OPERATORS DEFINED BY CONDITIONAL EXPECTATIONS AND RANDOM MEASURES

Daniel Thanyani Rambane, M.Sc.

Thesis submitted for the degree Doctor of Philosophy in Mathematics at the North-West University.

Promoter: Prof. J.J. Grobler

June 2004
Potchefstroom Campus
Abstract

This study revolves around operators defined by conditional expectations and operators generated by random measures.

Studies of operators in function spaces defined by conditional expectations first appeared in the mid 1950’s by S-T.C. Moy [22] and S. Sidak [26]. N. Kalton studied them in the setting of $L_p$-spaces $0 < p < 1$ in [15, 13] and in $L_1$-spaces, [14], while W. Arveson [5] studied them in $L_2$-spaces. Their averaging properties were studied by P.G. Dodds and C.B. Huijsmans and B. de Pagter in [7] and C.B. Huijsmans and B. de Pagter in [10]. A. Lambert [17] studied their relationship with multiplication operators in $C^*$-modules. It was shown by J.J. Grobler and B. de Pagter [8] that partial integral operators that were studied A.S. Kalitvin et al in [2, 4, 3, 11, 12] and the special cases of kernel operators that were, inter alia, studied by A.R. Schep in [25] were special cases of conditional expectation operators.

On the other hand, operators generated by random measures or pseudo-integral operators were studied by A. Sourour [28, 27] and L.W. Weis [29, 30], building on the studies of W. Arveson [5] and N. Kalton [14, 15], in the late 1970’s and early 1980’s.
In this thesis we extend the work of J.J. Grobler and B. de Pagter [8] on Multiplication Conditional Expectation-representable (MCE-representable) operators. We also generalize the result of A. Sourour [27] and show that order continuous linear maps between ideals of almost everywhere finite measurable functions on $\sigma$-finite measure spaces are MCE-representable. This fact enables us to easily deduce that sums and compositions of MCE-representable operators are again MCE-representable operators. We also show that operators generated by random measures are MCE-representable.

The first chapter gathers the definitions and introduces notions and concepts that are used throughout. In particular, we introduce Riesz spaces and operators therein, Riesz and Boolean homomorphisms, conditional expectation operators, kernel and absolute $\tau$-kernel operators.

In Chapter 2 we look at MCE-operators where we give a definition different from that given by J.J. Grobler and B. de Pagter in [8], but which we show to be equivalent.

Chapter 3 involves random measures and operators generated by random measures. We solve the problem (positively) that was posed by A. Sourour in [28] about the relationship of the lattice properties of operators generated by random measures and the lattice properties of their generating random measures. We show that the total variation of a random signed measure representing an order bounded operator $T$, it being the difference of two random measures, is again a random measure and represents $|T|$.

We also show that the set of all operators generated by a random measure
is a band in the Riesz space of all order bounded operators.

In Chapter 4 we investigate the relationship between operators generated by random measures and MCE-representable operators. It was shown by A. Sourour in [28, 27] that every order bounded order continuous linear operator acting between ideals of almost everywhere measurable functions is generated by a random measure, provided that the measure spaces involved are standard measure spaces. We prove an analogue of this theorem for the general case where the underlying measure spaces are $\sigma$-finite. We also, in this general setting, prove that every order continuous linear operator is MCE-representable. This rather surprising result enables us to easily show that sums, products and compositions of MCE-representable operator are again MCE-representable.

**Key words:** Riesz spaces, conditional expectations, multiplication conditional expectation-representable operators, random measures.
Opsomming

In hierdie studie word gekyk na operatore gedefinieer deur voorwaardelike verwagtings en operatore voortgebring deur stogastiese mate.


Andersyds is operatore voortgebring deur stogastiese mate (ook genoem pseudo-integraal operatore) deur A. Sourour [28, 27] en L.W. Weis [29, vi}
30], bestudeer voortbouend op studies onderneem deur W. Arveson [5] en N. Kalton, [14, 15], in die laat sewentiger en vroeg tagtiger jare.

In hierdie proefskrif brei ons die werk wat J.J. Grobler en B. de Pagter [8] oor vermenigvuldigings voorwaardelike verwagtings representeerbare (vvv-representeerbare) operatore gedoen het uit. Ons veralgemeen ook 'n resul- taat van Sourour, [27], en toon aan dat orde kontinue lineêre afbeeldings tussen ideale van byna-oral eindige meetbare funksies op σ-eindige maat- ruimtes vvv-representeerbaar is. Hierdie feit stel ons in staat om maklik te sien dat somme en samestellings van vvv-representeerbare operatore weer vvv-representeerbaar is. Ons toon ook aan dat operatore voortgebring deur stogastiese mate, vvv-representeerbaar is.

Die eerste hoofstuk dien as inleiding waarin ons die definisies en begrippe wat verder gebruik word saamvat. In die besonder gee ons aandag aan die teorie van Rieszruimtes en die operatore wat daarin 'n rol speel, naamlik Riesz- en Boole-homomorfismes, voorwaardelike verwagtingsoperatore, kern-operatore and absolute τ-kern operatore.

In Hoofstuk 2 bestudeer ons vvv-representeerbare operatore en ons gee 'n definisie daarvan wat verskil van die een wat deur Grobler en de Pagter in [8] gebruik word. Ons toon egter aan dat die twee definisies ekwivalent is.

Hoofstuk 3 bevat die teorie van stogastiese mate en die operatore voortgebring deur stogastiese mate. Ons los 'n probleem gestel deur Sourour in [28] positief op deur aan te toon dat die totale variasie van 'n betekende stogastiese maat wat 'n orde begrensde operator T voortbring en wat die
verskil van twee stogastiese mate is, weer 'n stogastiese maat is en dat dit die operator \( |T| \) voortbring.

Ons toon ook aan dat die versameling van alle operatore voortgebring deur 'n stogastiese maat 'n band is in die Rieszruimte van alle orde begrensde operatore.

In Hoofstuk 4 ondersoek ons die verband tussen operatore voortgebring deur stogastiese mate en vvv-representeerbare operatore. Sourour het in [28, 27] bewys dat elke orde begrensde ordekontinue lineêre operator wat ideale van byna-oral eindige meetbare funksies afbeeld in ideale van soortgelike funksies deur stogastiese mate voorgebring word mits die onderliggende maatriumtes standaard maatriumtes is. Ons bewys 'n analoog van hierdie stelling vir die algemene geval waar die onderliggende maatriumtes \( \sigma \)-eindig is. In hierdie algemene geval toon ons ook aan dat elke ordekontinue lineêre operator vvv-representeerbaar is. Hierdie verrassende resultaat stel ons in staat om sonder moeite te wys dat somme, produkte en samestellings van vvv-representeerbare operatore weer vvv-representeerbaar is.

**Sleuteltermes:** Rieszruimtes, voorwaardelike verwagtings, vermenigvuldigings voorwaardelike verwagtings representeerbare-operatore, stogastiese mate.
Acknowledgment

I would like to express my heartfelt thanks to the following people:

* My supervisor Prof. J.J. Grobler, for his professional way of supervising my studies. From him I learnt more than the mathematics in this thesis.

* The staff in the Department of Mathematics at North-West University (Potchefstroom). They were always there to lend a hand when I was stuck and for their financial assistance.

* The Opirif gang, the 19 Meyer gang, the Malherbe gang. Amongst whom Mr. Moeketsi D., Mr. Ndiitwani C., Mr. Kgoro C., Mr. Mojake S., Dr. Modise S.J., Ms. Mthembu K., Ms. Ditlopo N., Ms. Mtshatshe N., immediately come to mind. Those not mentioned are by no means less important.

* Members of the staff of the Department of Mathematics and Applied Mathematics at University of Venda.

* My personal friends who pushed me on: Mr. Tshivhandekano T.R., Mr. Siphugu M.V. and Mr. Maiwashe K.C., at times we suffered together.
* My brother, January; his wife Sara and the kids. Their house used to be my home whilst I was busy with this.

* My wife, Takalani; my son, Thendo and my daughter Tendani: this one is for you all as members of my family.

* The St. Francis AME Church in Ikageng, Potchefstroom and members of the Charles Rathogwa Memorial AME Church, Vuwani Circuit, for lending me a spiritual helping hand.

* GOD Almighy, for His wisdom and strength, part of which He gave me to enable me to accomplish this.
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Chapter 1

Preliminaries

1.1 Introduction

Operators in function spaces defined by conditional expectations were first studied by, among others, S - T.C. Moy [22], Z. Sidak [26] and H.D. Brunk in the setting of $L^p$ spaces. Conditional expectation operators on various function spaces exhibit a number of remarkable properties related to the underlying structure of the given function space or to the metric structure when the function space is equipped with the norm. P.G. Dodds, C.B. Huijsmans and B. de Pagter [7] linked these operators to averaging operators defined on abstract spaces earlier by J.L. Kelley [16], while A. Lambert [17] studied their link to classes of multiplication operators which form Hilbert $C^*$-modules. J.J. Grobler and B. de Pagter [8] showed that the classes of partial integral operators which were studied by A.S. Kalitvin and others (see [12], [2], [3] and [4]) were a special case of conditional expectation operators.

In this thesis we are going to discuss the notion of multiplication conditional expectation operators and extend the work by J.J. Grobler and B. de
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Pagter [8] to operators that can be represented by multiplication conditional expectation operators.

We also investigate results obtained by, among others, Sourour [28], [27] and Weis [29] and [30]. They worked on operators in ideals of almost everywhere finite measurable functions on standard measures spaces and showed that these can be generated by random measures. We generalize their results and show that order continuous linear maps between ideals of almost everywhere finite measurable functions on $\sigma$-finite measure spaces are Multiplication Conditional Expectation-representable (MCE-representable).

The first chapter gives the preliminaries and the background where definitions, basic concepts and notions that are needed in the sequel are stated. Chapter 2 focuses on characterizing conditional expectation operators as order continuous functions. It also identifies those operators that can be represented by conditional expectation operators. Chapter 3 looks at pseudo-integral operators or those operators that can be generated by random measures. In this chapter we also look at operators that are random measure representable. In Chapter 4 we investigate the relationship between random measure-representable operators and MCE-representable operators. Here we present our main result.

1.2 Riesz spaces

In this section we give a short introduction to the theory of Riesz spaces. We will concentrate only on definitions, notions and results that will be useful
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later. For deeper results we refer to W.A.J. Luxemburg and A.C. Zaanen [19], H.H. Schaefer [24], P. Meyer-Nieberg [21] and A.C. Zaanen [34].

Definition 1.2.1 A binary relation that is reflexive, anti-symmetric and transitive is called a partial order. We will denote a partial order by \( \leq \). A set in which a partial order has been defined is called a partially ordered set. A partially ordered set \( X \) with a partial order \( \leq \) will be denoted by \( (X, \leq) \).

If, however, the partial order \( \leq \) is obvious from the context, or if there is no fear of confusion, we will write \( X \) for a partially ordered set \( (X, \leq) \).

For elements \( x \) and \( y \) in a partially ordered set \( X \) we will sometimes write \( y \geq x \) if \( x \leq y \) and \( x < y \) to express the fact that \( x \leq y \) and \( x \neq y \), similarly for \( y > x \).

Definition 1.2.2 Let \( X \) be a partially ordered set. A set \( Y \subseteq X \) is said to be bounded from above if there is an element \( x \in X \) such that \( y \leq x \) for all \( y \in Y \). The element \( x \) is called an upper bound of \( Y \). An upper bound \( u \) of \( Y \) is called the least upper bound or supremum of \( Y \) if \( u \leq x \) for every upper bound \( x \) of \( Y \). The supremum of \( Y \) will be denoted by \( \sup Y \). The notions of bounded from below, lower bound and the greatest lower bound or infimum of the set \( Y \subseteq X \) are defined similarly with the inequalities reversed. The infimum of \( Y \) will be denoted by \( \inf Y \).

Definition 1.2.3 Let \( X \) be a partially ordered set. For \( x, y \in X \) with \( x \leq y \), the order interval \( [x, y] \) is defined as the subset

\[
[x, y] = \{ z \in X : x \leq z \leq y \}.
\]
A subset $A \subset X$ is said to be order bounded if $A$ is contained in an order interval.

From the definition of boundedness it is clear that every order bounded set is bounded from above and from below.

**Definition 1.2.4** A partially ordered set $X$ is called a lattice whenever $\sup \{x, y\} = x \vee y$ and $\inf \{x, y\} = x \wedge y$ exist for all $x, y \in X$.

**Definition 1.2.5** A real vector space $E$ which is partially ordered is called an ordered vector space if

\[ x \leq y \text{ implies that } x + y \leq y + z \text{ for all } x, y, z \in E \]

\[ x \leq y \text{ implies that } \lambda x \leq \lambda y \text{ for all } x, y \in E \text{ and } 0 < \lambda \in \mathbb{R}. \]

An ordered vector space is called a Riesz space or vector lattice if it is a lattice.

We list some examples of Riesz spaces which we will use further on. For more details on these we refer the reader to [8, 34].

**Example 1.2.6** (1) Let $(\Omega, \Sigma, \mu)$ be a measure space and consider $L^0 = L^0(\Omega, \Sigma, \mu)$, the set of all real $\mu$-a.e. finite measurable functions on $\Omega$. We identify functions which differ only on a set of measure zero, i.e., elements of $L^0$ are equivalence classes of functions, two functions being in the same equivalence class if and only if they differ only on a set of measure zero. If we define for $f, g \in L^0$, $f \leq g$ if $f(x) \leq g(x)$ $\mu$-a.e. on $\Omega$ then $L^0$ is a Riesz space.
For $1 \leq p < \infty$ define $L^p = L^p (\Omega, \Sigma, \mu)$ as the subset of $L^0$ consisting of all $f \in L^0$ such that $\int_\Omega |f|^p d\mu < \infty$. If we define the order in $L^p$ as that defined for $L^0$ then $L^p$ is a Riesz space.

Define the set of all essentially bounded $f \in L^0$ by $L^\infty = L^\infty (\Omega, \Sigma, \mu)$; here $f \in L^0$ is said to be essentially bounded if there exists a non-negative finite number $M$ such that $|f(x)| < M$ for $\mu$-almost every $x \in \Omega$. In other words $f \in L^\infty$ if $f \in L^0$ and $\text{ess sup} |f(x)| < \infty$. Again, if we define the order in $L^\infty$ as that in $L^0$ then $L^\infty$ is a Riesz space.

(2) Let $X$ be a compact Hausdorff space and $C(X)$ be a vector space of all real continuous functions on $X$. Define the order $\leq$ in $C(X)$ by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, this makes $C(X)$ a Riesz space.

(3) Let $\mathcal{F}$ be an algebra ($\sigma$-algebra) of subsets of a non-empty set $X$. Let $\mu$ be a bounded finitely additive signed measure on $\mathcal{F}$, i.e., for $A, B \in \mathcal{F}$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A$ is disjoint with $B$, $|\mu(A)| \leq K$ for some constant $K > 0$ and

$$||\mu|| = \sup\{|\mu(A)| : A \in \mathcal{F}\} < \infty.$$ 

Put $\mathcal{E}$ to be the set of all bounded finitely additive signed measures on $\mathcal{F}$, if we define the order on $\mathcal{E}$ by saying, for $\mu, \nu \in \mathcal{E}$, that $\mu \leq \nu$ whenever $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{F}$ then $\mathcal{E}$ is a Riesz space.
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Here for $\mu_1$ and $\mu_2$ in $\mathcal{E}$ we have that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are given by

$$\mu_1 \vee \mu_2(A) = \inf_{A \supseteq B} \{\mu_1(B) + \mu_2(A - B)\}$$
$$\mu_1 \wedge \mu_2(A) = \sup_{A \supseteq B} \{\mu_1(B) + \mu_2(A - B)\}$$

where $A, B \in \mathcal{F}$, see [19, 24, 34].

