter, two concrete phenomena that occur in applied sciences and applicable to fragmentation-coalescence models are presented and analysed. The aim is to establish a better understanding concerning the occurrence of these real phenomena, namely shattering and marine iron fertilisation.

6.2 Exact solutions of fragmentation equations with arbitrary fragmentation rates and separable particles distribution kernels

6.2.1 Introduction and preliminaries

We make use of Laplace transform techniques and the method of characteristics to solve fragmentation equations explicitly. The result is a breakthrough in the analysis of pure fragmentation equations as this is the first instance whereby an exact solution is provided for the fragmentation evolution equation with arbitrary fragmentation rates. Recall that fragmentation processes are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it, that is why, though linear, they display non-linear features such as phase transition which, in this case, is called "shattering" and consists in the formation of a "dust" of particles of zero size carrying, nevertheless, a non-zero mass. Quantitatively, one can identify this process by disappearance of mass from the system even though it is conserved in each fragmentation event. Probabilistically, shattering is an example of an explosive, or dishonest Markov process, see e.g. [3, 66]. So the analysis yields a key for resolving most of the open problems in fragmentation theory including shattering and the sudden appearance of infinitely many particles in some systems with initial finite particles number. Though mathematical study of fragmentation processes can be traced back to article by Melzak [62] (from the analytical point of view) and Filippov [41] (from the probabilistic one), it was not until the 1980s that a systematic investigation of them was undertaken, mainly by Ziff and his students, e.g. [91, 92], who provided explicit solutions to a large class of fragmentation equations of the form

$$\frac{\partial}{\partial t}u(t,x) = -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \quad x \geq 0, t > 0, \tag{6.1}$$

with power law fragmentation rates $a(x) = x^\alpha, \alpha \in (-\infty, \infty)$ and where $b(x|y)$, the distribution of particle masses x spawned by the fragmentation of a particle of mass $y > x$, also was given by a power law

$$b(x|y) = (\nu+2)\frac{x^\nu}{y^{\nu+1}}, \tag{6.2}$$

with $\nu \in (-2, 0]$ (see also [61] for a more detailed discussion of this case). Here, $u(t,x)$ is the density of particles having mass x at time t.

In the absence of any other mechanism, the mass of all daughter particles must be equal to the mass of the parent. This 'local' conservation mass principle mathematically is expressed by

$$\int_0^y xb(x|y)dx = y. \tag{6.3}$$

Similarly, the expected number of particles produced by a particle of mass y is given by

$$n(y) = \int_0^y b(x|y)dx. \tag{6.4}$$

We note that $n(y)$ may be infinite.

Local conservation principles (6.3) and (6.4) render formal conservation principles by integration of (6.1):

$$\frac{d}{dt}M(t) = \int_0^\infty \frac{\partial}{\partial t}u(t,x)xdx = 0, \tag{6.5}$$

$$\frac{d}{dt}N(t) = \int_0^\infty \frac{\partial}{\partial t}u(t,x)dx = \int_0^\infty a(x)(n(x)-1)u(t,x)dx. \tag{6.6}$$

In this study, we extend the class of power law fragmentation rates to arbitrary positive and continuous function on $(0,\infty)$. Furthermore, we assume that b can be written as

$$b(x|y) = \beta(x)\gamma(y) \tag{6.7}$$

where, to satisfy the local principle of mass conservation,

$$\gamma(y) = \frac{y}{\int_0^y s\beta(s)ds}. \tag{6.8}$$

We assume that β is a non-negative continuous function on $(0, \infty)$. Equation (6.7) is a natural generalisation of the power law b described in (6.2) and has the advantage of allowing the number of daughter particles

$$n(y) = \frac{y\int_0^y \beta(s)ds}{\int_0^y s\beta(s)ds}, \tag{6.9}$$

to vary with the parent size y, [18]. An important role in the analysis is played by the function

$$b(x|x) = \beta(x)\gamma(x) = \frac{x\beta(x)}{\int_0^x s\beta(s)ds} = \frac{d}{dx}\ln\int_0^x s\beta(s)ds. \tag{6.10}$$

Theorem 6.2.1. *[18] Assume that $\lim_{x\to 0^+} a(x)$ exists (finite or infinite). Then, the fragmentation equation 6.1 is conservative if and only if there exists $\delta > 0$ such that $b(x|x)/a(x) \notin L_1([0,\delta])$.*

Laplace transforms

Definition 6.2.2.

The laplace tansform of a piecewise continuous function $f(t)$, $0 \le t < +\infty$ is the function $F(s) = \mathcal{L}\{f(t)\}$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

The inverse Laplace transform of $F(s)$ is $f(t)$, $f(t) = \mathcal{L}^{-1}(F(s))$.

6.2.2 Solvability of the fragmentation equation

In this section, Laplace transform is used to solve the fragmentation equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \quad x \ge 0, t > 0, \\ u(0,x) &= u_0(x). \end{aligned} \tag{6.11}$$

Let $\widetilde{u}(s,x) = \mathcal{L}[u(t,x)]$. Clearly, we have that

$$\mathcal{L}\left\{\frac{\partial}{\partial t}u(t,x)\right\} = s\widetilde{u}(s,x) - u_0(x), \tag{6.12}$$

$$\mathcal{L}\{a(x)u(t,x)\} = a(x)\widetilde{u}(s,x) \tag{6.13}$$

and

$$\mathcal{L}\left\{\int_x^\infty a(y)b(x|y)u(t,y)dy\right\} = \int_x^\infty a(y)b(x|y)\widetilde{u}(s,y)dy. \tag{6.14}$$

Substituting into (6.26), we obtain the equation

$$s\widetilde{u}(s,x) - u_0(x) = -a(x)\widetilde{u}(s,x) + \int_x^\infty a(y)b(x|y)\widetilde{u}(s,y)dy$$

that is,

$$u_0(x) = (s + a(x))\widetilde{u}(s,x) - \int_x^\infty a(y)b(x|y)\widetilde{u}(s,y)dy. \tag{6.15}$$

Viewing s as a parameter, this is similar to the resolvent equation solved in 2010 (Banasiak and Noutchie). The solution reads as:

$$\widetilde{u}(s,x) = \frac{u_0(x)}{s+a(x)} + \frac{\beta(x)}{s+a(x)}e^{-\xi_s(x)}\int_x^\infty \frac{a(y)\gamma(y)}{s+a(y)}e^{\xi_s(y)}u_0(y)dy, \tag{6.16}$$

where

$$\xi_s(x) = \int_1^x \frac{a(\eta)b(\eta|\eta)}{s+a(\eta)}d\eta. \tag{6.17}$$

The solution $u(t,x)$ of (6.26) is the inverse Laplace transform of $\widetilde{u}(s,x)$. Clearly,

$$\mathcal{L}^{-1}\left\{\frac{u_0(x)}{s+a(x)}\right\} = u_0(x)\mathcal{L}^{-1}\left\{\frac{1}{s+a(x)}\right\} = u_0(x)e^{-ta(x)}$$

and

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\beta(x)}{s+a(x)}e^{-\xi_s(x)}\int_x^\infty \frac{a(y)\gamma(y)}{s+a(y)}e^{\xi_s(y)}u_0(y)dy\right\} \\
&= \int_x^\infty a(y)b(x|y)u_0(y)\mathcal{L}^{-1}\left\{\frac{1}{s+a(x)}\frac{1}{s+a(y)}e^{\{\xi_s(y)-\xi_s(x)\}}\right\}dy \\
&= \int_x^\infty a(y)b(x|y)u_0(y)\mathcal{L}^{-1}\{\Theta(s,x,y)\}dy,
\end{aligned}$$

where

$$\Theta(s,x,y) = \frac{1}{s+a(x)}\frac{1}{s+a(y)}\exp\left\{\int_x^y \frac{a(\eta)b(\eta|\eta)}{s+a(\eta)}d\eta\right\}. \tag{6.18}$$

Therefore, the solution of the fragmentation equation

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \quad x \geq 0, t > 0, \\
u(0,x) &= u_0(x).
\end{aligned} \tag{6.19}$$

is given by

$$u(t,x) = u_0(x)e^{-ta(x)} + \int_x^\infty a(y)b(x|y)u_0(y)\mathcal{L}^{-1}\{\Theta(s,x,y)\}dy. \tag{6.20}$$

6.2.3 Applications

In this section, we assume that

$$a(x) = x^{\alpha+1}, \qquad \alpha \in (-\infty, \infty) \tag{6.21}$$

and

$$b(x|y) = (2+\nu)\frac{x^{\nu}}{y^{\nu+1}} \tag{6.22}$$

with $\nu \in (-2, 0]$. We have

$$\int_x^y \frac{a(\eta)b(\eta|\eta)}{s+a(\eta)}d\eta = (2+\nu)\int_x^y \frac{\eta^{\alpha}}{s+\eta^{\alpha+1}}d\eta = \frac{2+\nu}{\alpha+1}\ln\left\{\frac{s+y^{\alpha+1}}{s+x^{\alpha+1}}\right\},$$

it follows that

$$\exp\left\{\int_x^y \frac{a(\eta)b(\eta|\eta)}{s+a(\eta)}d\eta\right\} = \left\{\frac{s+y^{\alpha+1}}{s+x^{\alpha+1}}\right\}^{\gamma},$$

where

$$\gamma = \frac{2+\nu}{\alpha+1}.$$

Thus,

$$\Theta_{\alpha,\nu}(s,x,y) = \frac{(s+y^{\alpha+1})^{\gamma-1}}{(s+x^{\alpha+1})^{\gamma+1}} = \left\{\frac{1}{s+x^{\alpha+1}}\right\}^{\gamma+1}\left\{s+y^{\alpha+1}\right\}^{\gamma-1}.$$

Therefore, the solution $u(t,x)$ is given by

$$u(t,x) = u_0(x)e^{-tx^{\alpha+1}} + (2+\nu)\int_x^{\infty}\left\{\frac{x}{y}\right\}^{\nu} y^{\alpha}u_0(y)\mathcal{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\}dy.$$

Case $\alpha = -3$ and $\nu = 0$

We want to solve the equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= -x^{-2}u(t,x) + 2\int_x^{\infty} y^{-3}u(t,y)dy, \\ u(0,x) &= u_0(x). \end{aligned} \tag{6.23}$$