**Definition 1.2.7** The subset $E^+ = \{x \in E : x \geq 0\}$ is called the positive cone of the ordered vector space $E$, $x \in E$ is said to be a positive element of $E$.

From this definition, it follows that the positive cone $E^+$ exhibits the following properties:

- If $x, y \in E^+$ then $x + y \in E^+$.
- If $x \in E^+$ and $0 \leq \lambda \in \mathbb{R}$ then $\lambda x \in E^+$.
- If $x \in E^+$ and $-x \in E^+$ then $x = 0$.

**Definition 1.2.8** Let $E$ be a Riesz space. For every $x \in E$ we define the positive part of $x$ by $x^+ = x \vee 0$, the negative part of $x$ by $x^- = (-x) \vee 0$ and the absolute value of $x$ by $|x| = x \vee (-x)$.

It is immediately clear that $x^+$ and $x^-$ are in $E^+$ and that $|x| = |(-x)|$. Also $(-x)^- = -(-x) \vee 0 = x \vee 0 = x^+$ and $(-x)^+ = (-x) \vee 0 = x^-$. The proof of the following properties can be found in Meyer-Nieberg [21].

**Proposition 1.2.9** Let $E$ be a Riesz space. Then the following hold for every $x \in E$
(i) \( x = x^+ - x^- , x^+ \land x^- = 0 \).

(ii) \( 0 \leq x^+ \leq |x| \).

(iii) \( -x^- \leq x \leq x^+ \).

(iv) \((\lambda x)^+ = \lambda x^+ \) and \((\lambda x)^- = \lambda x^- \) for \( \lambda \geq 0 \).

\( (\lambda x)^+ = -\lambda x^- \) and \((\lambda x)^- = -\lambda x^+ \) for \( \lambda \geq 0 \).

\(|\lambda x| = |\lambda||x| \) for \( \lambda \in \mathbb{R} \).

(v) \( x \leq y \) if and only if \( x^+ \leq y^+ \) and \( x^- \geq y^- \).

Consider the space \( E \) in Example 1.2.6 (3). We have that \( \mu^+ = \mu \lor 0 \), \( \mu^- = (-\mu) \lor 0 \) and \(|\mu| = (-\mu) \lor \mu \). Also \(|\mu| = \mu^+ + \mu^- \).

The above properties can be used to prove the following properties:

**Proposition 1.2.10** If \( E \) is a Riesz space then

\[ x \lor y = \frac{1}{2} (x + y + |x - y|) \quad \text{and} \quad x \land y = \frac{1}{2} (x + y - |x - y|) \]

and

\[ |x| \lor |y| = \frac{1}{2} (|x + y| + |x - y|) \quad \text{and} \quad |x| \land |y| = \frac{1}{2} (|x + y| - |x - y|) \].

We state a property of Riesz spaces whose importance is apparent when looking at the space of linear functionals on a Riesz space. It is known as the Riesz Decomposition Property, its proof can be found in Zaanen [32].

**Proposition 1.2.11** Let \( E \) be a Riesz space and let \( x, y \in E^+ \)
(i) If $0 \leq z \leq x + y$ there exist elements $0 \leq u \leq x$ and $0 \leq v \leq y$ such that $z = u + v$.

(ii) For all $x, y \in E^+$ we have that $[0, x + y] = [0, x] + [0, y]$.

**Definition 1.2.12** If $E$ is a Riesz space, $x$ and $y$ in $E$ are said to be *disjoint* whenever $|x| \land |y| = 0$ and we write $x \perp y$.

Two subsets $A$ and $B$ in $E$ are called *disjoint* whenever $a \perp b$ for every $a \in A$ and $b \in B$.

The set $A^d = \{x \in E : x \perp a \text{ for all } a \in A\}$ is called the *disjoint complement* of $A \subset E$.

A subset $S \subset E$ is called a *disjoint system* whenever $0 \in S$ and $x \perp y$ for every $x, y \in S$.

We present the following result, the proof of which can be found in Meyer-Nieberg [21].

**Proposition 1.2.13** If $x, y$ are elements of a Riesz space $E$ then the following are equivalent

(a) $x \perp y$.

(b) $|x| \lor |y| = |x + y| = |x - y|$.

(c) $|x| \lor |y| = |x| + |y|$.

The following definitions and notions are adapted from the book of Zaanen [34].
Let $\Delta$ be a non-empty set and $E$ be a Riesz space. Assume that for each $\alpha \in \Delta$ there exists an element $f_\alpha$ in $E$, i.e., there is some mapping $\alpha \mapsto f_\alpha$ from $\Delta$ into $E$. Put $A = \{f_\alpha : \alpha \in \Delta\}$. Here $\Delta$ is an indexing set for $A$.

**Definition 1.2.14** With the notation above, the set $A$ is said to be *upwards directed* if for any $\alpha$ and $\beta$ in $\Delta$ there exists a $\delta$ in $\Delta$ such that $f_\delta \geq f_\alpha \vee f_\beta$ and it is said to be *downwards directed* if there is a $\delta$ in $\Delta$ such that $f_\delta \leq f_\alpha \wedge f_\beta$.

We denote an upwards directed set $A$ by $A \uparrow$ and if $A$ is downwards directed we denote this by $A \downarrow$. If $A \uparrow$ and $\sup A = a$ we write $A \uparrow a$. Similarly, we write $A \downarrow a$ if $A \downarrow$ and $\inf A = a$.

We give, as a definition, a more specific case of a directed set.

**Definition 1.2.15** A sequence $(x_n)$ in a Riesz space $E$ is said to be *increasing* if $x_1 \leq x_2 \leq \cdots$ and it is said to be *decreasing* if $x_1 \geq x_2 \geq \cdots$. We denote an increasing sequence $(x_n)$ by $x_n \uparrow$ and a decreasing sequence $(x_n)$ by $x_n \downarrow$. If $x_n \uparrow$ and $x = \sup x_n$ exists in $E$ we write $x_n \uparrow x$ and $x_n \downarrow x$ whenever $x_n \downarrow$ and $x = \inf x_n$ exists in $E$. In the latter case we sometimes say that $(x_n)$ *converges monotonely* to $x$ (as $n$ tends to infinity).

We introduce a more general notion of convergence, which is convergence associated with the order structure of $E$. This kind of convergence is known as *order convergence*.

**Definition 1.2.16** A sequence $(x_n)$ in a Riesz space $E$ is said to *converge in order* to $x$ if there exists a sequence $(y_n) \downarrow 0$ such that $|x_n - x| \leq y_n$ for all
We shall denote that \((x_n)\) converges in order to \(x\) (or is *order convergent to* \(x\)) by \(x_n \to (o)x\) or by \(x_n \to x\) in order. In this case \(x\) is called the *order limit* of the sequence \((x_n)\).

We give a remark from [34] that the notion of a downwards directed sequence and that of a decreasing sequence are different. For instance, in the Riesz space \(\mathbb{R}\) consider the sequence

\[
(x_n) = \begin{cases} 
\frac{1}{n} & \text{if } n = 1, 3, 5, \ldots \\
\frac{1}{1 + \frac{1}{n}} & \text{if } n = 2, 4, 6, \ldots
\end{cases}
\]

We have that \((x_n)\) is not monotone but \(x_n \downarrow\), in fact \(x_n \downarrow 0\).

**Example 1.2.17** Let \((X, \Sigma, \mu)\) be a measure space and let \(E = M(X, \mu)\), the space of all \(\mu\)-a.e. finite functions on \(X\). Then a sequence \(f_n \downarrow 0\) if and only if \(f_n(t) \downarrow 0\) for almost every \(t \in X\). Therefore, \(f_n \to f\) in order if and only if there exists a positive \(g \in E\) such that \(|f_n| \leq g\) and \((f_n(t))\) converges to \(f(t)\) for almost every \(t \in X\).

We also put in a few non-topological properties of vector lattices. In particular we look at some subsets of Riesz spaces that will be encountered as we progress.

**Definition 1.2.18** Let \(E\) be a Riesz space

(i) \(E\) is called *Archimedean* if, for all \(x, y \in E\), it follows from \(nx \leq y\) for all \(n \in \mathbb{N}\) that \(x \leq 0\).

(ii) \(E\) is said to be *laterally complete* if every set of pairwise disjoint elements has a supremum.
(iii) $E$ is said to be Dedekind complete or order complete if every non-empty subset of $E$ that is bounded from above has a supremum, or equivalently, if every non-empty subset of $E$ that is bounded from below has an infimum.

(iv) $E$ is called $\sigma$--Dedekind complete if every non-empty finite or countable subset of $E$ that is bounded from above (bounded from below) has a supremum (infimum).

(v) $E$ is called order separable if every non-empty subset $D$ of $E$ that has a supremum contains a subset that is at most countable with the same supremum as $D$.

(vi) $E$ is called super Dedekind complete if it is Dedekind complete and order separable.

For $\sigma$-finite measure spaces $(X, \Sigma, \mu)$, the spaces $L^0(X, \Sigma, \mu)$, $M(X, \Sigma, \mu)$ and $L^p(X, \Sigma, \mu)$, with $0 \leq p < \infty$, are examples of super Dedekind complete spaces. (The space $E$ is Dedekind complete but in general not super Dedekind complete, see [19, Example 23.3]. $L^0(X, \Sigma, \mu)$ is an example of a laterally complete Riesz space.

**Definition 1.2.19** A linear subspace $A \subseteq E$ is called a Riesz subspace or a sublattice of the Riesz space $E$ if $x \vee y$ and $x \wedge y$ belong to $A$ for all $x, y \in A$.

(i) A subset $A$ is called order dense in $E$ whenever for each $0 < x \in E$ there exists some $y \in A$ that satisfies $0 < y \leq x$. 
(ii) A subset $A$ is called solid if $|x| < |y|$ and $y \in A$ implies that $x \in A$.

(iii) A solid linear subspace of the Riesz space $E$ is called an ideal.

(iv) An ideal $B$ is a band if for every subset $A \subset E$ we have $\sup A \in B$ whenever $\sup A$ exists in $E$. An ideal generated by a singleton set is called a principal ideal.

(v) A band $B$ is called a projection band if there exists a linear projection $P : E \rightarrow B$ satisfying $0 \leq Px \leq x$ for all $x \in E^+$. $P$ is then called a band projection.

From the definition it follows that the ideal $B$ is a band in $E$ if $0 \leq f_\alpha \uparrow f \in E$ with $f_\alpha \in B$ implies that $f \in B$ and that an ideal $A$ is order dense in $E$ if the band generated by $A$ is the whole $E$.

Example 1.2.20  (1) $L^p(\Omega, \Sigma, \mu)$ is an ideal in $L^0(\Omega, \Sigma, \mu)$ (both in Example 1.2.6).

(2) Again $L^p(\Omega, \Sigma, \mu)$ in Example 1.2.6 is an ideal in the space $M(\Omega, \Sigma, \mu)$, the space of all $\mu$-a.e. finite functions on $\Omega$, but it is not a band. $B \subset M(\Omega, \Sigma, \mu)$ is a band if and only if there exists a measurable set $E_B \subset \Omega$ such that

$$B = \{f \in M(\Omega, \Sigma, \mu) : f(x) = 0 \text{ for almost all } x \notin E_B\}.$$

We will, to a great extent, be concerned with ideals of measurable functions, i.e., ideals $L$ in the Riesz space $L^0(Y, \Lambda, \nu)$. The collection of all $\Lambda$-measurable functions into $[0, \infty]$ will be denoted by $M^+(Y, \Lambda, \nu)$.
If $L$ is an ideal in $L^0(Y, \Lambda, \nu)$, the set $Z \in \Lambda$ is called an $L$-zero set if every $f \in L$ vanishes $\nu$-a.e. on $Z$. There exists a maximal $L$-zero set $Z_1 \in \Lambda$ and the set $Y_1 = Y \setminus Z_1$ is called the carrier of the ideal $L$. Also, there exists a sequence $A_n \uparrow Y_1$ in $\Lambda$ such that $\nu(A_n) < \infty$ and $1_{A_n} \in L$ for all $n \in \mathbb{N}$. For proofs of these see [32].

We note that if the carrier of the ideal $L$ in $L^0(Y, \Lambda, \nu)$ is the whole set $Y$, i.e., $Y = Y_1$ then $L$ is order dense in $L^0(Y, \Lambda, \nu)$.

1.3 Operators in Riesz spaces

We now look at the basic theory of operators in Riesz spaces. We consider two Riesz spaces $E$ and $F$. We will denote the set of all linear operators from $E$ to $F$ by $\mathcal{L}(E, F)$. $\mathcal{L}(E, F)$ is an ordered space if, for $T, S \in \mathcal{L}(E, F)$, we define $S \leq T$ to mean $T - S \geq 0$.

**Definition 1.3.1** Let $E$ and $F$ be Riesz spaces and let $T \in \mathcal{L}(E, F)$.

(i) $T$ is called **order bounded** if it maps order bounded subsets into order bounded subsets.

(ii) $T$ is called **positive** whenever $T x \geq 0$ for all $x \geq 0$. If $T$ is positive we write $T \geq 0$.

(iii) $T$ is called a **Riesz homomorphism** or **lattice homomorphism** whenever $T(x \lor y) = Tx \lor Ty$. A bijective Riesz homomorphism is called a **Riesz isomorphism**.
(iv) $T$ is said to be order continuous whenever $\inf\{|Tx| : x \in D\} = 0$ in $F$ for every set $D$ such that $D \downarrow 0$ in $E$.

(v) $T$ is said to be $\sigma$-order continuous, if, for any monotone sequence $x_n \downarrow 0$ we have that $\inf\{|Tx_n|\} = 0$.

(vi) $T$ is said to be regular if it can be expressed as a difference of two positive linear operators from $E$ into $F$, i.e., if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive linear operators from $E$ into $F$.

Note that for a positive $T$, we have that $T$ is order continuous if and only if it follows that $T(D) \downarrow 0$ in $F$ for all sets $D \downarrow 0$ in $E$. In the case of $\sigma$-order continuity, we have that a positive $T$ is $\sigma$-order continuous if $x_n \downarrow 0$ implies that $Tx_n \downarrow 0$.

The set of order bounded linear operators from $E$ to $F$ will be denoted by $\mathcal{L}_b(E, F)$. The bounded order dual of $E$, which is $\mathcal{L}_b(E, \mathbb{R})$, will be denoted by $E^\sim$. Note that if $F$ is Dedekind complete then $\mathcal{L}_b(E, F)$ is Dedekind complete. We will denote the set of order continuous linear operators in $\mathcal{L}_b(E, F)$ by $\mathcal{L}_c(E, F)$ and in the case where $F = \mathbb{R}$ by $E_n^\sim$. The set of all $\sigma$-order continuous linear operators will be denoted by $\mathcal{L}_c(E, F)$ and for the case where $F = \mathbb{R}$ by $E_c^\sim$.

We also note that a positive operator is regular and that if $F$ is Dedekind complete then every operator $T \in \mathcal{L}(E, F)$ is regular if and only if it is order bounded, i.e., the set of regular operators and $\mathcal{L}_b(E, F)$ coincide if $F$ is Dedekind complete, see [32].
We state a few characteristics of positive operators whose proofs can be found in [24] and in [34].

**Lemma 1.3.2** Let $T \in \mathcal{L}_b(E,F)$. Then $T \in \mathcal{L}_n(E,F)$ if and only if $|T| \in \mathcal{L}_n(E,F)$, i.e., if $T^+, T^- \in \mathcal{L}_n(E,F)$. Similarly $T \in \mathcal{L}_c(E,F)$ if and only if $|T| \in \mathcal{L}_c(E,F)$.

**Proposition 1.3.3** (i) $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are bands in $\mathcal{L}_b(E,F)$.

(ii) $E_\sim^-$ and $E_\sim^+$ are bands in $E_\sim$.

**Proposition 1.3.4** Let $T \in \mathcal{L}(E,E)$, then

(i) $T$ is positive if and only if $|Tx| \leq T|x|$ for all $x \in E$.

(ii) $(Tf)^+ \leq Tf^+$

(iii) Every Riesz homomorphism is positive.

(iv) $T$ is a Riesz isomorphism if and only if $T$ and $T^{-1}$ are positive.

We will denote the range of $T \in \mathcal{L}(E,F)$ by ran$(T)$ and the kernel of $T$ by ker$(T)$. Recall that ker$(T) = \{f \in E : Tf = 0\}$.

**Definition 1.3.5** For an operator $T \in \mathcal{L}_b(E,F)$ the set

$$N_T = \{x \in E : |T|x| = 0\}$$

is called the null ideal of $T$ or the absolute kernel of $T$. 
We have that the absolute kernel of a positive operator is an ideal and that \( N_T \subset T^{-1}(0) \).

**Definition 1.3.6** The disjoint complement of the absolute kernel of \( T \in \mathcal{L}_b(E, F) \) is called the **carrier** of \( T \) and will be denoted by \( C_T \).

Note that if \( T \in \mathcal{L}_n(E, F) \) then \( N_T \) is a band and, furthermore, by the Riesz decomposition of Dedekind complete Riesz spaces we have that

\[
E = N_T \oplus C_T.
\]

The following result implies that every regular operator is norm bounded, its proof can be found in [34].

**Proposition 1.3.7** Let \( E \) be a Banach lattice and \( F \) a normed Riesz space. Then every positive linear operator from \( E \) into \( F \) is continuous.

### 1.4 Riesz and Boolean homomorphisms

In this section we insert some remarks concerning Riesz homomorphisms that will be used later on. We start by stating some results about order convergence and \( \sigma \)-order convergence.

**Lemma 1.4.1** If \( (f_n) \) is a sequence in \( L^0(X, \Sigma, \mu) \) such that for \( \mu \)-a.e. \( x \), \( f_n \to 0 \) then \( f(x) = \sup_n |f_n(x)| \) exists in \( L^0(X, \Sigma, \mu) \).

**Proof** Each \( f_n(x) \) is a.e. \( \mu \)-measurable and so \( f(x) \) is a.e. \( \mu \)-measurable.

We show that it is \( \mu \)-a.e. finite valued. We have that for \( \mu \)-a.e. \( x \),
$f_n \to 0$. Let $x_0$ be one of these points, then there exists a natural number $N(x_0)$ such that $|f_n(x_0)| \leq 1$ whenever $n > N(x_0)$. Put $M = \sup\{1, |f_1(x_0)|, |f_2(x_0)|, \ldots, |f_{N(x_0)}(x_0)|\}$. Then

$$f(x_0) = \sup_n |f_n(x_0)| \leq M.$$  

Hence $f(x_0)$ is finite. Thus $f(x)$ is $\mu$-a.e. finite valued on $X$. \qed

If $(f_n)$ is a sequence in $L^0(X, \Sigma, \mu)$ such that $f_n(x) \downarrow 0$ $\mu$-a.e., then there exists a sequence of positive real numbers $a_n \uparrow \infty$ such that $a_n f_n(x) \to 0$, see [19, Theorem 71.4]. Putting $f_0(x) = \sup_n a_n f_n(x)$, and using Lemma 1.4.1, we have that $f_0 \in L^0(X, \Sigma, \mu)$. But then $0 \leq f_n(x) \leq \frac{1}{a_n} f_0(x)$ and so $(f_n)$ converges $f_0$-uniformly to 0.

The above observation can be stated as

**Theorem 1.4.2** Order convergent sequences in $L^0(X, \Sigma, \mu)$ are relatively uniformly convergent.

We use this in the proof of the following

**Proposition 1.4.3** Let $(Y, \Lambda, \nu)$ and $(X, \Sigma, \mu)$ be $\sigma$-finite measure spaces. Suppose that $\phi$ is a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$. Then

(i) $\phi$ is order continuous;

(ii) if $\phi$ is interval preserving, $\text{ran}(\phi)$ is a band in $L^0(X, \Sigma, \mu)$;
(iii) If \( \phi \) is surjective, then it is interval preserving and in particular, if \( L \subset (Y, \Lambda, \nu) \) is an ideal, then \( \phi(L) \) is an ideal in \( L^0(X, \Sigma, \mu) \).

(iv) If \( L \subset L^0(Y, \Lambda, \nu) \) and \( M \subset L^0(X, \Sigma, \mu) \) are order dense ideals and if \( \phi : L \to M \) is an order continuous Riesz homomorphism, then it can be extended uniquely to a Riesz homomorphism from \( L^0(Y, \Lambda, \nu) \) into \( L^0(X, \Sigma, \mu) \).

**Proof**

(i) By Theorem 1.4.2 we have that an order convergent sequence in \( L^0(X, \Sigma, \mu) \) is relatively uniformly convergent. Since \( \phi \) is positive it is \( \sigma \)-order continuous. But \( L^0(X, \Sigma, \mu) \) is super Dedekind complete, and therefore order separable and so \( \phi \) is order continuous.

(ii) Assume that \( \phi \) is interval preserving. If \( G = \text{ran}(\phi) \), then \( G \) is an ideal in \( L^0(X, \Sigma, \mu) \). Since \( L^0(Y, \Lambda, \nu) \) is laterally complete and \( \phi \) is order continuous, \( G \) is laterally complete. We have that for every \( 0 \leq w \in G \), \( B_w = \{w\}^d \subset G \). Indeed, if we let \( 0 \leq f \in B_w \) and put

\[
    f_n = f1_{E_n},
\]

where

\[
    E_n = \{t \in X \mid nw(t) < f(t) \leq (n+1)w(t) \},
\]

and \( n = 1, 2, \cdots \), then \( f_n \) is a disjoint system in \( G \) and so \( \sup_n f_n \) belongs to \( G \). Since this supremum is equal to \( f \in L^0(X, \Sigma, \mu) \) we have that \( f \in G \).

Now, let \( \{w_\alpha\} \) be a maximal disjoint system in \( G \), then

\[
    w = \sup w_\alpha \in G
\]
and the band generated by $w$ is contained in $G$.

On the other hand, if $0 \leq v \in G$, we write $v = v_1 + v_2$ with $v_1 \in B_w$ and $v_2 \in B_w^d$. Since $v_2$ is disjoint to every $w_\alpha$ and since $w_\alpha$ is a maximal disjoint system in $G$, we have that $v_2 = 0$. Thus $G \subset B_w$ and so $G = B_w$ is a band in $L^0(X, \Sigma, \mu)$.

(iii) Let $\phi$ be a surjective Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ to $L^0(X, \Sigma, \mu)$ and let $0 \leq g \leq \phi(u)$ for some $0 < u \in L^0(Y, \Lambda, \nu)$. Let $w \in L^0(Y, \Lambda, \nu)$ be such that $\phi(w) = g$ and consider $w^+ \land u$. Then $0 \leq w^+ \land u \leq u$ and $0 \leq \phi(w^+ \land u) \leq \phi(u)$. This gives us that $0 \leq \phi(w^+) \land \phi(u) \leq u$ and so $\phi(w^+ \land u) = g$ since $\phi(w^+) = g^+ = g$.

(iv) Since $L^0(X, \Sigma, \mu)$ is laterally complete and $L \subset L^0(Y, \Lambda, \lambda)$ is Dedekind complete we have by Theorem 7.20 in [1], that $\phi$ can be extended into a Riesz homomorphism $\phi'$ from $L^0(Y, \Lambda, \mu)$ into $L^0(X, \Sigma, \mu)$ that satisfy the equation

$$\phi'(x) = \sup\{\phi(y) : y \in L \text{ and } 0 \leq y \leq x\},$$

for all $0 < x \in L^0(Y, \Lambda, \mu)$. Since every extension satisfies this formula it is unique. $\square$

Let $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$, be a Riesz homomorphism, we denote its null-ideal and carrier by $N_\phi$ and $C_\phi$, respectively, i.e.,

$$N_\phi = \{f \in L^0(Y, \Lambda, \nu) : \phi(f) = 0\} \text{ and } C_\phi = N_\phi^d.$$

Since $\phi$ is order continuous, $N_\phi$ is a band and so

$$L^0(Y, \Lambda, \nu) = C_\phi \oplus N_\phi.$$
CHAPTER 1. PRELIMINARIES

If $Y_1 \in \Lambda$ is the carrier of $C_\phi$, then

$$C_\phi = \{ f 1_{Y_1} : f \in L^0(Y, \Lambda, \nu) \} = L^0(Y_1, \Lambda_{Y_1}, \nu),$$

where $\Lambda_{Y_1} = \{ A \cap Y_1 : A \in \Lambda \}$.

Note that the restriction of $\phi$ to $L^0(Y_1, \Lambda_{Y_1}, \nu)$ is a Riesz isomorphism into $L^0(X, \Sigma, \mu)$.

We will denote by $\Lambda_\nu$ the measure algebra of $(Y, \Lambda, \nu)$ and by $\Sigma_\mu$ the measure algebra of $(X, \Sigma, \mu)$. $C$ will denote the equivalence class in either of the algebras to which the measurable set $C$ belongs.

Let $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$ be a Riesz homomorphism such that $\phi(1_Y)$ is an a.e. strict positive function on $X$. If $A \in \Lambda$ then $\phi(1_A) = \phi(1_Y) 1_B$, for some $B \in \Sigma$ which is uniquely determined modulo $\mu$–null sets by $A$. If we put $\widehat{\phi}(\hat{A}) = \hat{B}$ it follows that

$$\widehat{\phi} : \Lambda_\nu \to \Sigma_\mu$$

is an order continuous Boolean homomorphism that satisfies $\widehat{\phi}(\hat{Y}) = \hat{X}$. Put

$$\Sigma_\phi = \{ B \in \Sigma : \hat{B} \in \widehat{\phi}(\Lambda_\nu) \}$$

Then $\Sigma_\phi$ is a sub-$\sigma$-algebra of $\Sigma$ and we write $\Sigma_\phi = \widehat{\phi}(\Lambda)$. (In this case we do not distinguish between the $\sigma$-algebra $\Lambda$ and the measure algebra $\Lambda_\nu$).

We show that $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma_\phi, \mu)$ is interval preserving. To that end let $0 \leq u \in L^0(Y, \Lambda, \nu)$ and let $v \in L^0(X, \Sigma_\phi, \mu)$ be such that
$0 \leq v \leq \phi(u)$. For every step function $f(y) = \sum_{i=1}^{n} a_i 1_{B_i}(y)$ with $B_i = \widehat{\phi}(A_i) \in \Sigma_\phi$ we have

$$
\phi \left( \frac{1}{\phi(1_Y)} \sum_{i=1}^{n} a_i 1_{A_i} \right) = \frac{1}{\phi(1_Y)} \sum_{i=1}^{n} a_i \phi(1_Y) 1_{B_i} = \sum_{i=1}^{n} a_i 1_{B_i}.
$$

Let $(v_n)$ be a sequence of step functions such that $0 \leq v_n \uparrow v$ in $L^0(X, \Sigma_\phi, \mu)$. By the preceding argument there exists a sequence $(u_n)$ of step functions in $L^0(Y, \Lambda, \nu)$ such that $\phi(u_n) = v_n$. Since $\phi$ is a Riesz homomorphism we may assume, without loss of generality, that $0 \leq u_n \uparrow u$ [else we can successively replace $u_n$ firstly by $u_n^+$ (to get $\phi(u_n^+) = \phi(u_n \vee 0) = \phi(u_n) \vee 0 = v_n^+ = v_n$), secondly by $u_n \vee u_{n-1}$ (to get $\phi(u_n \vee u_{n-1}) = v_n \vee v_{n-1} = v_n$) and lastly by $u_n \wedge u$ (to get $\phi(u_n \wedge u) = v_n \wedge \phi(u) = v_n$)]. Let $0 \leq u_n \uparrow w$. Then

$$
\phi(w) = \sup \phi(u_n) = \sup(v_n) = v,
$$

which shows that $\phi$ is interval preserving.

By Proposition 1.4.3 (ii) we get that the ran($\phi$) is a band in $L^0(X, \Sigma_\phi, \mu)$ containing the weak order unit $\phi(1_Y)$. This band is the whole of $L^0(X, \Sigma_\phi, \mu)$ and so $\phi(L^0(Y, \Lambda, \nu)) = L^0(X, \Sigma_\phi, \mu)$.

On the other hand, if $\sigma : \Lambda_\nu \rightarrow \Sigma_\mu$ is an order continuous Boolean homomorphism with $\sigma(\mathcal{Y}) = \mathcal{X}$, then there exists a unique Riesz homomorphism $\phi : L^0(Y, \Lambda, \nu) \rightarrow L^0(X, \Sigma, \mu)$ with $\phi(1) = 1$ such that $\widehat{\sigma} = \phi$.

Now assume that $\tau : X \rightarrow Y$ is a $(\Sigma, \Lambda)$-measurable mapping which is null preserving (i.e., if $B \in \Lambda$ and $\nu(B) = 0$ then $\mu(\tau^{-1}(B)) = 0$). The
mapping $B \mapsto \tau^{-1}(B)$ defines an order continuous Boolean homomorphism

$$\tau_* : \Lambda_\nu \to \Sigma_\mu$$

with

$$\tau_*(\hat{Y}) = \hat{X}.$$ 

The associated Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$ will be denoted by $\phi_\tau$, i.e., $\widehat{\phi_\tau} = \tau_*$ and $\phi_\tau(1) = 1$. It is easily verified that

$$(\phi_\tau f)(x) = f(\tau x) \mu\text{-a.e. on } X$$

for all $f \in L^0(Y, \Lambda, \nu)$.

Let $L \subset L^0(Y, \Lambda, \nu)$ be an order dense ideal with order continuous dual $L_n^\sim$. As usual we identify $L_n^\sim$ with an ideal $L'$ of functions in $L^0(Y, \Lambda, \nu)$ and we will assume that $L'$ is again an order dense ideal (which is always the case if $L$ is a Banach function space; (see [33], Theorem 112.1 or [21], Theorem 2.6.4). Equivalent to this assumption is that $L_n^\sim$ separates the points of $L$.

The duality relation between $L$ and $L'$ is given by

$$\langle f, g \rangle = \int_Y fg d\nu \text{ for } f \in L \text{ and } g \in L',$$

(see [33] Section 86). Let $T \in \mathcal{L}_n(L, M)$ with $L$ and $M$ ideals of functions in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$ respectively. We define its order continuous adjoint $T' : M' \to L'$ by $\langle g, T'f \rangle = \langle Tg, f \rangle$ for all $f \in M'$ and $g \in L$ (see [33], Section 97). Then $T' \in \mathcal{L}_n(M', L')$.

The next result is from [8]. In it we gather some results relating to the adjoints of homomorphisms.
Lemma 1.4.4 Let \((X, \Sigma, \mu)\) and \((Y, \Lambda, \nu)\) be \(\sigma\)-finite measure spaces and let \(L \subseteq L^0(Y, \Lambda, \nu)\) and \(M \subseteq L^0(X, \Sigma, \mu)\) be order dense ideals for which \(L'\) and \(M'\) are order dense ideals as well. Let \(\phi : L \to M\) be an order continuous interval preserving Riesz homomorphism. Then,

(i) the adjoint \(\phi' : M' \to L'\) is an order continuous interval preserving Riesz homomorphism as well and \(\phi'\) extends uniquely to an order continuous interval preserving Riesz homomorphism \(\phi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)\);

(ii) if \(\phi(L)\) is order dense in \(L^0(X, \Sigma, \mu)\), then \(\phi'\) is injective;

(iii) if \(\phi\) is injective, then \(\phi'(1_X)\) is strictly positive and \(\phi'(M')\) is order dense in \(L'\);

(iv) if \(\phi\) is injective and \(\phi(L)\) is a band in \(M\) then \(\phi' : M' \to L'\) is surjective.