We have $\gamma = -1$, it follows that

$$\Theta_{-3,0}(s,x,y) = \left\{\frac{1}{s+x^{-2}}\right\}^{0}\left\{s+y^{-2}\right\}^{-2} = \left\{s+y^{-2}\right\}^{-2}.$$

Thus,

$$\mathcal{L}^{-1}\left\{\Theta_{-3,0}(s,x,y)\right\} = \mathcal{L}^{-1}\left\{(s+y^{-2})^{-2}\right\} = te^{-ty^{-2}}.$$

Therefore,

$$u(t,x) = u_0(x)e^{-tx^{-2}} + 2t\int_x^{\infty} y^{-3}e^{-ty^{-2}}u_0(y)dy. \tag{6.24}$$

Case $\alpha = -2$ and $\nu = 0$

We want to solve the equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= -x^{-1}u(t,x) + 2\int_x^\infty y^{-2}u(t,y)dy, \\ u(0,x) &= u_0(x). \end{aligned} \tag{6.25}$$

We have $\gamma = -2$, it follows that

$$\begin{aligned} \Theta_{-2,0}(s,x,y) &= \left\{\frac{1}{s+x^{-1}}\right\}^{-1}\left\{s+y^{-1}\right\}^{-3} \\ &= \frac{s+y^{-1}-y^{-1}+x^{-1}}{(s+y^{-1})^3} \\ &= \frac{1}{(s+y^{-1})^2} + \frac{(x^{-1}-y^{-1})}{(s+y^{-1})^3}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\Theta_{-2,0}(s,x,y)\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+y^{-1})^2}\right\} + (x^{-1}-y^{-1})\mathcal{L}^{-1}\left\{\frac{1}{(s+y^{-1})^3}\right\} \\ &= te^{-ty^{-1}} + (x^{-1}-y^{-1})e^{-ty^{-1}}\frac{t^2}{2}. \end{aligned}$$

Therefore,

$$u(x,t) = e^{-\frac{t}{x}}u_0(x) + 2t\int_x^\infty \frac{e^{-\frac{t}{y}}}{y^2}e^{-\frac{t}{y}}u_0(y)dy + t^2\int_x^\infty \frac{e^{-\frac{t}{y}}}{y^2}\left(\frac{1}{x}-\frac{1}{y}\right)u_0(y)dy.$$

General case $a(x) = x^{\alpha+1}$ and $b(x|y) = (2+\nu)\dfrac{x^\nu}{y^{\nu+1}}$

We want to solve

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= -x^{\alpha+1}u(t,x) + (2+\nu)x^\nu\int_x^\infty y^{\alpha-\nu}u(t,y)dy, \quad x \geq 0, t > 0, \\ u(0,x) &= u_0(x). \end{aligned} \tag{6.26}$$

From the previous section, the solution of this equation is

$$u(t,x) = u_0(x)e^{-tx^{\alpha+1}} + (2+\nu)\int_x^\infty \left\{\frac{x}{y}\right\}^\nu y^\alpha u_0(y)\mathcal{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\}dy.$$

Note that

$$\mathcal{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\} = \mathcal{L}^{-1}\left\{\left\{\frac{1}{s+x^{\alpha+1}}\right\}^{\gamma+1}\left\{s+y^{\alpha+1}\right\}^{\gamma-1}\right\}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\} &= \mathcal{L}^{-1}\left\{\left\{\frac{1}{s+x^{\alpha+1}}\right\}^{\gamma+1}\left\{s+y^{\alpha+1}\right\}^{\gamma-1}\right\} \\ &= t\exp(-tx^{\alpha+1})\,{}_1F_1(1-\gamma;\ 2;\ t(x^{\alpha+1}-y^{\alpha+1})).\end{aligned}$$

It follows that

$$\begin{aligned}u(t,x) &= u_0(x)e^{-tx^{\alpha+1}} + \\ &\quad (2+\nu)t\exp(-tx^{\alpha+1})\int_x^\infty \left\{\frac{x}{y}\right\}^\nu {}_1F_1(1-\gamma;\ 2;\ t(x^{\alpha+1}-y^{\alpha+1}))y^\alpha u_0(y)dy.\end{aligned}$$

We recover the results of Ziff and his students, which were exactly the expected results.

6.3 Analysis of the effects of large scale marine iron fertilisation

6.3.1 Introduction and motivation

In this section, a non-linear integro-differential equation is used to investigate the effects of ocean iron fertilisation on the evolution of the phytoplankton biomass. This equation contains terms responsible for fragmentation, coalescence, growth-decay, grazing and sinking of the phytoplankton aggregates. The evolution equation is analysed using the theory of semilinear dynamical systems and numerical simulations are performed. The results demonstrate the validity of the iron hypothesis in fighting climate change.

Phytoplankton are microscopic plant-like marine organisms that sit at the bottom of the food chain. They are food for other plankton and small fish, as well as larger animals such as whales. Phytoplankton get their energy from carbon dioxide through photosynthesis and so are very important in carbon cycling. Each year, as they transfer billions of tonnes of carbon from the atmosphere to the ocean reducing global warming in the process, they are of primary interest to oceanographers and Earth scientists around the world. According to researchers [40, 55], these tiny marine organisms, which are crucial components of marine ecosystems, have been slowly disappearing over the last century. The decline is worrying because it may have profound effects on marine life and climate change. The major decrease has been recorded in the High Nitrate Low Chlorophyll

(HNLC) Regions that are thought to represent about 20 percent of the areal extent of the world's oceans ([40] and references therein). These are generally regions characterised by more than 2 micromolar nitrate and less than 0.5 micrograms chlorophyll per litre. The major HNLC regions include the Subarctic Pacific, large regions of the Eastern Equatorial Pacific and the Southern ocean. These HNLC regions persist in areas which have high macronutrient concentrations, adequate light and physical characteristics required for phytoplankton growth, but have very low plant biomass. It is believed that phytoplankton growth in major nutrient-rich HNLC regions is limited by iron deficiency [40, 55]. The main purpose of this study is to show that global warming can be substantially reduced and to some extent, annihilated by fertilizing the HNLC areas of the oceans by a very modest amount of iron. The formation of large particles (aggregates) through multiple collision of smaller ones is a highly visible phenomenon in oceanic waters. Several authors have attempted to model the dynamics of phytoplankton in such a way as to exhibit this structure [2, 5, 10, 13, 26, 27, 28, 53, 58, 68, 69]. In this setting, the individual unit is an aggregate and aggregates are structured by their mass. One of the most efficient approaches to modeling the dynamics of phytoplankton aggregates is through a rate equation which describes the evolution of the distribution of interacting aggregates with respect to their mass. The evolution equation contains terms responsible for the coalescence, disaggregation, growth-decay, sinking to the seabed of the aggregates and their grazing by the zooplankton. The novelty in the model from a mathematical point of view is that we allow the kernels to vary according to the level of marine iron concentration. Next, we present a full description of the phytoplankton aggregates model used in this article and provide the assumptions. Then, we make use of the theory of semilinear abstract Cauchy problem used to analyse coagulation fragmentation processes with growth [5, 10, 12, 13, 28, 58, 62, 69, 68] or decay [12, 69] in order to show the well-posedness of the adopted model. In the last part of the study, numerical simulations are performed and the results are discussed.

6.3.2 Description of the model and assumptions

Following [10], we consider the following non-linear transport equation that contains terms responsible for the growth/decay of phytoplankton aggregates, their fragmentation, coagulation, grazing and sinking of aggregates into the seabed:

$$\begin{aligned}\frac{\partial}{\partial t}u_\zeta(t,x) &= -\frac{\partial}{\partial x}[r_\zeta(x)u_\zeta(t,x)] - s_\zeta(x)u_\zeta(t,x) - d(x)u_\zeta(t,x)\\ &\quad -a(x)u_\zeta(t,x) + \int_x^\infty a(y)b(x|y)u_\zeta(t,y)\,dy \qquad (6.27)\\ &\quad -u_\zeta(t,x)\int_0^\infty k(x,y)u_\zeta(t,y)dy + \frac{1}{2}\int_0^x k(x-y,y)u_\zeta(t,x-y)u_\zeta(t,y)dy,\end{aligned}$$

where ζ represents the iron concentration in the sea. The sinking rate and the growth-decay rate of the clusters are denoted by s_ζ and r_ζ respectively, they are ζ-dependent.

Here, $x \in \mathbb{R}_+$ represents the size of particles, t is the time variable and u_ζ is the density of particles of mass x. The fragmentation rate is denoted by a and b describes the distribution of masses x of particles spawned by the fragmentation of a particle of mass y. The removal of phytoplankton aggregate is carried out by the grazing of the population by the zooplankton and the clusters sinking into the seabed. The grazing rate is denoted by $d(x)$ and it is assumed that

$$d \in L_\infty(\mathbb{R}_+). \tag{6.28}$$

We introduce the following notation for formal expressions appearing in (6.27):

$$[\mathcal{T}_\zeta u_\zeta](x) = -\frac{\partial}{\partial x}[r_\zeta(x)u_\zeta(x)] - q_\zeta(x)u_\zeta(x), \tag{6.29}$$

$$[\mathcal{B}u_\zeta](x) = \int_x^\infty a(y)b(x|y)u_\zeta(y)\,dy, \tag{6.30}$$

$$[\mathcal{K}u_\zeta](x) = -u_\zeta(x)\int_0^\infty k(x,y)u_\zeta(y)dy + \frac{1}{2}\int_0^x k(x-y,y)u_\zeta(x-y)u_\zeta(y)dy, \tag{6.31}$$

where $q_\zeta = a + d + s_\zeta$.