Proof (i) It follows from [1], Theorem 7.7 that \(\phi'\) is a Riesz homomorphism and from [1] Theorem 7.8 that \(\phi'\) is an interval preserving. By Proposition 1.4.3, \(\phi'\) can be extended to an order continuous Riesz homomorphism \(\phi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)\). We show that \(\phi'\) is interval preserving: Let \(0 \leq u \in L^0(X, \Sigma, \mu) \supseteq M\) and \(v \in L^0(Y, \Lambda, \nu) \supseteq L\) be such that \(0 \leq v \leq \phi'(u)\). Since \(M'\) is order dense in \(L^0(X, \Sigma, \mu)\) there is a sequence \(0 \leq u_n \uparrow u\) in \(M'\) and so \(0 \leq \phi'(u_n) \uparrow \phi'(u)\). We have that \(\phi' : M' \to L'\) is interval preserving. Since \(0 \leq v \land \phi'(u_n) \leq \phi'(u_n)\) there exists, for each \(n = 1, 2, \ldots\), an element \(w_n \in M'\) such that \(\phi'(w_n) = v \land \phi'(u_n)\).
Again, since $\phi'$ is a Riesz homomorphism, we may assume without loss of generality that $0 \leq w_n \uparrow u$. Let $w_n \uparrow w$, then

$$\phi'(w) = \sup \phi'(w_n) = \sup (v \wedge \phi'(u_n)) = v \wedge \phi'(u) = v.$$  

This proves the assertion.

(ii) Now assume that $\phi(L)$ is order dense in $L^0(X, \Sigma, \mu)$. Let $0 \leq g \in M'$ be such that $\phi'(g) = 0$. Then

$$\langle \phi(u), g \rangle = \langle u, \phi'(g) \rangle = 0 \quad \forall \ 0 \leq u \in L,$$

and so $g = 0$ by order denseness of $\phi(L)$. Hence $\phi'$ is injective.

(iii) Now assume that $\phi$ is injective. To show that $\phi'(1_X)$ is strictly positive, take $0 \leq g \in L$ such that $g \wedge \phi'(1_X) = 0$. Take $X_n \in \Sigma$ such that $X_n \uparrow X$ and $1_{X_n} \in M'$ for all $n$. Then

$$\int_{X_n} \phi(g) \, d\mu = \langle \phi(g), 1_{X_n} \rangle = \langle g, \phi'(1_{X_n}) \rangle = 0$$

for all $n$, and so $\int_X \phi(g) \, d\mu = 0$. This implies that $\phi(g) = 0 \ \mu$-a.e. on $X$. Hence $g = 0$, which shows that $\phi'(1_X)$ is strictly positive, as $L$ is order dense in $L^0(Y, \Lambda, \nu)$.

In order to see that $\phi'(M')$ is order dense in $L'$, let $0 < h \in L'$. Since $\phi'(1_X)$ is strictly positive, $0 < \phi'(1_X) \wedge h \leq \phi'(1_X)$ and as $\phi'$ is interval preserving, there exists $0 < f \in L^0(X, \Sigma, \mu)$ such
that \( 0 < f \leq 1_X \) and \( \phi'(f) = \phi'(1_X) \wedge h \). But \( M' \) is order dense in \( L^0(X, \Sigma, \mu) \) and so for some \( g \in M' \), we have \( 0 < g \leq f \). By injectivity, \( 0 < \phi'(g) \leq \phi'(f) \leq h \) and it follows that \( \phi'(M') \) is order dense in \( L' \).

(iv) Assume now that \( \phi \) is injective and that \( \phi(L) \) is a band in \( M \). By (ii), \( \phi'(1_X) \) is strictly positive and since \( \phi' \) is interval preserving, it follows that \( \phi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu) \) is surjective. Let \( X_1 \in \Sigma \) be the carrier of \( \phi(L) \). By hypothesis, \( \phi(L) = \{ f1_{X_1} : f \in M \} \).

Furthermore it is easy to see that \( \phi'(g) = 0 \) for all \( g \in L^0(X, \Sigma, \mu) \) such that \( g = 0 \) on \( X_1 \).

Now let \( 0 < h \in L' \) be given. Then \( h = \phi'(g) \) for some \( 0 \leq g \in L^0(X, \Sigma, \mu) \), and we may assume that \( g = 0 \) on \( X\setminus X_1 \). It remains to show that \( 0 \leq g \in M' \), i.e., that \( \int_X g f \, d\mu < \infty \) for all \( 0 \leq f \in M \). To this end, take \( 0 \leq f \in M \). Then \( f1_{X_1} \in \phi(L) \), so \( f1_{X_1} = \phi(u) \) for some \( 0 \leq u \in L \). Let \( 0 \leq g_n \in M' \) be such that \( 0 \leq g_n \uparrow g \). We find that

\[
\int_X g f \, d\mu = \int_X g(f1_{X_1}) \, d\mu = \int_X g\phi(u) \, d\mu = \lim_{n \to \infty} \int_X g_n\phi(u) \, d\mu.
\]
Thus
\[
\int_X g f \ d\mu = \lim_{n \to \infty} \int_Y \phi'(g_n) u \ d\nu \\
= \int_Y \phi'(g) u \ d\nu \\
= \int_Y hu \ d\nu \\
< \infty,
\]
which shows that $0 \leq g \in M'$.

\[
\]

1.5 Conditional Expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e., $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$ and $\mathbb{P}$ a countably additive measure on $\mathcal{F}$ with $\mathbb{P}(\Omega) = 1$.

Let $\mathcal{G}$ be a $\sigma$-algebra (sub-$\sigma$-algebra) of subsets of $\Omega$ with $\mathcal{G} \subset \mathcal{F}$. For every $A \in \mathcal{G}$ the equation
\[
\phi(A) = \int_A f \ d\mathbb{P},
\]
where $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, defines a countably additive function $\phi$ on $\mathcal{G}$. By the Randon-Nikodym theorem there is an extended real valued function $g$ defined on $\Omega$ which is measurable with respect to $\mathcal{G}$ such that
\[
\phi(A) = \int_A g \ d\mathbb{P}
\]
for every $A \in \mathcal{G}$. If we denote $g$ by $\mathbb{E}(f | \mathcal{G})$ we get the following

**Definition 1.5.1** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. For $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ we denote by $\mathbb{E}(f | \mathcal{G})$ the $\mathbb{P}$-a.e. unique
\(\mathcal{G}\)-measurable function with the property that
\[
\int_A \mathbb{E}(f \mid \mathcal{G}) \, dP = \int_A f \, dP
\]
for all \(A \in \mathcal{G}\). The function \(\mathbb{E}(f \mid \mathcal{G})\) is called the conditional expectation of \(f\) with respect to (or given) \(\mathcal{G}\).

**Proposition 1.5.2** Let \(\mathbb{E}(\cdot \mid \mathcal{G})\) be a conditional expectation. Then \(\mathbb{E}(\cdot \mid \mathcal{G})\) can be extended from a mapping of \(L^1(\Omega, \mathcal{F}, P)\) into itself to a mapping from \(M^+(\Omega, \mathcal{F}, P)\) into itself.

**Proof** If \(f \in M^+(\Omega, \mathcal{F}, P)\), then, (see [23] Corollary I-2-10), we can, for a sequence \((f_n)\) such that \(0 \leq f_n \in L^1(\Omega, \mathcal{F}, P)\) satisfying \(0 \leq f_n \uparrow f\) \(P\)-a.e., define \(\mathbb{E}(f \mid \mathcal{G}) \in M^+(\Omega, \mathcal{F}, P)\) by

\[
\mathbb{E}(f \mid \mathcal{G}) = \sup_n \mathbb{E}(f_n \mid \mathcal{G}).
\]

Next we show that this definition is independent of the choice of the sequence \((f_n)\). To that end let \((g_m)\) be a sequence in \(L^1(\Omega, \mathcal{F}, P)\) such that \(0 \leq g_m \uparrow f\) \(P\)-a.e. Then \(g_m = \sup_n (f_n \wedge g_m)\) and so

\[
\mathbb{E}(g_m \mid \mathcal{G}) = \sup_n \mathbb{E}(f_n \wedge g_m \mid \mathcal{G}) \leq \sup_n \mathbb{E}(f_n \mid \mathcal{G}).
\]

Thus

\[
\sup_m \mathbb{E}(g_m \mid \mathcal{G}) \leq \sup_n \mathbb{E}(f_n \mid \mathcal{G}). \quad (1.5.1)
\]

On the other hand \(f_n = \sup_m (f_n \wedge g_m)\) and

\[
\mathbb{E}(f_n \mid \mathcal{G}) = \sup_m \mathbb{E}(f_n \wedge g_m \mid \mathcal{G}) \leq \sup_m \mathbb{E}(g_m \mid \mathcal{G}).
\]
Thus

\[ \sup_n \mathbb{E}(f_n | \mathcal{G}) \leq \sup_m \mathbb{E}(g_m | \mathcal{G}) , \quad (1.5.2) \]

Equation 1.5.1 and Equation 1.5.2 together give that

\[ \sup_n \mathbb{E}(f_n | \mathcal{G}) = \sup_m \mathbb{E}(g_m | \mathcal{G}) . \]

We can use these suprema and write

\[ \sup_n \mathbb{E}(f_n | \mathcal{G}) = \sup_m \mathbb{E}(g_m | \mathcal{G}) = \mathbb{E}(f | \mathcal{G}). \]

We list some properties of \( \mathbb{E} (\cdot | \mathcal{G}) \) and refer the reader to [23].

Proposition 1.5.3

(i) If \( f \in L^1 (\Omega, \mathcal{F}, \mathbb{P}) \) and \( g \in L^\infty (\Omega, \mathcal{G}, \mathbb{P}) \), then \( \mathbb{E} (g f | \mathcal{G}) = g \mathbb{E} (f | \mathcal{G}) . \)

(ii) \( \mathbb{E} (\alpha f + \beta g | \mathcal{G}) = \alpha \mathbb{E} (f | \mathcal{G}) + \beta \mathbb{E} (g | \mathcal{G}) \) for all \( f, g \in M^+ (\Omega, \mathcal{F}, \mathbb{P}) \) and all \( 0 \leq \alpha, \beta \in \mathbb{R} . \)

(iii) \( 0 \leq f \leq g \) in \( M^+ (\Omega, \mathcal{F}, \mathbb{P}) \) implies that \( 0 \leq \mathbb{E} (f | \mathcal{G}) \leq \mathbb{E} (g | \mathcal{G}) . \)

(iv) \( 0 \leq f_n \uparrow f \) \( \mathbb{P} \)-a.e. implies that \( 0 \leq \mathbb{E} (f_n | \mathcal{G}) \uparrow \mathbb{E} (f | \mathcal{G}) \) \( \mathbb{P} \)-a.e.

(v) For all \( f \in M^+ (\Omega, \mathcal{F}, \mathbb{P}) \) and all \( g \in M^+ (\Omega, \mathcal{G}, \mathbb{P}) \) we have \( \mathbb{E} (g f | \mathcal{G}) = g \mathbb{E} (f | \mathcal{G}) . \)

(vi) If \( g \in M^+ (\Omega, \mathcal{G}, \mathbb{P}) \) and \( f \in M^+ (\Omega, \mathcal{F}, \mathbb{P}) \) then \( \int_A g d\mathbb{P} = \int_A f d\mathbb{P} \) for all \( A \in \mathcal{G} \) if and only if \( g = \mathbb{E} (f | \mathcal{G}) \) \( \mathbb{P} \)-a.e.
(vii) If $\mathcal{G}$ and $\mathcal{H}$ are sub-$\sigma$-algebras of $\mathcal{F}$, such that $\mathcal{G} \subset \mathcal{H}$, then $\mathbb{E}(f \mid \mathcal{G}) = \mathbb{E}(\mathbb{E}(f \mid \mathcal{H}) \mid \mathcal{G})$ for all $0 \leq f \in M^+(\Omega, \mathcal{F}, \mathcal{P})$.

(viii) If $f \in M^+(\Omega, \mathcal{F}, \mathcal{P})$ is such that $\mathbb{E}(f \mid \mathcal{G}) \in L^0(\Omega, \mathcal{F}, \mathcal{P})$ then we also have that $f \in L^0(\Omega, \mathcal{F}, \mathcal{P})$.

**Proof** We prove (viii) only

If $\mathbb{E}(\cdot \mid \mathcal{G}) \in L^0(\Omega, \mathcal{F}, \mathcal{P})$ then there exists a sequence $(\Omega_n) \subset \mathcal{G}$ such that $\Omega_n \uparrow \Omega$ and $\int_{\Omega_n} \mathbb{E}(f \mid \mathcal{G}) \, d\mathcal{P} < \infty$. This implies that $\int_{\Omega_n} f \, d\mathcal{P} < \infty,$ so that $f \in L^0(\Omega, \mathcal{F}, \mathcal{P})$. $\square$

We remark that the converse of the above does not hold in general. For instance, let $\Omega = [0, 1]$ and $\mathcal{P}$ be a Lebesgue measure. Put

$$\mathcal{G} = \{\emptyset, [0, \frac{1}{2}], [\frac{1}{2}, 1], [0, 1]\}$$

and $f(x) = \frac{1}{x}$, where $0 \leq x \leq 1$. Then $\mathbb{E}(f \mid \mathcal{G}) = \infty$ on $[0, \frac{1}{2}]$. We therefore need the following:

**Definition 1.5.4** The **domain** of $\mathbb{E}(\cdot \mid \mathcal{G})$ is the set $\text{dom}\mathbb{E}(\cdot \mid \mathcal{G})$ given by

$$\text{dom}\mathbb{E}(\cdot \mid \mathcal{G}) = \{f \in L^0(\Omega, \mathcal{F}, \mathcal{P}) : \mathbb{E}(|f| \mid \mathcal{G}) \in L^0(\Omega, \mathcal{G}, \mathcal{P})\}.$$ 

It is clear that $\text{dom}\mathbb{E}(\cdot \mid \mathcal{G})$ is an ideal in $L^0(\Omega, \mathcal{F}, \mathcal{P})$ which contains $L^1(\Omega, \mathcal{F}, \mathcal{P})$ and therefore it is order dense in $L^0(\Omega, \mathcal{F}, \mathcal{P})$. For an element $f$ of $\text{dom}\mathbb{E}(\cdot \mid \mathcal{G})$ we define

$$\mathbb{E}(f \mid \mathcal{G}) = \mathbb{E}(f^+ \mid \mathcal{G}) - \mathbb{E}(f^- \mid \mathcal{G}).$$
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This defines a positive linear operator

\[ \mathbb{E} (\cdot | \mathcal{G}) : \text{dom} \mathbb{E} (\cdot | \mathcal{G}) \to L^0 (\Omega, \mathcal{G}, \mathbb{P}) \subset L^0 (\Omega, \mathcal{F}, \mathbb{P}). \]

**Example 1.5.5** Let \((\Omega_1, \mathcal{F}'', \mathbb{P}_1)\) and \((\Omega_2, \mathcal{F}'', \mathbb{P}_2)\) be two probability spaces and \(\Omega = \Omega_1 \times \Omega_2, \mathcal{F} = \mathcal{F}' \otimes \mathcal{F}''\) and \(\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2\). Put \(\mathcal{G} = \{ A \times \Omega_2 : A \in \mathcal{F}' \}\). Then \(\mathcal{G}\) is a sub-\(\sigma\)-algebra of \(\mathcal{F}\). A \(\mathcal{F}\)-measurable function \(g : \Omega_1 \times \Omega_2 \to \mathbb{R}\) is \(\mathcal{F}'\)-measurable if and only if \(g (x_1, x_2) = g (x_1)\), i.e., if and only if \(g\) is independent of \(x_2\). For \(f \in L^1 (\Omega, \mathcal{F}, \mathbb{P})\) we now have that \(\mathbb{E} (f|\mathcal{G}) (x_1, x_2) = \int_{\Omega_2} f (x_1, y) \, d\mathbb{P}_2 (y) \) \(\mathbb{P}\)-a.e. on \(\Omega\). From the way in which \(\mathbb{E} (\cdot | \mathcal{G})\) is extended to \(M^+ (\Omega, \mathcal{F}, \mathbb{P})\) it then follows that for \(f \in L^0 (\Omega, \mathcal{F}, \mathbb{P})\) we have \(f \in \text{dom} \mathbb{E} (\cdot | \mathcal{G})\) if and only if \(\int_{\Omega_2} |f (x_1, y)| \, d\mathbb{P}_2 (y) < \infty \) \(\mathbb{P}\)-a.e. on \(\Omega\). In this case we have that \(\mathbb{E} (f|\mathcal{G}) (x_1, x_2) = \int_{\Omega_2} f (x_1, y) \, d\mathbb{P}_2 (y) \) \(\mathbb{P}\)-a.e. on \(\Omega\). We also note that \(\int_{\Omega_2} |f (x_1, y)| \, d\mathbb{P}_2 (y) < \infty \) \(\mathbb{P}\)-a.e. on \(\Omega\) is equivalent to \(\int_{\Omega_2} |f (x_1, y)| \, d\mathbb{P}_2 (y) < \infty \) \(\mathbb{P}_1\)-a.e. on \(\Omega_1\).