Assumptions on the coefficients

The sinking function $s_\zeta \geq 0$ represents the removal rate of the aggregates of phytoplankton into the seabed; it is assumed that for any fixed $\zeta \in \mathbb{R}_+$,

$$s_\zeta \in L_\infty(\mathbb{R}_+). \tag{6.32}$$

We assume that the fragmentation rate a is essentially bounded on compact subintervals of $\overline{\mathbb{R}}_+$; i.e.

$$a \in L_{\infty,loc}(\overline{\mathbb{R}}_+). \tag{6.33}$$

Further, $b \geq 0$ is assumed to be a measurable function of two variables, satisfying

$$b(x|y) = 0; \quad \text{for} \quad x > y. \tag{6.34}$$

The local law of mass conservation requires

$$\int_0^y xb(x|y)dx = y, \quad \text{for each } y > 0. \tag{6.35}$$

The coagulation kernel $k(x,y)$ represents the likelihood of a particle of size x attaching itself to a particle of size y and we assume

$$0 \leq k \in L_\infty(\mathbb{R}_+^2). \tag{6.36}$$

The transport part is more tortuous. Our principal assumption is that clusters of phytoplankton grow ($r_\zeta > 0$) when the iron concentration ζ in the sea is bigger than a critical value ζ_c and they decay otherwise $(r_\zeta < 0)$. In phytoplankton models typically, we have $r_\zeta(x) \sim x$ as growth/decay is proportional to the number of particles (cells) in the aggregate. Thus, we assume that

$$|r_\zeta(x)| \leq \widetilde{r}x \tag{6.37}$$

for some constant $\widetilde{r} > 0$ and

$$r_\zeta \in AC(\mathbb{R}_+), \tag{6.38}$$

where $r_\zeta \in AC(\mathbb{R}_+)$ means that r_ζ is absolutely continuous on each compact subinterval of $\mathbb{R}_+$. Further assumptions on r_ζ depend on whether to deal with the decay, or growth, case. As we shall see, in the decay case there is no need for boundary conditions . On the other hand, depending on the integrability of r_ζ at $x = 0$, the transport equation describing growth may require a boundary condition at $x = 0$. In this study, we consider the general McKendrick-von Foerster renewal boundary condition

$$\lim_{x\to 0^+} r_\zeta(x)u_\zeta(t,x) = \int_0^\infty \beta_\zeta(y)u_\zeta(t,y)\,dy, \tag{6.39}$$

where β_ζ is a suitable positive measurable function for any $\zeta \neq 0$. If $\beta_\zeta \equiv 0$, then we have standard no-influx condition. If, however, $\beta_\zeta(y) > 0$, then it describes the rate at which an aggregate of size y sheds monomers of the smallest 'zero' size which then re-enter the system as new aggregates and start to grow. The non-linear integro-differential equation (6.27) will be supplemented with an initial condition.

6.3.3 Analysis of the problem

The approach in this study is to analyse the evolution equation in the Banach space

$$X_{0,1} := L_1((0,\infty),(1+x)dx) = \left\{\phi;\ \int_0^\infty |\phi(x)|(1+x)dx < +\infty\right\}$$

in which both the total mass and the number of particles are controlled. In order to ensure the validity of the general McKendrick-von Foerster renewal boundary condition, we further assume that

$$\beta \in X_{\infty,1}, \quad \text{and} \quad r_\zeta \in X_{\infty,1} \quad \text{for any } \zeta > 0,$$

where $X_{\infty,1}$ is the dual space of $X_{0,1}$. It consists of measurable functions f for which

$$||f||_{\infty,1} = \operatorname*{ess\ sup}_{x\in\mathbb{R}_+} \frac{|f(x)|}{1+x} < \infty.$$

The duality pairing is given by the integral

$$< f, g >= \int_0^\infty f(x)g(x)dx.$$

In this section, we make use of the theory of semilinear abstract Cauchy problems. The idea is to show that the linear operator induced by fragmentation, growth and sinking of the aggregates generates a strongly continuous semigroup. This linear operator is then perturbed by the non-linear operator induced by coagulation of the aggregates and yields the existence and uniqueness of a local solution to the evolution equation. Global existence is thereafter obtained by showing that the local solution does not blow up in finite time.

Analysis of the linear part

Let us denote by T_ζ the realisation of $\mathcal{T}_\zeta$ (defined *via* (6.29)) in the domain

$$D_\zeta = \{\psi \in X_{0,1};\ q\psi \in X_{0,1},\ \ r_\zeta\psi \in AC((x_0,\infty))\ \ (r_\zeta\psi)_x \in X_{0,1}\} \tag{6.40}$$

if r_ζ^{-1} non-integrable at 0, and in the domain

$$D_{\beta,\zeta} = \left\{\psi \in D_\zeta :\quad \lim_{x\to 0^+} r_\zeta(x)\psi(x) = \int_0^\infty \beta(y)\psi(y)\,dy\right\}, \tag{6.41}$$

otherwise. In addition, let B be the realisation of $\mathcal{B}$ (defined *via* (6.30)) in the domain

$$D(B) = D(T_\zeta) = \{\psi \in X_{0,1};\ q\psi \in X_{0,1},\ \ r\psi \in AC((0,\infty))\ \ (r\psi)_x \in X_{0,1}\}.$$

For further use we, define for any iron concentration $\zeta > 0$

$$R_\zeta(x) := \int_1^x \frac{1}{r_\zeta(s)}ds, \quad Q_\zeta(x) := \int_1^x \frac{q_\zeta(s)}{r_\zeta(s)}ds. \tag{6.42}$$

Theorem 6.3.1. *The operator $(T_\zeta; D(T_\zeta))$ with the resolvent given by*

$$(Res_\zeta(\lambda)f)(x) = \frac{e^{\lambda R_\zeta(x)+Q_\zeta(x)}}{r_\zeta(x)}\int_x^\infty \frac{e^{\lambda R_\zeta(y)+Q_\zeta(y)}}{r_\zeta(y)}f(y)dy, \tag{6.43}$$

for any $\lambda > 0$ and $f \in X_{0,1}$ is the generator of a strongly continuous positive semigroup of contractions, say $\{S_{T_\zeta}(t)\}_{t\geq 0}$ on $X_{0,1}$.

Proof. The case $r_\zeta > 0$ representing fragmentation with growth is similar to [68] and the case $r_\zeta < 0$ representing decay can be found in detail in [69]. In both cases, the expression of the resolvent is obtained and Hille-Yosida's inequality is proven to be satisfied. □

Theorem 6.3.2. *There exists an extension $(G_\zeta; D(G_\zeta))$ of the operator $(T_\zeta + B; D(T_\zeta))$ which generates a positive strongly continuous semigroup $(S_{G_\zeta}(t))_{t\geq 0}$ in $X_{0,1}$. Moreover, the generator G_ζ is characterised by:*

$$(\lambda I - G_\zeta)^{-1}\psi = \sum_{n=0}^{\infty}(\lambda I - T_\zeta)^{-1}[B(\lambda I - T_\zeta)^{-1}]^n\psi, \tag{6.44}$$

for $\psi \in X_{0,1}$ and $\lambda > 0$.

Proof. The proof is a generalisation of a similar result for the space $X_{0,1}$, obtained in [10] by assuming that the fragmentation rate a is linearly bounded. The analysis in [10] can be easily extended to general fragmentation rates because the fragmentation equation behaves well in the bigger space $X_1 := L_1((0,\infty), xdx) = \{\phi;\ \int_0^\infty |\phi(x)|xdx < +\infty\}$. A complete proof of this thm is available in [69]. □

Theorem 6.3.3. *Assume* $\lim_{x\to x_0} q_\varsigma(x) = \lim_{x\to 0} a(x)+d(x)+s_\varsigma(x) < +\infty$, *then the generator of the semigroup* $(S_{G_\varsigma}(t))_{t\geq 0}$ *is given by*

$$G_\varsigma = \overline{T_\varsigma + B}.$$

Proof. The theory of extension of operator is instrumental in the proof of this thm. In the case r^{-1} non-integrable at x_0, the assumption made is not necessary. The semigroup $(S_{G_\varsigma}(t))_{t\geq 0}$ is honest for arbitrary fragmentation rate $a \in L_{\infty,loc}((0,\infty))$ and grazing rate $d \in L_\infty((0,\infty))$. The proof is analogous to the analysis for honesty performed in [12]. For r^{-1} integrable at 0, the proof is obtained in a similar way as in [10] where honesty was investigated in the space $X_{0,1}$. □

Global solutions of the transport equation with fragmentation and coagulation

In this section we show the existence of a global solution to the full evolution problem (6.27) endowed with its initial and boundary conditions. This evolution equation is represented by the following semilinear astract cauchy problem:

$$\begin{aligned} \frac{du_\varsigma}{dt}(t) &= [G_\varsigma + K]u_\varsigma(t) \\ u_\varsigma(0) &= u_0, \end{aligned} \tag{6.45}$$

where K is the realisation of the expression

$$[\mathcal{K}\psi](x) = \frac{1}{2}\int_0^x k(x-y,y)\psi(x-y)\psi(y)dy - \psi(x)\int_0^\infty k(x,y)\psi(y)dy, \tag{6.46}$$

for non-zero ψ on the space $X_{0,1}$ and $K(0) = 0$. Since the linear semigroups $(S_{G_\varsigma}(t))_{t\geq 0}$ is positive, we shall work in the positive cone of $X_{0,1}$, denoted by $X_{0,1}^+$.

Theorem 6.3.4. *Let* $u_0 \in X_{0,1}^+$, *then the Cauchy problem*

$$\frac{du_\varsigma}{dt}(t) = G_\varsigma[u(t)] + K[u(t)], \qquad u_\varsigma(0) = u_0 \tag{6.47}$$

has a unique global solution.

Proof. In order to prove that (6.45) has a solution which is global in time, we shall proceed in a usual way [10] by converting it to the integral equation

$$u_\zeta(t) = S_G(t)u_0 + \int_0^t S_G(t-s)K[u_\zeta(s)]\,ds, \qquad t \geq 0, \tag{6.48}$$

where $(S_G(t))_{t\geq 0}$ is the semigroup generated by G. We use the fact that $X_{0,1+}$ is a complete metric space as a closed subspace of a Banach space, see [74, Theorem 6.1.2]. The method is analogous to the proof of global existence in [10] with the space $X_{0,1}$. □

6.3.4 Numerical simulations

This section provides a prediction of the phytoplankton biomass

$$N_\zeta(t) = \int_0^\infty x u_\zeta(t,x)dx$$

from 2010 to 2030. A numerical method will be used and numerical simulations performed over the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t}u_\zeta(t,x) &= -\frac{\partial}{\partial x}[r_\zeta(x)u(t,x)] - s_\zeta(x)u(t,x) - d(x)u(t,x) \\ &\quad -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)\,dy \\ &\quad -u(t,x)\int_0^\infty k(x,y)u(t,y)dy + \frac{1}{2}\int_0^x k(x-y,y)u(t,x-y)u(t,y)dy \end{aligned}$$

describing the dynamics of the phytoplankton population.

Empirical data and estimation of kernels

Let the initial conditions set to be $u(x,0) = 10^8\chi e^{-x}$, where χ is a positive real number. To determine the effects of ocean iron concentration ζ on the dynamics of the population, we follow [89] and make use of Runge–Kutta methods extended with a quadrature technique (Pouzet type) in order to simulate the corresponding non-dimensionalised evolution equation. We investigate the dynamics of the plankton population in the HNLC regions to predict the evolution corresponding to some specified values of iron concentration. The kernels used in the simulations are summarised in the following table:

Parameter values were estimated from available experimental information. In the event where no observational data could be obtained, parameter values were picked out to provide the best qualitative numerical simulation results. This is in line with previous studies successfully simulating the dynamics of phytoplankton [2, 55].

Table 6.1: Kernels used in the simulations

Description	Kernels
Fragmentation rate	$a(x) = x^{\alpha}$
Daughter particles distribution rate	$b(x,y) = \dfrac{(\nu+2)x^{\nu}}{y^{\nu+1}}$
Grazing rate	$d(x) = d$
Aggregation rate	$k(x,y) = k$
Growth rate	$r_{\zeta}(x) = (\zeta - \zeta_c)x$
Sinking rate	$s_{\zeta}(x) = \zeta x$
Renewal rate	$\beta_{\zeta}(x) = \zeta x$

Table 6.2: Parameter values used in the simulations

Parameter	Value	Source
ζ_c	2	[2, 55]
ζ	$(0.02, 10)$	fitted
d	20	fitted
k	500	fitted
ν	0	[55, 69]
χ	$(1.8, 2.2)$	fitted
α	0.75	[2, 55, 69]

Computational simulations

The critical iron concentration value that determines the growth or decay of phytoplankton aggregates is approximately 2 nanomole (nM) per litre in the High Nitrate Low Chlorophyll (HNLC) regions of the oceans [69]. Simulations are performed for iron concentration values around this critical value:

$$\zeta \in \{0.02, 0.1, 0.5, 1, 1.4, 1.6, 2.1, 2.4, 2.5, 3, 5, 10\}$$

and arbitrary χ values in the range $(1.8, 2.2)$. The fragmentation daughter particle distribution kernel is chosen to be binary $\nu = 0$ and the fragmentation rate $a(x) = x^{0.75}$ is chosen to be linearly bounded. The coalescence rate k and the grazing rate d are taken to be 500 and 20 respectively. The simulation results are summarised in Figures 6.1, 6.2 and 6.3 respectively.

Interpretation of the results and discussions

The simulations results suggest that iron (Fe) availability is the primary factor controlling phytoplankton production in HNLC regions of the oceans. The population biomass is

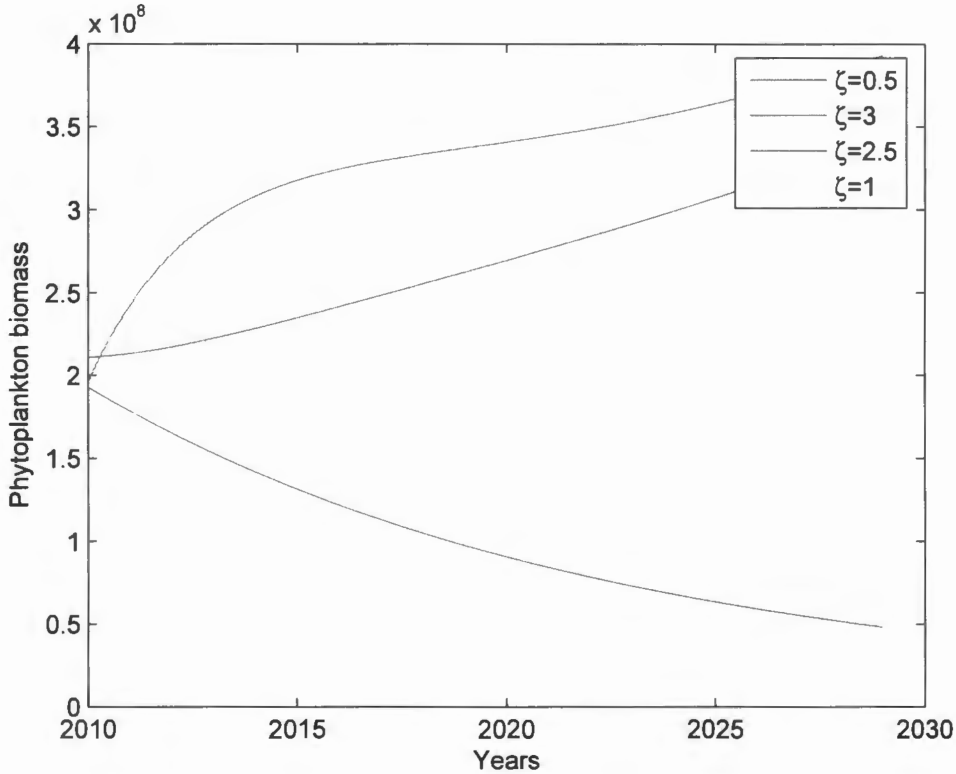

Figure 6.1: Comprehensive prediction of the phytoplankton biomass.

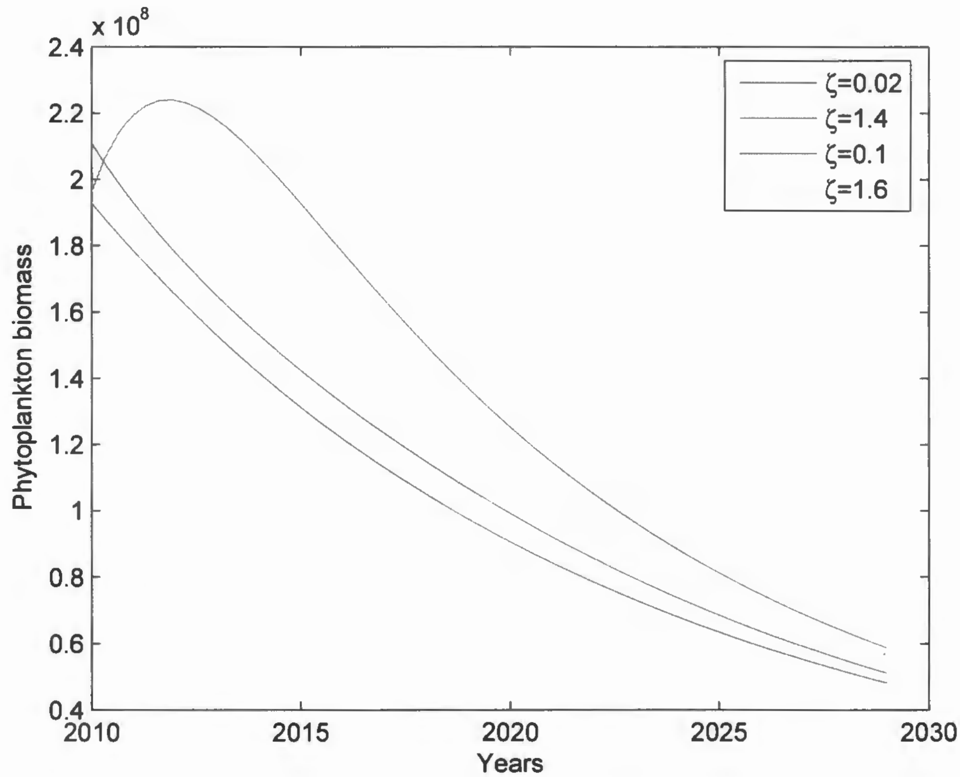

Figure 6.2: Prediction of the phytoplankton biomass with $\zeta < \zeta_c$.

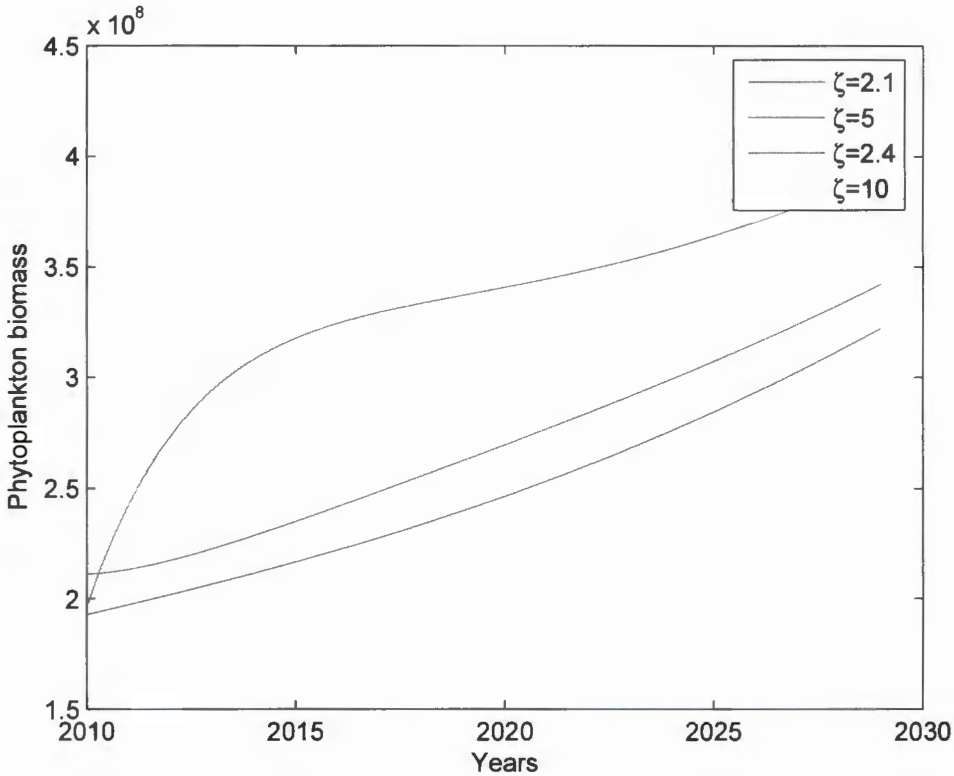

Figure 6.3: Prediction of the phytoplankton biomass with $\zeta > \zeta_c$.

seen to increase with $\zeta > \zeta_c$ and decrease otherwise over a long period of time. The increment in the biomass observed for $\zeta = 1.4, 1.6 < \zeta_c$ in the first 10 year interval can be explained by the fact that HNLC regions are in general nutrient-rich areas and plankton population may grow with a very little amount of iron available ($\zeta < 0.03$ nM). However, the population reduces substantially in the long run because the acquisition of sufficient iron (Fe) for phytoplankton syntheses of Chl and nitrate reductase needed by them to use the abundant major nutrients becomes a serious problem [40, 55]. Another important feature in Figure 6.2 is the suggestion that maintaining the level of iron concentration just a little above the critical value ζ_c has the potential to ensure a long term satisfactory level of phytoplankton biomass. It also clearly shows that raising the level of marine iron too much above ζ_c does not present any technical advantage in the long run. The results of this study indicate that iron hypothesis can be implemented in a very cost-effective way and produce impressive results. In summary we have made used of mathematical and computational techniques in order to present a very efficient method to increase the world phytoplankton biomass. This method may be recommended for future use in order to systematically reduce the effects of global warming.