We give the following characterization of conditional expectation:

**Proposition 1.5.6**

(i) If \(f \in \text{dom} \mathbb{E} (\cdot | \mathcal{G})\) and \(g \in L^0 (\Omega, \mathcal{G}, \mathbb{P})\), then \(gf \in \text{dom} \mathbb{E} (\cdot | \mathcal{G})\) and

\[ \mathbb{E} (gf|\mathcal{G}) = g \mathbb{E} (f|\mathcal{G}) \]

(ii) If \(f \in L^0 (\Omega, \mathcal{F}, \mathbb{P})\), then \(f \in \text{dom} \mathbb{E} (\cdot | \mathcal{G})\) if and only if there is a sequence \(\{A_n\}\) in \(\mathcal{G}\) such that \(A_n \uparrow \Omega\) and

\[ \int_{A_n} |f| \, d\mathbb{P} < \infty \text{ for } n = 1, 2, \cdots. \]
Moreover, if \( f \in \text{domE}(\cdot|\mathcal{G}) \), then

\[
\int_A E(f|\mathcal{G})\,d\mathbb{P} = \int_A f\,d\mathbb{P}
\]

for all \( A \in \mathcal{G} \) with \( \int_A |f|\,d\mathbb{P} < \infty \).

**Proof** (i) Since \( E(gf|\mathcal{G}) = gE(f|\mathcal{G}) \) for all \( f \in M^+(\Omega, \mathfrak{F}, \mathbb{P}) \) and \( g \in M^+(\Omega, \mathfrak{G}, \mathbb{P}) \) we get that

\[
E(|gf|\mathcal{G}) = |g|E(|f|\mathcal{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P}).
\]

This implies that \( gf \in \text{domE}(\cdot|\mathcal{G}) \). If we take \( g = g^+ - g^- \) and \( f = f^+ - f^- \) we have that \( gf = (g^+ - g^-)(f^+ - f^-) \) and it then follows that

\[
E(gf|\mathcal{G}) = gE(f|\mathcal{G}).
\]

(ii) We first assume that there exists a sequence \( A_n \subset \mathcal{G} \) such that

\( A_n \uparrow \Omega \) and \( \int_{A_n} |f|\,d\mathbb{P} < \infty \) for \( n = 1, 2, \cdots \). We have that if \( f \in M^+(\Omega, \mathfrak{F}, \mathbb{P}) \) then \( \int_A E(f|\mathcal{G})\,d\mathbb{P} = \int_A f\,d\mathbb{P} \) for all \( A \in \mathcal{G} \). It then follows that

\[
\int_{A_n} E(|f|\mathcal{G})\,d\mathbb{P} = \int_{A_n} |f|\,d\mathbb{P} < \infty \text{ for } n = 1, 2, \cdots.
\]

Hence \( E(|f|\mathcal{G}) < \infty \mathbb{P}-\text{a.e. on } A_n \) for all \( n = 1, 2, \cdots \). Since \( A_n \uparrow \Omega \), this implies that \( E(|f|\mathcal{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P}) \), i.e., \( f \in \text{domE}(\cdot|\mathcal{G}) \).

Conversely, assume that \( f \in \text{domE}(\cdot|\mathcal{G}) \). Then \( E(|f|\mathcal{G}) \) is in \( L^0(\Omega, \mathfrak{G}, \mathbb{P}) \) and so there is a sequence \( A_n \in \mathcal{G} \) with \( A_n \uparrow \Omega \) and

\[
\int_{A_n} |f|\,d\mathbb{P} = \int_{A_n} E(|f|\mathcal{G})\,d\mathbb{P} < \infty \text{ for } n = 1, 2, \cdots.
\]
Since \( E(f|\mathcal{G}) \), \( E(f^+|\mathcal{G}) \) and \( E(f^-|\mathcal{G}) \) are all less than or equal to \( E(f|\mathcal{G}) \) it then follows that \( E(f|\mathcal{G}) \) is integrable over \( A \) and that

\[
\int_A E(f|\mathcal{G}) \, d\mathbb{P} = \int_A E(f^+|\mathcal{G}) \, d\mathbb{P} - \int_A E(f^-|\mathcal{G}) \, d\mathbb{P}
\]

\[
= \int_A f^+ \, d\mathbb{P} - \int_A f^- \, d\mathbb{P}
\]

\[
= \int_A f \, d\mathbb{P}.
\]

\( \square \)

### 1.6 Kernel Operators

In this section we give an introduction into kernel operators and a summary of their properties. We will mention only those concepts that will be useful later. We refer the reader to [33, sections 93, 94 and 95] as well as [25].

Let \((X, \Sigma, \mu)\) and \((Y, \Lambda, \nu)\) be \(\sigma\)-finite measure spaces and let \( t(x, y) \) be a real valued \( \mu \otimes \nu \)-measurable function on \( X \times Y \), where \( \mu \otimes \nu \) denotes the product measure of \( \mu \) and \( \nu \). For any \( f \in L^0(Y, \Lambda, \nu) \) the function \( t(x, y) f(y) \) is \( \mu \otimes \nu \)-measurable, which implies that for almost every \( x \in X \) the function \( t(x, y) f(y) \) is \( \nu \)-measurable as a function of \( y \). It then follows that

\[
h(x) = \int_Y |t(x, y) f(y)| \, d\nu(y)
\]

(1.6.1)

makes sense for all \( x \in X \) such that \( t(x, y) f(y) \) is \( \nu \)-measurable as a function of \( y \). By Fubini's theorem \( h(x) \) is a \( \mu \)-measurable function on \( X \). The set of all \( f \in L^0(Y, \Lambda, \nu) \) for which the function \( h(x) \) is finite valued \( \mu \)-a.e. on \( X \)
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will be called the $Y$-domain of $t(x, y)$ and will be denoted by $\text{dom}_Y(t)$. For $f \in \text{dom}_Y(t)$ the function $h$ is finite $\mu$-a.e. on $X$ and so

$$g(x) = \int_Y t(x, y) f(y) \, d\nu(y)$$

is also finite $\mu$-a.e on $X$. The function $g$ is $\mu$-measurable since

$$g(x) = \int_Y [t(x, y) f(y)]^+ \, d\nu(y) - \int_Y [t(x, y) f(y)]^- \, d\nu(y), \quad (1.6.2)$$

where, $\int_Y [t(x, y) f(y)]^+ \, d\nu(y)$ and $\int_Y [t(x, y) f(y)]^- \, d\nu(y)$ are, by Fubini's theorem, $\mu$-measurable. Equation (1.6.2) then defines a linear operator $T : f \to g$ from $\text{dom}_Y(t)$ into $L^0(X, \Sigma, \mu)$. The operator $T$ is called a kernel operator or integral operator. The function $t(x, y)$ is called the kernel of $T$.

The set of kernel operators with kernel $k$ will be denoted by $\mathcal{L}_k(L, M)$.

If $T$ is a kernel operator with kernel $t(x, y)$ and if $L$ and $M$ are ideals in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$, respectively, then $T$ is said to be a kernel operator from $L$ into $M$ if $L \subset \text{dom}_Y(t)$ and $\int_Y t(x, y) f(y) \, d\nu(y) \in M$, for all $f \in L$. In this case $\int_Y |t(x, y) f(y)| \, d\nu(y) \in M$ and

$$(T_1 f)(x) = \int_Y t^+(x, y) f(y) \, d\nu(y)$$

$$(T_2 f)(x) = \int_Y t^-(x, y) f(y) \, d\nu(y)$$

are also kernel operators from $L$ into $M$. Hence we have that $T = T_1 - T_2$ and $T_1$ and $T_2$ are positive operators. Thus the set of all kernel operators from $L$ into $M$ is a linear subspace of $\mathcal{L}_k(L, M)$.

We extend this notion to the concept of $\tau$-kernel operators. For this we refer the reader to [8, Section 5]
Let \((X, \Sigma, \mu), (Y, \Lambda, \nu)\) and \((Z, \Gamma, \gamma)\) be \(\sigma\)-finite measure spaces and suppose that \(L \subseteq L^0(Y, \Lambda, \nu)\) and \(M \subseteq L^0(X, \Sigma, \mu)\) are ideals with carriers \(Y\) and \(X\) respectively. Let \(\tau : X \times Z \to Y\) be a \((\Sigma \otimes \Gamma, \Lambda)\)-measurable null-preserving mapping with respect to \(\mu \otimes \gamma\) and \(\nu\).

**Definition 1.6.1** A function \(k \in L^0(X \times Z, \Sigma \otimes \Gamma, \mu \otimes \gamma)\) is called an absolute \(\tau\)-kernel for \(L\) and \(M\) if

\[
\int_Z |k(\cdot, z) f(\tau(\cdot, z))| d\gamma(z) \in M
\]

for all \(f \in L\).

The collection of all such \(\tau\)-kernels will be denoted by \(\mathcal{R}(L, M)\). For \(k \in \mathcal{R}(L, M)\) and \(f \in L\) the operator

\[
Kf(x) = \int_Z k(x, z) f(\tau(x, z)) d\gamma(z)
\]

is \(\mu\)-a.e well defined on \(X\) and \(Kf \in M\). This defines a linear, order bounded, order continuous operator \(K\) from \(L\) into \(M\), i.e., \(K \in \mathcal{L}_n(L, M)\). The operator \(K\) thus defined is called an absolute \(\tau\)-kernel operator with kernel \(k(x, z)\). We will denote the collection of absolute \(\tau\)-kernel operators by \(\mathcal{L}_{\tau,k}(L, M)\). We have that \(\mathcal{L}_{\tau,k}(L, M) \subseteq \mathcal{L}_n(L, M)\) and that if \(k \geq 0\), \(\mu \otimes \gamma\)-a.e. on \(X \times Z\) then \(K \geq 0\).

We note that if \((Z, \Gamma, \gamma) = (Y, \Lambda, \nu)\) and we choose \(\tau(x, z) = z\) then the absolute \(\tau\)-kernel operator \(K\) is an absolute kernel operator.

As an example of an absolute \(\tau\)-kernel operator we consider the partial integral operator as introduced by Kalitvin and Zabrejko in [12] and the
previous mentioned authors together with Appell in [4]. The reader is also referred to [3], [2], and [11]:

Consider \( \sigma \)-finite measure spaces \((X, \Sigma, \mu)\) and \((\overline{Y}, \Lambda, \nu)\). Let \( L \) and \( M \) be ideals in \( L^0(\overline{X} \times \overline{Y}, \Sigma \otimes \Lambda, \mu \otimes \nu) \) with carrier \( \overline{X} \times \overline{Y} \).

**Definition 1.6.2** For a function \( k \in L^0(\overline{X} \times \overline{Y} \times \overline{X}, \Sigma \otimes \Lambda \otimes \Sigma, \mu \otimes \nu \otimes \mu) \) such that

\[
\int_{\overline{X}} |k(\cdot, \cdot, z)| f(\cdot, z) \, d\mu(z) \in M \quad \text{for all } f \in L
\]

we define the operator \( K : L \to M \) by

\[
K(f)(x, y) = \int_{\overline{X}} |k(x, y, z)| f(x, z) \, d\mu(z) \quad \mu \otimes \nu \text{ a.e on } \overline{X} \times \overline{Y} \text{ for all } f \in L.
\]

(1.6.3)

We call such an operator \( K \) an **absolute partial integral operator**. We will denote the collection of all partial integral operators of the form (1.6.3) by \( \mathcal{P}_X(L, M) \).

If in the above definition we put \( X = Y = \overline{X} \times \overline{Y}, Z = \overline{X} \) and \( \tau : (X \times Z \to Y) \) defined by \( \tau(x, y, z) = (z, y) \) we get that partial integral operators defined by (1.6.3) become \( \tau \)-kernel operators.

Similarly, if \( t \in L^0(\overline{X} \times \overline{Y} \times \overline{Y}, \Sigma \otimes \Lambda \otimes \Lambda, \mu \otimes \nu \otimes \nu) \) is such that

\[
\int_{\overline{Y}} |t(\cdot, \cdot, z)| f(\cdot, z) \, d\nu(z) \in M \quad \text{for all } f \in L,
\]

The corresponding partial integral operator is defined by

\[
T(f)(x, y) = \int_{\overline{Y}} |t(x, y, z)| f(x, z) \, d\nu(z) \quad \mu \otimes \nu \text{ a.e on } \overline{X} \times \overline{Y} \text{ for all } f \in L.
\]

(1.6.4)
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We will denote the collection of all operators of the form (1.6.4) by \( \mathcal{P}_F(L, M) \). Again these operators are \( \tau \)-integral operators if we take \( X = Y = \overline{X} \times \overline{Y}, Z = \overline{Y} \) and \( \tau : X \times Z \to Y \) defined by \( \tau(x,y,z) = (x,z) \).

We list a number of properties of absolute kernel operators proofs of which can be found in [33, Chapter 13] and [25].

**Theorem 1.6.3** Let \( (Y, \Lambda, \nu) \) and \( (X, \Sigma, \mu) \) be \( \sigma \)-finite spaces, \( L \) and \( M \) ideals in \( L^0(Y, \Lambda, \nu) \) and \( L^0(X, \Sigma, \mu) \), respectively. Denote the carrier of \( L \) by \( Y_L \). If \( T \) is a kernel operator from \( L \) into \( M \) with kernel \( t(x,y) \) then the following hold:

(i) \( T \) is positive if and only if \( t(x,y) \geq 0 \) \( \mu \otimes \nu \)-a.e. on \( X \times Y_L \)

(ii) \( T \) is the null operator, 0, if and only if \( t(x,y) = 0 \) \( \mu \otimes \nu \)-a.e. on \( X \times Y_L \)

(iii) If \( 0 \leq S \leq T \) in \( \mathcal{L}_k(L, M) \) then \( S \) is a kernel operator.

(iv) If \( S \) and \( T \) are kernel operators from \( L \) to \( M \) with kernels \( s(x,y) \) and \( t(x,y) \), respectively, then \( \sup(T, S) \) is a kernel operator with its kernel given by the pointwise a.e. supremum of \( s(x,y) \) and \( t(x,y) \).