Chapter 7

Discrete Non-local Fragmentation Dynamics

7.1 Introduction

In this chapter, an investigation of the honesty in non-local and discrete fragmentation dynamics is done. In the process, the major problem always arises when each fragmentation rate becomes infinite at infinity. That is why in this chapter, we consider a discrete Cauchy problem describing multiple fragmentation processes that is investigated by means of parameter-dependent operators together with the theory of substochastic semigroups with a parameter. Focus is on the case where fragmentation rates are size and position dependent and where new particles are also spatially randomly distributed according to a certain probabilistic law. The existence of semigroups is established for each parameter and "glued"together in order to obtain a semigroup to the full space. The cases of discrete models with bounded and unbounded fragmentation rates are both treated (see also the published articles [36, 71]). We use Kato's theorem in L_1 [12] and the dominated convergence theorem [21] to show existence of the infinitesimal generator of a positive semigroup of contractions and give sufficient conditions for honesty in the case of unbounded fragmentation rates. Essentially, we demonstrate that even in discrete and non-local case, the process is conservative if at infinity, daughter particles tend to go back into the system with a high probability.

7.2 Motivation and models' description

In the process of fragmentation dynamics, when it is supposed that every group of size $n \in \mathbb{N}$ (one n-group) in a system of particles clusters consists of n identical fundamental units (monomers), then the mass of every group is simply a multiple positive integer of the mass of the monomer. We focus here on clusters that are discrete; that is, they consist of a finite number of elementary (unbreakable) particles which are assumed to be of unit mass. The state at a given time t is the repartition at that time of all aggregates according to their size n and their position x. The evolution of such particle-mass-position distribution is given by an integrodifferential [9] equation as we will see in this thesis. Recall that various types of fragmentation equations have been comprehensively analysed in numerous studies (see, e.g., [12, 18, 17, 19, 11, 43, 54, 86, 91, 92]). But discrete fragmentation processes have not yet been widely investigated. In [9], a discrete model with the concentration depending only on the size n of clusters and time t is analysed and the author used compactness of the semigroups to analyse their long time behaviour and proved that they have the asynchronous growth property.

In this chapter, a technique called the method of semigroups with a parameter [12] is exploited to analyse discrete fragmentation models with the concentration of particles depending not only on the size n of clusters and time t, but also on the random position x of the clusters in the space.

Let us recall some useful assumptions already used in previous chapters. We focus on models with discrete size; that is, we assume that the mass of a particle can be an arbitrary positive integer. The starting point is to describe the state variable of the problem. $p_n = p(t, x, n)$ which characterises the state of the system at any moment t is the particle-mass-position distribution defined as $p : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{N} \to \mathbb{R}_+$. For the sake of simplicity, $p(t, x, n) = p_n(t, x)$ or simply p_n will sometimes be used. During the unit time, a fraction $a_n(x) = a(x, n)$ (or simply a_n) of aggregates of size n and located at x are undergoing breakup. We assume that for each $n \in \mathbb{N}$, a_n satisfies the condition (3.29), that is, there are two constants $0 < \theta_1$ and θ_2 such that

$$\theta_1\alpha_n \leq a_n(x) \leq \theta_2\alpha_n, \tag{7.1}$$

with $\alpha_n \in \mathbb{R}_+$ and independent of the state variable x. For the same reasons as in (3.3), we also require

$$a_1(x) = 0 \tag{7.2}$$

for every $x \in \mathbb{R}^3$. Once a group of size m and position x breaks, the expected average number of n-group produced upon the splitting is a non-negative measurable function $b_{n,m}(x) = b(x, n, m)$ defined on $\mathbb{R}^3 \times \mathbb{N} \times \mathbb{N}$ with support in the set

$$\mathbb{R}^3 \times \{(n, m) \in \mathbb{N} \times \mathbb{N} : m > n\}.$$

The sum of all individuals obtained by fragmentation of a n-group should obviously be n, hence it follows, as in (3.3), that for any $n \in \mathbb{N}$, $x \in \mathbb{R}^3$

$$\sum_{m=1}^{n-1} mb(x, m, n) = n. \tag{7.3}$$

Since a group of size $m \leq n$ cannot split to form a group of size n, we require

$$b_{n,m} = 0 \text{ for all } m \leq n. \tag{7.4}$$

Furthermore, the expected number of daughter particles produced by fragmentation of a n-group (with position x) is, by definition, given by $\sum_{m=1}^{n-1} b(x,m,n)$. In case of binary fragmentation [5, 52], it is straightforward that for a.a $x \in \mathbb{R}^3$, $b(x,m,n) = b(x,n-m,n)$ for all $n, m, n > m$, and $\sum_{m=1}^{n-1} mb(x,n,m) = 2$ for all $n \in \mathbb{N}$.

We also assume that the condition (4.2) holds. Therefore, the centres of new groups issued from cluster fragmentation are distributed according to the given probabilistic law $h(\cdot\,, m, m, y)$ verifying (4.2).

The equation describing the evolution of the particle-mass-size distribution function for a discrete system undergoing fragmentation can be derived by balancing loss and gain of clusters of size n (with position x) over a short period of time and is given by

$$\begin{aligned}\frac{\partial p}{\partial t}(t,x,n) &= -a_n(x)p_n \\ &+ \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)dy, \quad n = 1,2,3,\cdots\end{aligned} \tag{7.5}$$

where in terms of n and x, the state of the system is characterised at any moment t by the *density* (or *concentration*) of particles $p_n \equiv p(t,x,n)$.

7.3 Well posedness of the fragmentation problem

Since $p_n = p(t,x,n)$ is the number density of groups of size n at the position x and that mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$\mathcal{X}_1 := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n|g_n(x)|dx < \infty \right\}$$

in which the total number of individuals is finite. We assume that for each $t \geq 0$, the function $(x,n) \longrightarrow p(t,x,n) = p_n(t,x)$ is from the space $\mathcal{X}_1$. In order to make use of the semigroup theory of linear operators, we need to complement (7.5) with the initial mass-position distribution

$$p(0,x,n) = \overset{\circ}{p}_n(x), \qquad n = 1,2,3,... \tag{7.6}$$

where the function $\overset{\circ}{p}_n$ is integrable with respect to x over the full space $\mathbb{R}^3$, this integral multiplied by n is summable and the sum is finite.

In $\mathcal{X}_1$, (7.5) and (7.6) can be rewritten in more compact form,

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{p} &= \mathfrak{A}\mathbf{p} + \mathfrak{B}\mathbf{p} \\ \mathbf{p}_{|t=0} &= \overset{\circ}{\mathbf{p}}\end{aligned} \tag{7.7}$$

Here, $\mathbf{p}$ is the vector $(p(t,x,n))_{n\in\mathbb{N}}$, $\mathfrak{A}$ is the diagonal matrix $(a_n(x))_{n\in\mathbb{N}}$, $\mathfrak{B}$ is defined by the expression

$$\mathfrak{B}\mathbf{p} = \left(\sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y) b_{n,m}(y) h(x,n,m,y) p_m(y) dy \right)_{n=1}^{\infty}, \tag{7.8}$$

$\overset{\circ}{\mathbf{p}}$ the initial vector $(\overset{\circ}{p}_n(x))_{n\in\mathbb{N}}$ which belongs to $\mathcal{X}_1$

Operators $\mathbf{A}$ and $\mathbf{B}$ are introduced in $\mathcal{X}_1$ and defined by

$$\begin{array}{ll} [\mathbf{Ap}](x,n) = [\mathfrak{A}\mathbf{p}](x,n), & D(\mathbf{A}) = \{\mathbf{g} \in \mathcal{X}_1; \quad \mathbf{ag} \in \mathcal{X}_1\} \\ [\mathbf{Bp}](x,n) = [\mathfrak{B}\mathbf{p}](x,n), & D(\mathbf{B}) := D(\mathbf{A}) \end{array} \tag{7.9}$$

Lemma 7.3.1. *The operator $(\boldsymbol{A}+\boldsymbol{B},\ D(\boldsymbol{A}))$ is well-defined.*

Proof. We need to show that $\mathbf{B}D(\mathbf{A}) \subset \mathcal{X}_1$. For every $\mathbf{g} \in D(\mathbf{A})$,

$$\begin{aligned} \|\mathbf{Bg}\|_1 &= \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y) b_{n,m}(y) h(x,n,m,y) |g(y,m)| dy \right) dx \\ &= \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{\infty} a_m(y) b_{n,m}(y) |g(y,m)| \right) dy \\ &= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m(y) |g(y,m)| \left(\sum_{n=1}^{\infty} n b_{n,m}(y) \right) dy \\ &= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m(y) |g(y,m)| \left(\sum_{n=1}^{m-1} n b_{n,m}(y) \right) dy \\ &= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} m a_m(y) |g(y,m)| dy \\ &= \int_{\mathbb{R}^3} \sum_{m=1}^{\infty} m a_m(y) |g(y,m)| dy \\ &= \|\mathbf{Ag}\|_1 \\ &< \infty, \end{aligned}$$

where (4.2), (7.3) and (7.4) have been used respectively. Then, $\|\mathbf{Bg}\|_1 = \mathbf{Ag}\|_1, \quad \forall \mathbf{g} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and $(\mathbf{A}+\mathbf{B},\ D(\mathbf{A}))$ is well-defined. □

7.3.1 Mathematical setting and analysis

The context of this analysis is the same as the one given in Section 3.4.2. In fact, the operators on the right-hand side of (7.9) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other a variable. Thus, we also need to work with parameter-dependent operators that can be "glued" together in such a way that the resulting operator inherits the properties of the individual ones. A framework for such a technique was provided in Section 3.4.2. According to that context, we consider here, the space $\mathcal{X} := L_p(V, X)$ where $1 \leq p < \infty$, (V, dm) is a measure space and and X a Banach space. Then, we can take $\mathbb{A} = \mathbf{A}$, $\mathcal{X} = \mathcal{X}_1 = L_1(\mathbb{N}, X_x) = L_1(\Lambda, d\mu d\varsigma) = L_1(\mathbb{R}^3 \times \mathbb{N}, d\mu d\varsigma)$, where

$$X_x := L_1(\mathbb{R}^3, dx) := \{\psi : \|\psi\| = \int_{\mathbb{R}^3} |\psi(x, n)| dx < \infty\},$$

$\mathbb{A}$ is defined in (3.15)–(3.16) and $\mathbb{N}$ is equipped with the counting measure $d\varsigma$ and $d\mu = dx$ is the Lebesgue measure in $\mathbb{R}^3$. In X_x, we define the operators $(\mathcal{A}_n, D(\mathcal{A}_n))$ as

$$\begin{aligned} \mathcal{A}_n p(t, x, n) &= a_n(x) p(t, x, n), \\ D(\mathcal{A}_n) &:= \{p_n \in X_x, \ \ \mathcal{A}_n p_n \in X_x\}, \quad n \in \mathbb{N}. \end{aligned} \tag{7.10}$$

Obviously, $(\mathbb{N}, d\varsigma)$ is likened to (V, dm), X_x is likened to X and A_v likened to $\mathcal{A}_n$ in Proposition 3.4.1, therefore, $(\mathcal{A}_n, D(\mathcal{A}_n))_{n \in \mathbb{N}}$ is a family of operators in X_x and using (3.16), we have

$$(\mathbf{A}\mathbf{p})_n := \mathcal{A}_n p_n. \tag{7.11}$$

Theorem 7.3.2. *There is an extension K of $\boldsymbol{A}+\boldsymbol{B}$ that generates a positive semigroup of contractions $(S_K(t))_{t \geq 0}$ on $\mathcal{X}_1$. Moreover, for each $\mathring{\mathbf{p}} = (\mathring{p}_n(x))_{n \in \mathbb{N}} \in D(K)$, there is a measurable representation $\boldsymbol{p}$ of $S_K(t)\mathring{\mathbf{p}}$ which is absolutely continuous with respect to $t \geq 0$ for almost any (x, n) and such that (7.7) is satisfied almost everywhere.*

Proof. To prove the first part of the theorem, it is necessary to show that for each $n \in \mathbb{N}$, $\mathcal{A}_n$ generates a positive semigroup of contractions. In fact, because the operator $\mathcal{A}_n$ is a multiplication operator on X_x induced by the measurable function a, it is closed and densely defined [38]. Also, for any $\lambda > 0$, it is obvious that $\lambda I - \mathcal{A}_n$ is bijective and the resolvent $R(\lambda, \mathcal{A}_n)$ of $\mathcal{A}_n$ satisfies the estimate

$$\|R(\lambda, \mathcal{A}_n)\psi\| \leq \frac{1}{\lambda}\|\psi\| \tag{7.12}$$

for any $\psi \in X_x$. Furthermore, for any positive λ, the operator $R(\lambda, \mathcal{A}_n)$ is non-negative. Therefore, $(\mathcal{A}_n, D(\mathcal{A}_n))$ generates a positive semigroup of contractions. Thus, by the relation (3.17), we claim that $(\mathbf{A}, D(\mathbf{A}))$ also generates a positive semigroup of contractions.

It is clear that $(\mathbf{B}, D(\mathbf{B}))$ is positive. Furthermore, for any $\mathbf{p} \in D(\mathbf{A})$, from the calculations in the Lemma (7.3.1), we have $\|\mathbf{Ag}\|_1 = \|\mathbf{Bg}\|_1$ and

$$\int_{\Lambda} (-\mathbf{Ap} + \mathbf{Bp})\, d\mu d\varsigma \tag{7.13}$$

$$= -\int_{\mathbb{R}^3} \sum_{m=1}^{\infty} m a_m(y)|g(y,m)|dy \tag{7.14}$$

$$+ \int_{\mathbb{R}^3} \left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y) b_{n,m}(y) h(x,n,m,y)|g(y,m)|dy \right) dx$$

$$= 0.$$

Thus, the assumptions of Theorem 2.3.5 are satisfied. Therefore, there is an extension K of $\mathbf{A}+\mathbf{B}$ generating a substochastic semigroup $(G_K(t))_{t\geq 0}$. Let K_n be the n^{th} component of K according to (7.11) and Proposition 3.4.1, then from (3.17), it suffices to prove the assertions of the theorem for each K_n, $n \in \mathbb{N}$. For any $\overset{o}{p}_n \in D(K_n)$, the function $t \to G_{K_n}(t)\overset{o}{p}_n$ is a C^1-function in the norm of X_x and satisfies the equation

$$\frac{d}{dt} G_{K_n}(t)\overset{o}{p}_n = K_n G_{K_n}(t)\overset{o}{p}_n, \tag{7.15}$$

where the equality holds for any $t > 0$ in the sense of equality in X_x. The initial condition is satisfied in the following sense

$$\lim_{t\to 0^+} G_{K_n}(t)\overset{o}{p}_n = \overset{o}{p}_n, \tag{7.16}$$

where the convergence is in the X_x-norm.

In order to prove the second part of this theorem we make use of the theory of extensions and the theory of L spaces [12]. Let Θ be the set of finite almost everywhere measurable functions defined on $\mathbb{R}^3$. Recall that Θ is a lattice with respect to the usual relation ($\leq$ almost everywhere), $X_x \subset \Theta$ and X_x is a sublattice of Θ. We denote by $(X_x)_+$ and Θ_+ the positive cones of X_x and Θ respectively. For each $n \in \mathbb{N}$, we introduce the operator D_n defined such that for any nondecreasing sequence $(\psi_k)_{k\in\mathbb{N}}$ in $(X_x)_+$ with $\sup_{k\in\mathbb{N}} \psi_k = \psi \in \Theta_+$,

$$D_n\psi := \sup_{k\in\mathbb{N}} \mathcal{B}_n \psi_k. \tag{7.17}$$

where $\mathcal{B}_n$ is given by $\mathfrak{B}\mathbf{p} = (\mathcal{B}_n)_{n=1}^{\infty}$ defined in 7.8. Since $\mathcal{B}_n$ is an integral operator with positive kernel, Lebesgue's monotone convergence theorem yields that $D_n = \mathcal{B}_n$. Thus, [12, Theorem 6.20] yields $K_n \subset \mathcal{A}_n + \mathcal{B}_n$. Hence, $G_{K_n}(t)\overset{o}{p}_n$ satisfies

$$\left[\frac{d}{dt} G_{K_n}(t)\overset{o}{p}_n\right](x,n) = [\mathcal{A}_n G_{K_n}(t)\overset{o}{p}_n](x,n) + [\mathcal{B}_n G_{K_n}(t)\overset{o}{p}_n](x,n), \tag{7.18}$$

for each fixed $t > 0$, where the right hand side does not depend (in the sense of equality almost everywhere) on what representation of the solution $G_{K_n}(t)\overset{o}{p}_n$ is considered. Making use of the fact that X_x is an L-space, from [46, Theorem 3.4.2], we have that since the function $G_{K_n}(t)\overset{o}{p}_n$ is strongly differentiable, there is a representation $p(t,x,n) = p_n$ of $G_{K_n}(t)\overset{o}{p}_n$ that is absolutely continuous with respect to $t \geq 0$ for almost any $(x,n) \in \mathbb{R}_+ \times \mathbb{R}^3$, and that satisfies $\frac{\partial}{\partial t}p(t,x,n) = \left[\frac{d}{dt}G_{K_n}(t)\overset{o}{p}_n\right](x,n)$ for any $t \geq 0$ and almost any (x,n). Hence, taking this representation, we obtain that

$$\begin{aligned} \frac{\partial p}{\partial t}(t,x,n) &= -a_n(x)p_n \\ &+ \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)dy \end{aligned} \tag{7.19}$$

holds almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^3$. Moreover, the continuity of p_n with respect to t for almost every (x,n) shows that $\lim_{t\to 0^+} p_n = \overline{p}(x,n)$ exists almost everywhere. From (7.16) we see that there is a sequence $(t_k)_{k\in\mathbb{N}}$ converging to 0 such that $\lim_{k\to\infty} p(t_k,x,n) = \overset{o}{p}_n(x,n)$, for almost every (x,n). Here, we can use the same representation as above because we are dealing with a (countable) sequence. Indeed, changing the representation on a set of measure zero for each n and further taking the union of all these sets still produces a set of measure zero. Thus, $\overset{o}{p}_n = \overline{p}_n$ almost everywhere. □

In general, for each $n \in \mathbb{N}$, the function $G_{K_n}(t)\overset{o}{p}_n$ is not differentiable if $\overset{o}{p}_n \in X_x \backslash D(K_n)$. Therefore, it cannot be a classical solution of the Cauchy problem (7.15), (7.16). However, it is a mild solution, that is, it is a continuous function such that $\int_0^t p_n(\tau)d\tau \in D(K_n)$ for any $t \geq 0$, satisfying the integrated version of (7.15), (7.16):

$$p_n(t) = \overset{o}{p}_n + K_n \int_0^t p_n(\tau)d\tau. \tag{7.20}$$

Corollary 7.3.3. *If $\overset{o}{p}_n \in X_x \backslash D(K_n)$, then $p_n = [G_{K_n}(t)\overset{o}{p}_n](x,n)$ satisfies the equation*

$$\begin{aligned} p(t,x,n) &= \overset{o}{p}_n(x,n) - a_n(x)\int_0^t p(\tau,x,n)d\tau \\ &+ \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)\left(\int_0^t p(\tau,y,n)d\tau\right)dy. \end{aligned} \tag{7.21}$$

Proof. Because p_n is continuous in the norm of $X_x = L_1(\mathbb{R}^3, dx)$, we can use the fact that X_x is of type L, see [12, Theorem 2.39], to claim that $aa_n(x)\int_0^t p(\tau,x,n)d\tau$ is defined for almost any (x,n) and any t, and hence, we can write

$$\begin{aligned} \left[(\mathcal{A}_n + \mathcal{B}_n)\int_0^t p(\tau)d\tau\right](x,n) &= -a_n(x)\int_0^t p(\tau,x,n)d\tau \\ + \sum_{m=n+1}^{\infty} \int_{\mathbb{R}^3} a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)&\left(\int_0^t p(\tau,y,n)d\tau\right)dy. \end{aligned} \tag{7.22}$$

Thus, combining the result used in the previous theorem, that is, $K_n \subset \mathcal{A}_n + \mathcal{B}_n$ with (7.20), we obtain (7.21) and the proof ends. □

Next, we provide a fairly general condition for honesty of $(G_{K_n}(t))_{t\geq 0}$.