(v) The set \( \mathcal{L}_k(L, M) \) is a band in \( \mathcal{L}_b(L, M) \).

(vi) If \( Y \) does not contain any atom then the identity operator \( I \) is disjoint to the band \( \mathcal{L}_k(L, L) \) in \( \mathcal{L}_b(L, L) \). It then follows that \( I \) is not a kernel operator.

Again let \( (Y, \Lambda, \nu) \) be a \( \sigma \)-finite measure space and \( L \) an ideal in \( L^0(Y, \Lambda, \nu) \) such that the carrier of \( L \) is \( Y \). Let \( L^\sim_n \) be the band of order continuous
linear functionals on $L$ which we identify, as usual with $L'$ as discussed in page 22. Let $(X, \Sigma, \mu)$ be another $\sigma$-finite measure space and $M$ be an ideal in $L^0(X, \Sigma, \mu)$. Consider the functions $g \in L'$ and $h \in M$, then the $\mu \otimes \nu-$measurable function $t(x, y) = h(x)g(y)$ is the kernel of an absolute kernel operator from $L$ into $M$, for, for every $f \in L$

$$\int_Y |t(x, y)f(y)|d\nu(y) = |h(x)|\int_Y |g(y)f(y)|d\nu(y) \in M.$$ 

Any finite linear combination of kernel operators of this elementary kind is called a kernel operator of finite rank. We will use the notation $L_n^\sim \otimes M$ for the set of kernel operators of finite rank. The set $L_n^\sim \otimes M$ is a linear subspace of $L_b(L, M)$. The band generated by $L_n^\sim \otimes M$ in $L_b(L, M)$ equals $(L_n^\sim \otimes M)^{dd}$. The band $(L_n^\sim \otimes M)^{dd}$ is contained in the band of all absolute kernel operators since $L_n^\sim \otimes M$ consists of absolute kernel operators.

The following is from [33]:

**Theorem 1.6.4** If the carrier of $L'$ is the whole of $Y$ then $L_b(L, M) = (L_n^\sim \otimes M)^{dd}$, i.e., the set of absolute kernel operators is the band generated by the kernel operators of finite rank.

We now take a look at continuity of kernel operators. We assume that $Y$ is the carrier of both $L$ and $L'$. Given a sequence $f_n$ of measurable functions in $L$ and a measurable subset $E$ of $Y$. Let $\epsilon > 0$ be given and put $\overline{E} := \{y \in E : |f_n(y)| \geq \epsilon\}$. We say that $(f_n)$ converges in measure on $E$ to $0$ if

$$\lim_{n \to \infty} \nu(\overline{E}) = 0$$
for every $\epsilon > 0$. It is said that $(f_n)$ star-converges to 0 if every subsequence of $(f_n)$ contains a subsequence that converges pointwise to 0 a.e. on $Y$. We will denote this by $f_n \overset{\ast}{\rightharpoondown} 0$. We state the following theorem, taken from [33, 25], that shows the connection between convergence in measure and star-convergence.

**Theorem 1.6.5** Let $0 \leq u_n \leq u$ in $L$. Then the following are equivalent.

(a) $u_n \rightharpoondown 0$ as $n \to \infty$.

(b) $(u_n)$ converges to 0 in measure on every subset of $Y$ of finite measure.

(c) For every $E \subset Y$ such that $1_E \in L'$ we have

$$\int_E u_n \, dv \to 0 \text{ as } n \to \infty.$$ 

We note that the conditions (a), (b) and (c) in the previous theorem are weaker than the condition that $u_n(y) \to 0$ a.e. on $Y$. We state the following result that relates kernel operators to star-convergence, again see [33, 25].

**Theorem 1.6.6 (Bukhvalov)** For a positive linear operator $T : L \to M$ and a sequence $0 \leq u_n \leq u$ in $L$ the following are equivalent:

(a) $T$ is a kernel operator

(b) $u_n \rightharpoondown 0$ implies that $Tu_n(x) \to 0$ a.e. on $X$.

The condition (b) in the above theorem can be weakened further. We state the following as in [25].
Theorem 1.6.7 For a positive linear operator $T : L \to M$, a sequence $0 \leq u_n \leq u$ in $L$ and a sequence of sets $(A_n)$ in $Y$ we have that $T$ is a kernel operator if and only if

(i) $u_n(y) \to 0$ for almost every $y$ in $Y$ implies that $Tu_n(x) \to 0$ for almost every $x$ in $X$.

(ii) $0 \leq 1_{A_n} \leq u$ in $L$ and $\nu(A_n) \to 0$ implies that $T1_{A_n}(x) \to 0$ for every $x$ in $X$. 
Chapter 2

MCE operators

In this chapter we explore the notion of Multiplication Conditional Expectation operators (MCE operators) and their characteristics. We will look at those operators that can be represented as Multiplication Conditional Expectation operators or (MCE-representable operators). We will show that Riesz homomorphisms and Conditional Expectation operators are the most fundamental operators in the theory of Riesz spaces mainly because a large class of operators can be expressed as products of Riesz homomorphisms and Conditional Expectation operators.

Grobler and de Pagter in [8] defined MCE-representable operators using Riesz homomorphisms that operate between $\sigma$-finite measure spaces. We give an alternate definition of MCE-representable operators, thus in a way, giving necessary and sufficient conditions for Riesz homomorphisms and ideals in those $\sigma$-finite measure spaces to be used to define MCE-representable operators.

We first have a look at these operators and their properties on ideals of
measurable spaces. The bulk of this is from [8].

2.1 MCE operators on ideals

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(\mathcal{F}_L\) and \(\mathcal{F}_M\) be sub-\(\sigma\)-algebras of \(\mathcal{F}\) and let \(L \subset L^0(\Omega, \mathcal{F}_L, \mathbb{P})\) and \(M \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P})\) be ideals, both with carriers \(\Omega\). Put

\[
\mathcal{M}(L, M) = \{m \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : \mathbb{E}(|mf||\mathcal{F}_M) \in M \forall f \in L\}.
\]

For all \(m \in \mathcal{M}(L, M)\) and \(f \in L\) we have that \(mf \in \text{dom}\mathbb{E}(\cdot|\mathcal{F}_M)\) since \(M \subset L^0(X, \Sigma, \mu)\). For any \(m\) in \(\mathcal{M}(L, M)\) define an operator \(S_m : L \rightarrow M\) by

\[
S_m f = \mathbb{E}(mf|\mathcal{F}_M) \quad \forall f \in L.
\]

We have that \(\mathcal{M}(L, M)\) is an ideal in \(L^0(\Omega, \mathcal{F}, \mathbb{P})\) since \(\text{dom}\mathbb{E}(\cdot|\mathcal{F}_M)\) is an ideal in \(L^0(\Omega, \mathcal{F}, \mathbb{P})\). Indeed if, \(0 \leq |p| \leq |m|, m \in \mathcal{M}(L, M)\) then

\[
\mathbb{E}(|pf||\mathcal{F}_M) \leq \mathbb{E}(|mf||\mathcal{F}_M) \in M,
\]

thus \(\mathbb{E}(|pf||\mathcal{F}_M) \in M\), so \(p \in \mathcal{M}(L, M)\). Thus \(\mathcal{M}(L, M)\) is solid and hence an ideal.

We have that \(S_m\) is a well defined operator. If \(0 \leq m \in \mathcal{M}(M, L)\) then \(S_m \geq 0\) and if \(m \in \mathcal{M}(M, L)\) then

\[
|S_m f| = |\mathbb{E}(mf|\mathcal{G})| \\
\leq \mathbb{E}(|m||f||\mathcal{G}) \\
= S_m |f|
\]
and so

$$|S_m| \leq S_{m^i}.$$ 

Hence $S_m$ is order bounded.

Define $L_m(L, M) = \{S_m \mid m \in \mathcal{M}(L, M)\}$. We have thus constructed a space of functions $\mathcal{M}(L, M) \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ such that for an $m \in \mathcal{M}(L, M)$ there exists an operator $S_m$ which is order bounded and order continuous, hence $S_m \in L_m(L, M) \subset L^0_{n}(L, M)$. The mapping $m \mapsto S_m$ is positive, i.e., $m \geq 0 \Rightarrow S_m \geq 0$.

The following property is from Grobler and de Pagter, [8].

**Proposition 2.1.1** Let $\mathcal{F}_0 = \sigma(\mathcal{F}_M, \mathcal{F}_L)$ be $\sigma$-algebra generated by $\mathcal{F}_M$ and $\mathcal{F}_L$. Let $\mathcal{M}_0(L, M) = L^0(\Omega, \mathcal{F}_0, \mathbb{P}) \cap \mathcal{M}(L, M)$. Then we have

$$\mathcal{L}_{\mathcal{M}}(L, M) = \mathcal{L}_{\mathcal{M}_0}(L, M).$$

**Proof** It will suffice to show that given $0 \leq m \in \mathcal{M}(M, L)$ there exists $0 \leq m_0 \in \mathcal{M}_0(L, M)$ such that

$$S_m f = S_{m_0} f \text{ for all } 0 \leq f \in L.$$ 

Let $0 \leq m \in \mathcal{M}(M, L)$ be given and set

$$m_0 = \mathbb{E}(m|\mathcal{F}_0) \in M^+(\Omega, \mathcal{F}_0, \mathbb{P}).$$

For $0 \leq f \in L$ we have

$$S_m f = \mathbb{E}(mf|\mathcal{F}_M)$$

$$= \mathbb{E}(\mathbb{E}(mf|\mathcal{F}_0)|\mathcal{F}_M).$$
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Note that \( f \) is \( \mathcal{F}_M \) measurable and \( \mathcal{F}_M \subset \mathcal{F}_0 \). Thus \( f \) is \( \mathcal{F}_0 \) measurable and so \( \mathbb{E}(mf|\mathcal{F}_0) = \mathbb{E}(m|\mathcal{F}_0)f \). This gives us that

\[
S_m f = \mathbb{E}(\mathbb{E}(m|\mathcal{F}_0)f|\mathcal{F}_M) = \mathbb{E}(m_0 f|\mathcal{F}_M) = S_{m_0} f \text{ for all } 0 \leq f \in L.
\]

In particular, \( \mathbb{E}(m_0 f|\mathcal{F}_M) \in L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \) and thus \( m_0 f \in L^0(\Omega, \mathcal{F}_0, \mathbb{P}) \) for all \( 0 \leq f \in L \). Since the carrier of \( L \) is \( \Omega \), this implies that \( 0 \leq m_0 \in L^0(\Omega, \mathcal{F}_0, \mathbb{P}) \). We then have that \( 0 \leq m_0 \in \mathcal{M}_0(L, M) \) and \( S_m f = S_{m_0} f \) for all \( 0 \leq f \in L \).

This result shows that, for the study of the space \( \mathcal{L}_{\mathcal{M}} \), we may assume, without loss of generality, that \( \mathcal{F} = \sigma(\mathcal{F}_M, \mathcal{F}_L) \).

Consider a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and two sub-algebras of \( \mathcal{F}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \), such that \( \mathcal{F} = \sigma(\mathcal{F}_1, \mathcal{F}_2) \). Let

\[
\Gamma = \{ A \cap B \mid A \in \mathcal{F}_1, B \in \mathcal{F}_2 \}.
\]

Since \( \mathcal{F}_1, \mathcal{F}_2 \subset \Gamma \), we have that \( \sigma(\Gamma) = \mathcal{F} \) and \( \Gamma \) is a semi-ring. Therefore \( \mathcal{F} \) is a monotone class generated by the finite union of sets in \( \Gamma \). Thus if \( f \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) such that \( \int_G f d\mathbb{P} \geq 0 \) for all \( G \in \Gamma \) it follows that \( \int_E f d\mathbb{P} \geq 0 \) for all \( E \in \mathcal{F} \) since \( \mathcal{M} = \{ C \in \mathcal{F} \mid \int_C f d\mathbb{P} \geq 0 \} \) is a monotone class which contains all finite disjoint unions of sets in \( \Gamma \). Therefore, if \( \int_{A \cap B} f d\mathbb{P} \geq 0 \), for all \( A \in \mathcal{F}_1 \) and \( B \in \mathcal{F}_2 \), then \( f \geq 0 \) \( \mathbb{P} \)-a.e. on \( \Gamma \).

The following result is also from [8].
Lemma 2.1.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $L \subset L^0(\Omega, \mathcal{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P})$ be order dense ideals of measurable functions and suppose that $\mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M)$. For $m \in \mathcal{M}(L, M)$ we have

(i) if $S_{m^*} \geq 0$ then $m \geq 0 \text{ P-a.e. on } \Omega$,

(ii) if $S_{m^*} = 0$ then $m = 0 \text{ P-a.e. on } \Omega$.

Proof

(i) Put $A_0 \in \mathcal{F}_L$ such that $1_{A_0} \in L$. Let $B_0 \in \mathcal{F}_M$ be such that $\int_{B_0} |m1_{A_0}|d\mathbb{P} < \infty$, this is possible since the carrier of $L$ is the whole of $\Omega$.

We show that $m \geq 0$ P-a.e. on $A_0 \cap B_0$. For $A \in \mathcal{F}_L$ with $A \subset A_0$ and $B \in \mathcal{F}_M$ with $B \subset B_0$ we have that

$$\int_B |m1_A|d\mathbb{P} < \infty.$$ 

Thus

$$\int_B m1_A d\mathbb{P} = \int_B \mathbb{E}(m1_A|\mathcal{F}_M) d\mathbb{P}$$

$$= \int_B S_m(1) d\mathbb{P} \geq 0.$$

This shows that $\int_{A \cap B} md\mathbb{P} \geq 0$ for all such $A$ and $B$.

Now define the following $\sigma$-algebras of subsets of $A_0 \cap B_0$

$$\mathcal{F}_M^0 = \{A_0 \cap B \mid B \in \mathcal{F}_M, B \subseteq B_0\},$$

$$\mathcal{F}_L^0 = \{A \cap B_0 \mid A \in \mathcal{F}_L, A \subseteq A_0\},$$

$$\mathcal{F}^0 = \{C \in \mathcal{F} \mid C \subseteq A_0 \cap B_0\}.$$
We have that $\sigma(\mathcal{F}_M^0, \mathcal{F}_L^0) = \mathcal{F}^0$ and $\int_{A \cap B} md\mathbb{P} \geq 0$ for all $A \in \mathcal{F}_L^0$ and $B \in \mathcal{F}_M^0$.

From the remarks preceding this lemma, it follows that $\int C \, md\mathbb{P} \geq 0$ for all $C \in \mathcal{F}^0$, i.e., $m \geq 0$ $\mathbb{P}$-a.e. on $A_0 \cap B_0$. Now fix $A_0 \in \mathcal{F}_L$ such that $1_{A_0} \in L$. Since $m1_{A_0} \in \text{dom} \mathbb{E}(\cdot | \mathcal{F}_M)$, by Proposition 1.5.6 on page 30, there is a sequence $B_n \in \mathcal{F}_M$ with $B_n \uparrow \Omega$ such that $\int_{B_n} |m1_{A_0}| d\mathbb{P} < \infty$. We have that $m \geq 0$ $\mathbb{P}$-a.e. on $B_n \cap A_0$ for each $n = 1, 2, \ldots$. Thus $m \geq 0$ $\mathbb{P}$-a.e. on $\Omega \cap A_0$ since $\bigcup B_n = \Omega$. Hence $m \geq 0$ $\mathbb{P}$-a.e. on $A_0$. Since the carrier of $L$ is $\Omega$ there exists a sequence $A_n \in \mathcal{F}_L$ such that $A_n \uparrow \Omega$ and $1_{A_0} \in L$. Therefore $m \geq 0$ $\mathbb{P}$-a.e. on $A_n$, and hence, $m \geq 0$ $\mathbb{P}$-a.e. on $\Omega$.

(ii) From $m \mapsto S_m$ we get that $-m \mapsto S_{-m}$. Thus $\mathbb{E}(-mf|\mathcal{F}_M) = -\mathbb{E}(mf|\mathcal{F}_M)$. We already have that if $S_m \geq 0$ then $m \geq 0$. Now let $S_m = 0$, then $-S_m \geq 0$. This implies that $S_{-m} \geq 0$ and so $-m \geq 0$, which gives that $m \leq 0$. Thus $m = 0$. 

We also look at this result which is also from [8]:

Lemma 2.1.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $L \subset L^0(\Omega, \mathcal{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P})$ be ideals of measurable functions. If $S_\alpha \in \mathcal{L}(L, M)$ is an upwards directed net and if $0 \leq S_\alpha \uparrow S$ in $L_+(L, M)$ then $S \in \mathcal{L}(L, M)$.

Proof Without loss of generality, we may assume that $\mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M)$. Write $S_\alpha = S_{m_\alpha}$ with $m_\alpha \in \mathcal{M}(L, M)$. By the preceeding proposition we have that $0 \leq m_\alpha \uparrow$ in $M^+(\Omega, \mathcal{F}, \mathbb{P})$. Then $m := \sup_\alpha m_\alpha$ exists in
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$M^+(\Omega, \mathcal{F}, \mathbb{P})$ and there exists a sequence $(m_n)$ composed of terms of
$(m_n)$ such that $m_n \uparrow m$ (this is a slight variation of Lemma 94.4 in [33]).

For each $0 \leq f \in L$ we have

$$Sf \geq S_{m_n}f = \mathbb{E}(m_n f | \mathcal{F}_M) \uparrow \mathbb{E}(mf | \mathcal{F}_M) \in M^+(\Omega, \mathcal{F}_M, \mathbb{P}).$$

Now, $Sf \in M \subseteq L^0(\Omega, \mathcal{F}_M, \mathbb{P})$ implies $\mathbb{E}(mf | \mathcal{F}_M) \in L^0(\Omega, \mathcal{F}_M, \mathbb{P})$ and consequently, $mf \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. Since $L$ has carrier $\Omega$, this implies that $m \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ and hence $m \in \mathcal{M}(L, M)$. By Proposition 1.5.3 (iv), $m_n \uparrow m$ in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ implies that $S_{m_n} \uparrow S_m$ in $\mathcal{L}_b(L, M)$ and hence $S_m \leq S$. On the other hand, $m \geq m_\alpha$ and so $S_m \geq S_{m_\alpha}$ for all $\alpha$ and thus $S_m \geq S$. Hence, $S = S_m \in \mathcal{L}_m(L, M)$. \hfill \Box

The following result is also found in [8]

**Lemma 2.1.4** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $L \subseteq L^0(\Omega, \mathcal{F}_L, \mathbb{P})$
and $M \subseteq L^0(\Omega, \mathcal{F}_M, \mathbb{P})$ be ideals of measurable functions. If $S \in \mathcal{L}_n(L, M)$
such that $0 \leq S \leq S_m$ for some $m \in \mathcal{M}(L, M)$, then $S \in \mathcal{L}_m(L, M)$.

**Proof** We first prove the proposition under the additional assumption that
$\mathcal{F}_L = \mathcal{F}$. Then $L$ is an ideal in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ and the operator $S_m$ is interval
preserving (i.e., it has the Maharam property). Indeed, if $f \in L$ and $0 \leq g \leq S_m f$ in $M$, define the function $\sigma$ by

$$\sigma = \begin{cases} 
g/S_m f & \text{if } S_m f \neq 0 \\
0 & \text{otherwise.} \end{cases}$$
Since $0 \leq \sigma \leq 1$ and $\sigma \in L^0(\Omega, \mathcal{F}_M, \mathbb{P})$ we have $0 \leq \sigma f \leq f$ in $L$ and

$$S_m(\sigma f) = \mathbb{E}(\sigma mf | \mathcal{F}_M) = \sigma \mathbb{E}(mf | \mathcal{F}_M) = \sigma S_m f = g.$$ 

It follows from the Luxemburg-Schep Radon-Nikodym theorem (see [18]) that $S = S_m \pi$ for some $0 \leq \pi \leq I$ in the centre $Z(L)$ of $L$. Now $\pi$ is multiplication by some function $0 \leq p \leq 1$ on $\Omega$ (see [33] Example 141.3) and, since $\mathcal{M}(L, M)$ is an ideal in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, it follows that $S = S_{pm} \in \mathcal{L}\mathcal{M}(L, M)$.

For general $L$, let $I(L)$ be the ideal generated in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ by $L$. We have that $\mathcal{M}(I(L), M) = \mathcal{M}(L, M)$. Indeed, if $m \in \mathcal{M}(I(L), M)$ then $\mathbb{E}(|mg| | \mathcal{F}_M) \in M$ for all $g \in I(L)$. But, since $I(L) \supset L$, $\mathbb{E}(|mg| | \mathcal{F}_M) \in M$ holds for all $g \in L$; hence $m \in \mathcal{M}(L, M)$. On the other hand, if $m \in \mathcal{M}(L, M)$ then $\mathbb{E}(|mf| | \mathcal{F}_M) \in \mathcal{F}_M$ for all $f \in L$. Let $g \in I(L)$, then there is some $f \in L$ such that $|g| \leq |f|$. This then gives us that $\mathbb{E}(|mg| | \mathcal{F}_M) \leq \mathbb{E}(|mf| | \mathcal{F}_M) \in M$. Thus $m \in \mathcal{M}(I(L), M)$. Define $\bar{S}_m : I(L) \to M$ by $\bar{S}_m f = \mathbb{E}(mf | \mathcal{F}_M)$ for all $f \in I(L)$. Then $\bar{S}_m$ is an extension of $S_m$ and $\bar{S}_m \in \mathcal{L}\mathcal{M}(I(L), M)$. It follows from the Kantorovich extension theorem (see [1] Theorem I.2.2), that there exists an extension $0 \leq \bar{S} \in L_0(I(L), M)$ of $S$ such that $0 \leq \bar{S} \leq \bar{S}_m$. By the first part of the proof $\bar{S} \in \mathcal{L}\mathcal{M}(I(L), M)$. Since $\mathcal{M}(I(L), M) =$
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\[ \mathcal{M}(L, M), \] the restriction of \( \mathcal{F} \) to \( L \) belongs to \( \mathcal{L}(L, M) \), i.e., \( S \in \mathcal{L}(L, M) \). This completes the proof. 

We combine the three previous results to prove the following, as was done by Grobler and de Pagter in [8].

\textbf{Proposition 2.1.5} Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and ideals of measurable functions \( L \subset L^0(\Omega, \mathcal{F}_L, \mathbb{P}) \) and \( M \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P}). \) Then \( \mathcal{L}(L, M) \) is a band in \( \mathcal{L}(L, M) \). Moreover, assuming that \( \mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M) \), the mapping \( m \rightarrow S_m \) is a Riesz isomorphism from the ideal \( \mathcal{M}(L, M) \) onto \( \mathcal{L}(L, M) \).

\textbf{Proof} First we show that \( \mathcal{L}(L, M) \) is a Riesz subspace of \( \mathcal{L}(L, M) \). Take \( S_m \in \mathcal{L}(L, M) \). As observed earlier, \( |S_m| \leq S_m \). By Lemma 2.1.4 this implies that \( |S_m| \in \mathcal{L}(L, M) \). Hence, \( \mathcal{L}(L, M) \) is a Riesz subspace. Using the above Lemma 2.1.4 once more, we see that \( \mathcal{L}(L, M) \) is actually an ideal in \( \mathcal{L}(L, M) \). Moreover, Lemma 2.1.3 yields that \( \mathcal{L}(L, M) \) is a band in \( \mathcal{L}(L, M) \). Assuming that \( \mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M) \), it follows from Lemma 2.1.2 that the mapping \( m \rightarrow S_m \) is a bi-positive bijection from \( \mathcal{M}(L, M) \) onto \( \mathcal{L}(L, M) \), consequently this mapping is a Riesz isomorphism. 

\textbf{2.2 MCE-representable operators}

We now investigate a class of operators which factorizes through MCE operators. Our aim is to extend the MCE operators to the case of \( \sigma \)-finite measure
spaces. This class of operators includes operators such as kernel operators, order continuous Riesz homomorphisms as well as $\tau$-kernel operators.

We consider $\sigma$-finite measure spaces $(X, \Sigma, \mu)$ and $(Y, \Lambda, \nu)$. Let $L \subseteq L^0(Y, \Lambda, \nu)$ be an ideal with carrier $Y$ and $M \subseteq L^0(X, \Sigma, \mu)$ be an ideal with carrier $X$.

**Definition 2.2.1** A linear operator $T : L \to M$ is called Multiplication Conditional Expectation representable or (MCE-representable) if there exist

(i) a probability space $(\Omega, \mathcal{F}, P)$ and sub-$\sigma$-algebras $\mathcal{F}_L$ and $\mathcal{F}_M$ such that $\mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M)$,

(ii) order dense ideals $L_\Omega \subseteq L^0(\Omega, \mathcal{F}_L, P)$ and $M_\Omega \subseteq L^0(\Omega, \mathcal{F}_M, P)$ and order continuous Riesz homomorphisms $\phi_L : L \to L_\Omega$ and $\phi_M : M_\Omega \to M$ with $\phi_L$ surjective and $\phi_M$ a Riesz isomorphism onto an ideal in $M$ and

(iii) $m \in \mathcal{M}(L_\Omega, M_\Omega)$ such that

$$T = \psi_M s_m \phi_L,$$

i.e., such that the following diagram commutes

$$\begin{array}{ccc}
L & \xrightarrow{T} & M \\
\phi_L \downarrow & & \uparrow \psi_M \\
L_\Omega & \xrightarrow{s_m} & M_\Omega
\end{array}$$

$\Phi = ((\Omega, \mathcal{F}, P), \phi_L, \psi_M)$ is called a representation triple for the operator $T$ and $m$ a $\Phi$-kernel of $T$. The set of all linear operators $T : L \to M$ which
are MCE-representable via a fixed triple \( \Phi \) will be denoted by \( \mathcal{L}_\Phi(L, M) \) and we will also say that \( \Phi \) is a representation triple for the class \( \mathcal{L}_\Phi(L, M) \). An operator \( T \in \mathcal{L}_\Phi(L, M) \) will be called \( \Phi \)-representable.

Consider a \( \sigma \)-finite measure space \((X, \Sigma, \mu)\) and let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. If \( \phi \) is a \( \sigma \)-order continuous Riesz isomorphism from \( L^0(X, \Sigma, \mu) \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) such that \( \phi(1) = 1 \) then the restriction of \( \phi \) to the characteristic functions of elements of \( \Sigma \) gives a Boolean \( \sigma \)-homomorphism \( \hat{\phi} : \Sigma \to \mathcal{F} \) such that \( \mathbb{P}(\hat{\phi}(A)) = 0 \) if and only if \( \mu(A) = 0 \) for \( A \in \Sigma \). Such a mapping is called a bi-null preserving mapping or non-singular. If \( \mathcal{F}_L = \phi(\Sigma) \), then \( \mathcal{F}_L \) is a sub-\( \sigma \)-algebra of \( \mathcal{F} \) and \( \text{ran}(\phi) = L^0(\Omega, \mathcal{F}_L, \mathbb{P}) \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}) \).

**Lemma 2.2.2** Let \( L \) be an ideal in \( L^0(X, \Sigma, \mu) \) with carrier of \( L \) being \( X \) and \( \phi \) a \( \sigma \)-order continuous Riesz isomorphism from \( L^0(X, \Sigma, \mu) \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) such that \( \phi(1) = 1 \). Put

\[
L_\Omega = \phi(L) \subseteq L^0(\Omega, \mathcal{F}_L, \mathbb{P}) \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}).
\]

Then \( L_\Omega \) is order dense.

**Proof** Since \( \phi \) is a Riesz isomorphism from \( L^0(X, \Sigma, \mu) \) onto \( L^0(\Omega, \mathcal{F}_L, \mathbb{P}) \), we have, by Proposition 1.4.3 (iii), that \( L_\Omega \) is an ideal in \( L^0(\Omega, \mathcal{F}_L, \mathbb{P}) \).

We have that the carrier of \( L \) is \( X \) and therefore there exists a sequence of sets \( X_n \uparrow X \) such that \( 1_{X_n} \in L \) and so \( \phi(1_{X_n}) \in L_\Omega \). We have that \( 1_{X_n} \uparrow 1_X \) and since \( \phi \) is \( \sigma \)-order continuous we get that \( \phi(1_{X_n}) \uparrow \phi(1_X) \).

If we write \( \phi(1_{X_n}) = \phi(1_X)1_{\Omega_n} \in L_n \) we have that

\[
\Omega_n = \hat{\phi}(X_n) \uparrow \hat{\phi}(X) = \Omega.
\]
This implies that the carrier of \( L_\Omega \) is \( \Omega \) and so \( L_\Omega \) is order dense.

Grobler and de Pagter in [8] gave a somewhat different definition which is as follows.

**Definition 2.2.3** Let \( (X, \Sigma, \mu) \) and \( (Y, \Lambda, \nu) \) be \( \sigma \)-finite measure spaces and let \( L \subseteq L^0(Y, \Lambda, \nu) \) and \( M \subseteq L^0(X, \Sigma, \mu) \) be ideals with carriers \( Y \) and \( X \) respectively. A linear operator \( T : L \rightarrow M \) is called *Multiplication Conditional Expectation representable* or MCE-representable if there exist:

1. a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \);
2. a Riesz homomorphism \( \phi_L : L^0(Y, \Lambda, \nu) \rightarrow L^0(\Omega, \mathcal{F}, \mathbb{P}) \) with \( \phi_L(1) \mathbb{P} \)-a.e. strictly positive on \( \Omega \);
3. a sub-\( \sigma \)-algebra \( \mathcal{F}_M \subset \mathcal{F} \) and an interval preserving Riesz isomorphism \( \psi_M : L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \rightarrow L^0(X, \Sigma, \mu) \);
4. a function \( m \in \mathcal{M}(L_\Omega, M_\Omega) \),

such that:

\[
T = \psi_M S_m \phi_L,
\]

where \( L_\Omega = \phi_L(L) \), \( M_\Omega = \psi_M^{-1}(M) \) and \( S_m : L_\Omega \rightarrow M_\Omega \) is given by

\[
S_m f = \mathbb{E}(mf | \mathcal{F}_M).
\]

According to this definition the following diagram commutes

\[
\begin{array}{ccc}
L^0(Y, \Lambda, \nu) & \supset L & \xrightarrow{T} & M \subset L^0(X, \Sigma, \mu) \\
\phi_L \downarrow & & \uparrow \psi_M \\
L^0(\Omega, \mathcal{F}_L, \mathbb{P}) & \supset L_\Omega & \xrightarrow{s_m} & M_\Omega \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P})
\end{array}
\]
In the next two results we show that Definition 2.2.1 and Definition 2.2.3 are equivalent.

**Proposition 2.2.4** Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), sub-\(\sigma\)-algebras \(\mathcal{F}_L\) and \(\mathcal{F}_M\) such that \(\mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M)\). Let \(L_\Omega\) and \(M_\Omega\) be order dense ideals in \(L^0(\Omega, \mathcal{F}_L, \mathbb{P})\) and \(L^0(\Omega, \mathcal{F}_M, \mathbb{P})\), respectively, \(\phi_L : L \subseteq L^0(Y, \Lambda, \nu) \to L_\Omega\) and \(\psi_M : M_\Omega \to M \subseteq L^0(X, \Sigma, \mu)\) be order continuous Riesz homomorphisms with \(\phi_L\) surjective and \(\psi_M\) a Riesz isomorphism onto an ideal in \(M\). Then

(i) \(\phi_L\) and \(\psi_M\) can be extended to order continuous Riesz homomorphisms that respectively map \(L^0(Y, \Lambda, \nu)\) into \(L^0(\Omega, \mathcal{F}_L, \mathbb{P})\) and \(L^0(\Omega, \mathcal{F}_M, \mathbb{P})\) into \(L^0(X, \Omega, \mu)\).