7.4 Honesty

Because the total number of individuals in a population is not modified by interactions (fragmentation) among groups, the following conservation law is supposed to be satisfied throughout the evolution:

$$\frac{d}{dt}U(t) = 0 \tag{7.23}$$

where $U(t) = \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} np(t,x,n)dx = \sum_{n=1}^{\infty} n \int_{\mathbb{R}^3} p(t,x,n)dx$ is the total number of particles (total mass) in the system. This is formally expressed by (7.5) as the mass rate equation can be found by multiplying (7.5) by n, integrating over $\mathbb{R}^3$, summing from $n=1$ to ∞ and using (7.3), which agrees with the physics of the process as fragmentation should simply rearrange the distribution of masses of the particles without altering the total mass of the system. However, the validity of (7.23) depends on certain properties of the solution p that we tacitly assumed during the integration and which are far from obvious. In fact, by analysing models with specific coefficients, several authors have observed that the local version of (7.23) is not valid [90]. In other words, there occurs an unexpected mass loss in the system. In local fragmentation, the unaccounted for mass loss was termed *shattering fragmentation* (defined in Chapter 6) and was attributed to the phase transition in which a dust of particles with zero size and non-zero mass is formed. The presence of x in (7.23) suggests that honesty in non-local discrete fragmentation depends also on the spatial properties of the fragmentation kernels. In this section, we provide sufficient conditions for the discrete fragmentation semigroup to be honest for general coefficients.

Lemma 7.4.1. *Assume that for any $\boldsymbol{p} = (p_n)_{n=1}^{\infty} \in (\mathcal{X}_1)_+$ such that $-\boldsymbol{A}\boldsymbol{p} + \boldsymbol{B}\boldsymbol{p} \in \mathcal{X}_1$ we have the inequality*

$$\int_{\Lambda} (-\boldsymbol{A}\boldsymbol{p} + \boldsymbol{B}\boldsymbol{p})\, d\mu d\varsigma \geq 0, \tag{7.24}$$

then $K = \overline{\boldsymbol{A} + \boldsymbol{B}}$. Thus, the solution $(p_n)_{n=1}^{\infty} = \boldsymbol{p} = G_K(t)\mathring{\boldsymbol{p}} = (G_{K_n}(t)\mathring{p}_n)_{n=1}^{\infty}$ satisfies

$$\frac{d}{dt}\sum_{n=1}^{\infty}\int_{\mathbb{R}^3} G_{K_n}(t)\mathring{p}_n(x,n)ndx = \frac{d}{dt}\|G_{K_n}(t)\mathring{p}_n\| = 0$$

and for any $\mathring{\boldsymbol{p}}_n = (\mathring{p})_{n=1}^{\infty} \in D(K)_+$. In other words, the semigroup $(G_K(t))_{t\geq 0}$ is honest.

Proof. The method employed is analogous to that used in [12, Theorem 6.22]. Assume that for any $\mathbf{p} = (p_n)_{n=1}^{\infty} \in (\mathcal{X}_1)_+$ such that $-\mathbf{A}\mathbf{p} + \mathbf{B}\mathbf{p} \in \mathcal{X}_1$ the inequality (7.24) holds, then we have

$$-\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} na(y,n)g(y,n)dy$$
$$+\int_{\mathbb{R}^3}\left(\sum_{n=1}^{\infty} n \sum_{m=n+1}^{\infty}\int_{\mathbb{R}^3} a(y,m)b_{n,m}(y)h(x,n,m,y)g(y,m)dy\right)dx \geq 0.$$

According to Proposition 3.4.1 and [12, Theorem 6.13 and 6.22], it suffices to show, for each $n \in \mathbb{N}$, that any $f_n(x) = f(x,n) \in F_{n+}$ such that $-f_n + \mathcal{B}_n L f_n \in X_x$ the following inequality holds,

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^3} [\mathrm{L} f_n](x) n dx + \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} (-f_n(x) + [\mathcal{B}_n L f_n](x)) n dx \geq 0,$$

where $\mathrm{F}_n := \{\psi \in \Theta;\ (1 + a_n)^{-1}\psi \in X_x\}$, $L : (F_n)_+ \to X_x$ is defined such that $\mathrm{L} f_n := (1 + a_n)^{-1} f_n$ and $\mathcal{B}_n$ is given by $\mathfrak{B}\mathbf{p} = (\mathcal{B}_n p_n)_{n=1}^{\infty}$ defined in (7.8). Now, let $f_n \in F_{n+}$ such that $-f_n + B_n \mathrm{L} f_n \in X_x$, let us set $g_n := \mathrm{L} f_n$, it is clear that $g_n \in (X_x)_+$. Furthermore,

$$-a_n g_n + B_n g_n = -a_n \mathrm{L} f_n + B_n \mathrm{L} f_n = \mathrm{L} f_n + (-f_n + B_n \mathrm{L} f_n) \in X_x.$$

Since g_n satisfies the assumption then,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{\mathbb{R}^3} [L f_n](x) n dx + \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} (-f(x,n) + [B_n L f_n](x)) n dx \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} (g_n(x) - (1 + a_n(x)) g_n(x) + [B_n g_n](x)) n dx \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} (-a_n(x) g_n(x) + [B_n g_n](x)) n dx \geq 0.
\end{aligned}$$

The second part of the lemma follows from (7.23). □

Theorem 7.4.2. *Assume that the condition (7.1) is satisfied for almost all $(x,n) \in \mathbb{R}^3 \times \mathbb{N}$, that is $\theta_1 \alpha_n \leq a_n(x) \leq \theta_2 \alpha_n$, then the semigroup $(G_K(t))_{t \geq 0}$ is honest.*

Proof. Using the previous lemma, it is enough to prove that for any $\mathbf{p} = (p_n)_{n=1}^{\infty} \in (\mathcal{X}_1)_+$ such that $-\mathbf{Ap} + \mathbf{Bp} \in \mathcal{X}_1$, the inequality $\int_{\Lambda} (-\mathbf{Ap} + \mathbf{Bp})\, d\mu d\varsigma \geq 0$ is satisfied. Then,

$$\begin{aligned}
&\int_{\Lambda} (-\mathbf{Ap} + \mathbf{Bp})\, d\mu d\varsigma \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} ((-a(x,n) p_n(x,n) + [\mathcal{B}_n p_n](x)) n dx)\, dx \\
&= \lim_{N \to \infty} \left(\sum_{n=1}^{N} \int_{\mathbb{R}^3} -a(x,n) p_n(x) n dx + \sum_{n=1}^{N} \int_{\mathbb{R}^3} [\mathcal{B}_n p_n](x) n dx \right).
\end{aligned}$$

Also, according to (4.2),

$$\begin{aligned}
&\sum_{n=1}^{N}\int_{\mathbb{R}^3}[\mathcal{B}_n p_n](x)n dx\\
&= \sum_{n=1}^{N} n \sum_{m=n+1}^{\infty}\int_{\mathbb{R}^3} a(y,m)b_{n,m}(y)\left(\int_{\mathbb{R}^3} h(x,n,m,y)dx\right)p_m(y)dy\\
&= \int_{\mathbb{R}^3}\left(\sum_{n=1}^{N} n \sum_{m=n+1}^{\infty} a(y,m)b_{n,m}(y)p_m(y)\right)dy.
\end{aligned}$$

Furthermore, with reference to (7.3), we have for almost all $y \in \mathbb{R}^3$

$$\begin{aligned}
&\sum_{n=1}^{N} n \sum_{m=n+1}^{\infty} a(y,m)b_{n,m}(y)p_m(y)\\
&= W_N(y) + \sum_{m=1}^{N}\sum_{n=1}^{m-1} n a(y,m)b_{n,m}(y)p_m(y)\\
&= W_N(y) + \sum_{m=1}^{N} m a(y,m)p_m(y),
\end{aligned}$$

where

$$W_N(y) = \sum_{m=N+1}^{\infty}\sum_{n=1}^{N} n a(y,m)b_{n,m}(y)p_m(y) \geq 0.$$

Combining, for any $N > 0$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{N}\int_{\mathbb{R}^3}(-a(x,n)p_n(x) + [\mathcal{B}_n p_n](x))n dx\\
&= \sum_{n=1}^{N}\int_{\mathbb{R}^3}(-[ap](x,n)n dx + \int_{\mathbb{R}^3}\left(W_N(y) + \sum_{m=1}^{N} m[ap](y,m)\right)dy\\
&= \int_{\mathbb{R}^3} W_N(y)dy \geq 0.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}^3}(-a(x,n)p_n(x) + [\mathcal{B}_n p_n](x))n dx = \lim_{N\to\infty}\int_{\mathbb{R}^3} W_N(y)dy \geq 0,$$

and the theorem is proven. □

7.5 Honesty in discrete, non-local and randomly position structured fragmentation model with unbounded rates

As stated in the introduction, the conservation of total mass is not always satisfied in the system. In fact, by analysing models with specific coefficients, several authors have observed that the local version of the conservation law (7.23) is not valid [90]. That was attributed to the phase transition in which a dust of particles with zero size and non-zero mass is formed (*shattering*). The previous theorem shows that when each discrete fragmentation rate a_n is bounded by a size-only dependent function, the spatial and random distribution of the particles has no influence on the conservativeness of the system. In other words, non-local discrete models with each $a_n(x)$ bounded as $|x|$ approaches infinity always behave like local models, therefore, are conservative provided that the fragmentation rate a_n is bounded as n approaches zero. However, there is a major complication [12] that arises when, in the discrete case, each fragmentation rate $a_n(x)$ becomes infinite as $|x|$ is close to infinity. The next theorem gives sufficient conditions for conservativeness in that case.

Theorem 7.5.1. *Assume that for each $n \in \mathbb{N}$ we have*

$$a_n \in L_{\infty,loc}(\mathbb{R}^3) \tag{7.25}$$

and there exists $K > 0$ such that

$$a_m(y) \int_{|x|>|y|} h(x,n,m,y)dx < K. \tag{7.26}$$

is satisfied for almost all $(x,m) \in \mathbb{R}^3 \times \mathbb{N}$, then the semigroup $(G_K(t))_{t\geq 0}$ is honest.

Proof. The proof is based on [12, Theorem 6.13]. Let $\mathbf{p} = (p_n)_{n=1}^{\infty} \in (\mathcal{X}_1)_+$, by (7.25), for any $0 < N_1 < \infty$ we have that $a_n p_n \in L_1(B(O,N_1), ndx)$, where $B(O,N_1)$ represents the ball $\{x \in \mathbb{R}^3; |x| \leq N_1\}$. Because $-\mathbf{Ap} + \mathbf{Bp} \in \mathcal{X}_1$, we also have $\mathcal{B}_n p_n \in L_1(B(O,N_1), ndx)$. So, making use of Lemma ??, it is enough to prove that the inequality $\int\limits_{\Lambda} (-\mathbf{Ap} + \mathbf{Bp})\, d\mu d\varsigma \geq 0$ is satisfied. Then,

$$\int\limits_{\Lambda} (-\mathbf{Ap} + \mathbf{Bp})\, d\mu d\varsigma = \sum_{n=1}^{\infty} \int_{\mathbb{R}^3} (-a(x,n)p_n(x) + [\mathcal{B}_n p_n](x))\, ndx$$

$$= \lim_{N,N_1\to\infty} \left(\sum_{n=1}^{N} \int_{B(O,N_1)} -a(x,n)p_n(x)ndx + \sum_{n=1}^{N} \int_{B(O,N_1)} [\mathcal{B}_n p_n](x)ndx \right).$$

We have

$$\begin{aligned}&\sum_{n=1}^{N}\int_{B(O,N_1)}[\mathcal{B}_n p_n](x)ndx\\&=\sum_{n=1}^{N}\int_{B(O,N_1)}\left(\sum_{m=n+1}^{\infty}\int_{\mathbb{R}^3}a_m(y)b_{n,m}(y)\left(h(x,n,m,y)\right)p_m(y)dy\right)ndx\\&=Q(N,N_1)+\sum_{m=1}^{N}\int_{\mathbb{R}^3}\sum_{n=1}^{m-1}\int_{B(O,N_1)}a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)ndxdy,\end{aligned}$$

where

$$Q(N,N_1)=\sum_{m=N+1}^{\infty}\int_{\mathbb{R}^3}\sum_{n=1}^{N}\int_{B(O,N_1)}a_m(y)b_{n,m}(y)h(x,n,m,y)p_m(y)ndxdy\geq 0,$$

with h defined by (4.2). It follows that

$$\begin{aligned}&\sum_{n=1}^{N}\int_{B(O,N_1)}[\mathcal{B}_n p_n](x)ndx\\&\geq\sum_{m=1}^{N}\int_{\mathbb{R}^3}a_m(y)p_m(y)\left(\sum_{n=1}^{m-1}\int_{B(O,N_1)}b_{n,m}(y)h(x,n,m,y)ndx\right)dy\\&\geq\sum_{m=1}^{N}\int_{B(O,N_1)}a_m(y)p_m(y)\left(\sum_{n=1}^{m-1}\int_{B(O,N_1)}b_{n,m}(y)h(x,n,m,y)ndx\right)dy.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{N}\int_{B(O,N_1)}[\mathcal{B}_n p_n](x)ndx&\geq\sum_{m=1}^{N}\int_{B(O,N_1)}a_m(y)p_m(y)mdy\\&-\sum_{m=1}^{N}\int_{B(O,N_1)}a_m(y)p_m(y)\left(\sum_{n=1}^{m-1}\int_{|x|>N_1}b_{n,m}(y)h(x,n,m,y)ndx\right)dy.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{n=1}^{N}\int_{B(O,N_1)}(-a(x,n)p_n(x))ndx+\sum_{n=1}^{N}\int_{B(O,N_1)}[\mathcal{B}_n p_n](x)ndx\\&\geq-\sum_{m=1}^{N}\int_{B(O,N_1)}a_m(y)p_m(y)\left(\sum_{n=1}^{m-1}\int_{|x|>N_1}b_{n,m}(y)h(x,n,m,y)ndx\right)dy.\end{aligned}$$

Following the assumption (7.26), for any $y\in B(O,N_1)$, we have

$$a_m(y)\int_{|x|>N_1}h(x,n,m,y)dx\leq a_m(y)\int_{|x|>|y|}h(x,n,m,y)dx<K.$$

Using (7.3), this implies that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\int_{B(O,N_1)} a_m(y)p_m(y)\left(\sum_{n=1}^{m-1}\int_{|x|>N_1} b_{n,m}(y)h(x,n,m,y)ndx\right)dy \\
&\leq K\sum_{m=1}^{\infty}\int_{\mathbb{R}^3} p_m(y)\left(\sum_{n=1}^{m-1} nb_{n,m}(y)\right)dy \\
&\leq K\sum_{m=1}^{\infty}\int_{\mathbb{R}^3} mp_m(y)dy < \infty.
\end{aligned}$$

According to the dominated convergence theorem [21] and using (4.2),

$$\begin{aligned}
&\lim_{N,N_1\to\infty}\sum_{m=1}^{N}\int_{B(O,N_1)} a_m(y)p_m(y)\left(\sum_{n=1}^{m-1}\int_{|x|>N_1} b_{n,m}(y)h(x,n,m,y)ndx\right)dy \\
&=\sum_{m=1}^{\infty}\int_{\mathbb{R}^3}\sum_{n=1}^{m-1} na_m(y)p_m(y)b_{n,m}(y)\left(1-\lim_{N_1\to\infty}\int_{B(O,N_1)} h(x,n,m,y)dx\right)dy \\
&=0.
\end{aligned}$$

Therefore,

$$\sum_{m=1}^{\infty}\int_{\mathbb{R}^3}(-a(x,n)p_n(x)+[\mathcal{B}_n p_n](x))ndx \geq 0,$$

which concludes the proof. □

7.6 Concluding remarks and discussion

The processes of fragmentation with rates both finite and infinite at infinity have been investigated by means of the theory of substochastic semigroups with a parameter and parameter-dependent operators. We succeeded to combine a discrete model with a non-local multiple fragmentation process with fragmentation rate depending on size and position and where new particles are spatially randomly distributed according a given probabilistic law. We used Kato's Theorem and the dominated convergence theorem to get the main results here, that are conditions (7.25) and (7.26) which guarantee existence and conservativeness for the non-local discrete model described above and where each fragmentation rate $a_n(x)$ becomes infinite as $|x|$ is close to infinity. The physical interpretation is that the process is conservative if at infinity, daughter particles tend to move back into the system with a high probability described by (7.26).

Chapter 8

General Conclusion

As we discussed earlier in this study, fragmentation (or coagulation) and transport processes combined in the same model are still barely touched in the domain of mathematical and abstract analysis. The great achievement in this research was to explore less known aspects characterising the multiple combination of mathematical models that arise in fragmentation-coagulation-transport (non-local or non-autonomous) theory. Various techniques and methods were exploited throughout. In chapter 3, the theory of strongly continuous semigroups of operators was used to analyse the well-posedness of an integro-differential equation modelling convection-fragmentation processes, and in chapter 5, a similar analysis was repeated, but this time for non-linear convection-coagulation models. We showed that the combined fragmentation (or coagulation)-transportation operator is the infinitesimal generator of a strongly continuous stochastic semigroup, thereby addressing the problem of existence of solutions for these models. Moreover, we established the particularities of conservativeness for the fragmentation process and uniqueness for the coagulation process. Non-local and non-autonomous fragmentation processes with transport dynamics were analysed, by using an equivalent norm approach on one side, approximation and truncation techniques on other side, to obtain a similar result of existence. These results are a prowess in the sense that, they generalize previous studies with the inclusion of the spatial transportation kernel which was not considered before.

Another great feat of the present study was to address more concrete and applicable problems like 'Shattering' or the effects of ocean iron fertilisation on the evolution of the phytoplankton biomass. The former is known as the formation of a 'dust' of particles of zero size carrying, nevertheless, a non-zero mass. We succeeded to derive an analytic expression for the resolvent of the fragmentation operator, hereby extending the work by Banasiak *et al.* [14], where a similar problem was solved for fragmentation with separable kernels) and which was an open problem in pure fragmentation theory. Therefore, the resolvent can be used to derive the spectral properties of the fragmentation operator, which fully explain the phenomenon of shattering. For the second problem concerning the analysis of effects of ocean iron fertilisation on the evolution of the phytoplankton biomass, mathematical and computational techniques were used to present a very efficient method to increase the world phytoplankton biomass. We clearly showed, analytically and with simulations that, raising the level of marine iron too much above ζ_c does not present any technical advantage in the long run. Hence, we can conclude that iron hypothesis can be implemented in a very cost-effective way and produce impressive results. This result is also a prowess since the method can be recommended in future use in order to systematically reduce the effects of global warming.

Finally we addressed the major problem that arises the process of discrete and non-local aggregation when each fragmentation rate becomes infinite at infinity. After giving sufficient conditions for honesty, we demonstrated that even in discrete and non-local case, the process is conservative if at infinity daughter particles tend to go back into the system with a high probability.

Although some progress have been made in the use of (sub)stochastic semigroups techniques to analyse and better understand the evolution of (non-local or non-autonomous) fragmentation (or coagulation) dynamics in moving media, there are still many areas in which further investigation could prove fruitful. For instance, the full identification of the generator and characterisation of its domain for the integro-differential equation modelling convection-fragmentation processes. This may help analyse in the same way a model with combined coagulation-fragmentation-transport-direction changing whose the full identification of the generator and characterisation of the domain is still ongoing. It would also be interesting to provide, in the context of non-local and non-autonomous fragmentation process, a reasonable physical interpretation of the phenomenon of 'shattering'. The future will guide us.

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