(ii) For the extended \(\phi_L\) we have that \(\phi_L(1)\) is \(\mathbb{P}\)-a.e. strictly positive on \(\Omega\).

(iii) The extended \(\psi_M\) is an interval preserving Riesz isomorphism.

**Proof** (i) Since the carrier of the ideal \(L \subseteq L^0(Y, \Lambda, \nu)\) is \(Y\) we have that \(L\) is order dense in \(L^0(Y, \Lambda, \nu)\). Similarly \(M\) is order dense in \(L^0(X, \Sigma, \mu)\). Also, \(\phi_L\) and \(\psi_M\) are order continuous Riesz homomorphisms. So by Proposition 1.4.3 (iv) \(\phi_L\) and \(\psi_M\) can be extended, uniquely, to order continuous Riesz homomorphisms that respectively map \(L^0(Y, \Lambda, \nu)\) into \(L^0(\Omega, \mathcal{F}_L, \mathbb{P})\) and \(L^0(\Omega, \mathcal{F}_M, \mathbb{P})\) into \(L^0(X, \Omega, \mu)\).

(ii) Suppose that \(\phi_L(1)\) is not strictly positive. Then there exists some \(B \subset \Omega\) such that \(1_B \neq 0\) in \(L^0(\Omega, \mathcal{F}_L, \mathbb{P})\) and \(1_B \wedge \phi(1) = 0\). \(L_\Omega\)
is order dense in $L^0(\Omega, \mathcal{F}_\mu, \mathbb{P})$ and so there exists $0 < h \in L_\Omega$ such that $0 < h \leq 1_B$. Now $\phi_L$ is a surjection from $L$ to $L_\Omega$ and so there is some $0 \neq f \in L$ such that $\phi_L(f) = h$. Since $\phi_L$ is a Riesz homomorphism we have
\[
\phi_L(f^+) = \phi_L(f \lor 0) \\
= \phi_L(f) \lor \phi_L(0) \\
= h.
\]
Thus we have that there is $0 < f \in L$ such that $\phi_L(f) = h$.

From $f \land n \uparrow f$ we get, by order continuity of $\phi_L$, that
\[
\phi_L(f \land n) \uparrow \phi_L(f) = h > 0.
\]
On the other hand, for all $n$
\[
\phi_L(f) \land \phi_L(n1) = h \land n\phi_L(1) = 0.
\]
Therefore $\phi_L(1)$ is $\mathbb{P}$ strictly positive on $\Omega$.

(iii) We first show that the extended $\psi_M$ is a $1 - 1$ and thus a Riesz isomorphism. Let $f \in L^0(\Omega, \mathcal{F}_\mu, \mathbb{P})$ be such that $\psi_M(f) = 0$. This gives us that $0 = |\psi_M(f)| = \psi_M(|f|)$. We may assume that $f \geq 0$. Since $M_\Omega$ is order dense there exists a sequence $(f_n)$ in $M_\Omega$ such that $f_n \uparrow f$. Since $\psi_M$ is order continuous we have that $0 \leq \psi_M(f_n) \uparrow \psi_M(f) = 0$. Thus $\psi_M(f_n) = 0$ for all $n$. Since $\psi_M$ is injective we get that $f_n = 0$ for all $n$. Thus $f = 0$. Hence $\psi_M$ is a Riesz isomorphism of $L^0(\Omega, \mathcal{F}_\mu, \mathbb{P})$ into $L^0(X, \Sigma, \mu)$. 
Next we show that \( \psi_M \) is interval preserving. We want to show that if \([u, v]\) is an interval in \( L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \) and \( f \in [\psi_M(u), \psi_M(v)] \) in \( L^0(X, \Sigma, \mu) \) then there exists some \( t \in [u, v] \) such that \( \psi_M(t) = f \). To that end, we first consider the interval \([0, v]\) in \( L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \) and let \( f \in [0, \psi_M(v)] \).

Since \( M_\Omega \) is order dense there exists a sequence \((v_n)\) in \( M_\Omega \) such that \( 0 \leq v_n \uparrow v \). We have that \( \psi_M(M_\Omega) \) is an ideal and so there exists a sequence \((g_n)\) in \( M_\Omega \) such that \( \psi_M(g_n) = f \land \psi_M(v_n) \) for each \( n \). \( \psi_M \) is a homomorphism and so \( \psi_M(g_n \lor v_n) = \psi_M(g_n) \). We may therefore assume that \( 0 \leq g_n \leq v_n \). Also, for \( m \geq n \), \( \psi_M(g_n \lor g_m) = \psi_M(g_m) \) and so we may assume that \( g_n \uparrow \). Let \( g = \sup g_n \). Then \( 0 \leq g \leq v \), which implies that \( g \in L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \). By the order continuity of \( \psi_M \) we get that \( \psi_M(g_n) \uparrow \psi_M(g) \); but
\[
\psi_M(g_n) = f \land \psi_M(v_n) \uparrow f \land \psi_M(v) = f.
\]

Hence \( \psi_M(g) = f \). Thus for the interval \([0, v] \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \) and \( f \in [0, \psi_M(v)] \) there exists a \( g \in [0, v] \) such that \( \psi_M(g) = f \).

Now let us consider the interval \([u, v]\) in \( L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \) with \( u < v \). Put \( f \in [\psi_M(u), \psi_M(v)] \), thus \( f - \psi_M(u) \in [0, \psi_M(v - u)] \). By the preceding argument there exists a \( g \in [0, v - u] \) such that \( \psi_M(g) = f - \psi_M(u) \). But \( g \in [0, v - u] \) implies that \( g + u \in [u, v] \).

The proof is completed by putting \( t = g + u \).

**Proposition 2.2.5** Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), sub-\( \sigma \)-algebras \( \mathcal{F}_L \) and \( \mathcal{F}_M \) such that \( \mathcal{F} = \sigma(\mathcal{F}_L, \mathcal{F}_M) \) and ideals \( L \subseteq L^0(Y, \Lambda, \nu) \) and \( M \subseteq \)
$L^0(X,\Sigma,\nu)$ with carriers $Y$ and $X$ respectively. Let $\phi_L : L^0(Y,\Lambda,\nu) \to L^0(\Omega,\mathcal{F},\mathbb{P})$ be a Riesz homomorphism such that $\phi_L(1)$ is $\mathbb{P}$-a.e. strictly positive on $\Omega$ and $\psi_M : L^0(\Omega,\mathcal{F}_M,\mathbb{P}) \to L^0(X,\Sigma,\mu)$ be an interval preserving Riesz isomorphism. Let $L_\Omega = \phi_L(L)$ and $M_\Omega = \psi_M^{-1}(M)$. Then

(i) $\phi_L$ and $\psi_M$ can be restricted to order continuous Riesz homomorphisms mapping $L$ to $L_\Omega \subset L^0(\Omega,\mathcal{F}_L,\mathbb{P})$ and $M_\Omega \subset L^0(\Omega,\mathcal{F}_M,\mathbb{P})$ to $M$, respectively, with the restricted $\phi_L$ surjective and the restricted $\psi_M$ a Riesz isomorphism onto an ideal in $M$.

(ii) The restricted $\psi_M$ is injective.

(iii) $L_\Omega$ and $M_\Omega$ are order dense.

Proof (i) We have that $\phi_L$ is a Riesz homomorphism from $L^0(Y,\Lambda,\nu)$ into $L^0(\Omega,\mathcal{F}_L,\mathbb{P})$ (which are $\sigma$-finite measure spaces) and so it is order continuous by Proposition 1.4.3 (i). Its restriction to a mapping from $L$ to $L_\Omega$ will be an order continuous Riesz homomorphism as well, which is, moreover, surjective.

We show that $\psi_M(M_\Omega)$ is an ideal. Let $0 \leq f \in \psi_M(M_\Omega)$ and $0 \leq |g| \leq f$. Now $f = \psi_M(v)$ for some $v \in M_\Omega^+$. Since $\psi_M$ is interval preserving, there exists some $t \in M_\Omega$ with $0 \leq t \leq v$ such that $|g| = \psi_M(t)$. Thus $|g| \in \psi_M(M_\Omega)$. Similar argument shows, since, $g^+ \leq |g|$ and $g^- \leq |g|$, that $g^+ \in \psi_M(M_\Omega)$ and $g^- \in \psi_M(M_\Omega)$. We then have that $g = g^+ - g^- \in \psi_M(M_\Omega)$. 
(ii) Since $\psi_M$ is a Riesz isomorphism, it is injective and so its restriction will be injective as well.

(iii) $L$ is an ideal in $L^0(Y, \Lambda, \mu)$ and $\phi_L$ is a Riesz homomorphism from $L^0(X, \Sigma, \mu)$ to $L^0(\Omega, \mathcal{F}_L, \mathbb{P})$ with carrier of $L$ being $Y$. By Lemma 2.2.2 we then have that $L_\Omega = \phi_L(L)$ is order dense.

Let $0 < f \in L^0(\Omega, \mathcal{F}_M, \mathbb{P})$. Since $\psi_M$ is $1-1$, we have that $0 < \psi_M(f) \in L^0(X, \Sigma, \mu)$. But $M$ is dense in $L^0(X, \Sigma, \mu)$ and so there exists an element $g \in M$ such that $0 < g \leq \psi_M(f)$. Also, since $\psi_M$ is interval preserving, there is an element $0 < f_1$ such that $\psi_M(f_1) = g$ and $0 < f_1 \leq f$. Hence, since $\psi_M(f_1) \in M$, we have that $f_1 \in \psi_M^{-1}(M) = M_\Omega$. Hence, $M_\Omega$ is order dense in $L^0(\Omega, \mathcal{F}_M, \mathbb{P})$. □

We note that Proposition 2.2.4 and Proposition 2.2.5 give necessary and sufficient conditions that Riesz homomorphisms $\phi_L$ and $\psi_M$ and ideals $L, L_\Omega, M$ and $M_\Omega$ should satisfy for them to be used in defining MCE-representable operators.

We conclude by showing that order continuous Riesz homomorphisms, kernel operators and $\tau$-kernel operators are amongst the class of operators that are MCE-representable.

Example 2.2.6  (1) Let $L \subset L^0(Y, \Lambda, \nu)$ be an ideal of measurable functions on the $\sigma$-finite measure space $(Y, \Lambda, \nu)$ and $M \subset L^0(X, \Sigma, \mu)$ be an ideal of measurable functions on the $\sigma$-finite measure space $(X, \Sigma, \mu)$
such that \( L \) and \( M \) have carriers \( Y \) and \( X \) respectively. Let \( T : L \to M \) be an order continuous Riesz homomorphism. We want to show that \( T \) is MCE-representable. We first use Proposition 1.4.3 (iv) to extend \( T \) into an order continuous Riesz homomorphism \( \phi_L : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu) \). We have that \( \{\phi_L(1_Y)\}^{dd} \) is a band and so there exists some \( X_T \in \Sigma \) such that \( \{\phi_L(1_Y)\}^{dd} = \{(1_{X_T})\}^{dd} \). Let \( w_M \) be a strictly positive function on \( X_T \) such that \( \int_{X_T} w_M d\mu = 1 \). Such a function exists, indeed, let \( X_n \uparrow X_T \) with \( X_n \in \Sigma \) and \( \mu(X_n) < \infty \) since \( \Sigma \) is \( \sigma \)-finite. Let \( w_M \) be given by \( \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(X_n)} 1_{X_n} \). For \( x \in X_T \) we have

\[
w_M(x) \geq \frac{1}{2^n} \frac{1}{\mu(X_n)} > 0.
\]

and

\[
\int_{X_T} w_M d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(X_n)} \int_{X_T} 1_{X_n} d\mu \\
= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(X_n)} \mu(X_n) \\
= \sum_{n=1}^{\infty} \frac{1}{2^n} \\
= 1.
\]

Put \( \Omega = X_T, \mathbb{P} = w_M \mu, \mathcal{F} = \{ A \cap X_T : A \in \Sigma \} \) and \( \mathcal{F}_M = \mathcal{F} \). Let \( \psi_M \) be the canonical embedding of \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) into \( L^0(X, \Sigma, \mu) \). We have that \( \phi_L \) is a Riesz homomorphism from \( L^0(Y, \Lambda, \nu) \) into \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) and is such that \( \phi_L(1) \) is strictly positive on \( \Omega \). Let \( m = 1 \). Then for all
(2) We show that kernel operators are MCE-representable. Consider σ-finite measure spaces $(X, \Sigma, \mu)$ and $(Y, \Lambda, \nu)$ and let $L \subseteq L^0(Y, \Lambda, \nu)$ and $M \subseteq L^0(X, \Sigma, \mu)$ be ideals with carriers $Y$ and $X$ respectively. Let $K : L \rightarrow M$ be the absolute kernel operator from $L$ into $M$ defined by

$$Kf(\omega) = \int_Y k(\omega, x) f(x) d\nu(x) \text{ for all } f \in L,$$

where $k \in L^0(X \times Y, \Sigma \otimes \Lambda, \mu \otimes \nu)$ is such that

$$\int_Y |k(\cdot, x)f(x)|d\nu(x) \in M \text{ for all } f \in L.$$

We show that $K$ is MCE-representable:

Let $w_1 \in L^1(X, \Sigma, \mu)$ and $w_2 \in L^1(Y, \Lambda, \nu)$ be strictly positive and satisfy $\int_X w_1 d\mu = \int_Y w_2 d\nu = 1.$

Taking $\Omega := X \times Y$, $\mathcal{F} := \Sigma \otimes \Lambda$, and putting $P := w_1 \mu \otimes w_2 \nu$, we have
that \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, for,

\[
\int_{\Omega} d\mathbb{P} = \int_{X \times Y} w_1 \mu \otimes w_2 \nu d\mu(x)d\nu(y)
\]

\[
= \int_{X} w_1(x) \left( \int_{Y} w_2(y)d\nu(y) \right) d\mu(x)
\]

\[
= \int_{X} w_1(x)d\mu(x) \int_{Y} w_2d\nu(y)
\]

\[
= 1.
\]

Define \((\phi_L f)(x, y) := f(y)1_X(x)\) for all \(f \in L\) and \(\mathcal{F}_M = \{A \times Y : A \in \Sigma\}\). Then \(L^0(\Omega, \mathcal{F}_M, \mathbb{P}) = \{f \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : f(x, y) = g(x)1_Y(y)\text{ for some } g \in L^0(X, \Sigma, \mu)\}\). If \(f = g1_Y \in L^0(\Omega, \mathcal{F}_M, \mathbb{P})\), define \(\psi_M(f) = g\). We have \(\phi_L(1) = 1\) and \(\psi_M\) is a bijection from \(L^0(\Omega, \mathcal{F}, \mathbb{P})\) onto \(L^0(X, \Sigma, \mu)\) and therefore interval preserving. Taking \(m := w_2^{-1}k\) we get that, for all \(f \in L\),

\[
E(mf|\mathcal{F}_M)(x, y) = E(w_2^{-1}k f|\mathcal{F}_M)(x, y)
\]

\[
= \int_Y w_2^{-1}(y)k(x, y)f(y)1_X(x)w_2(y)d\nu(y)
\]

\[
= \int_Y k(x, y)f(y)d\nu(y)1_Y(y)
\]

\[
= \int_Y k(x, y)f(y)d\nu(y)
\]

Hence, the triple \(\Phi := ((\Omega, \mathcal{F}, \mathbb{P}), \phi_L, \psi_M)\) represents \(K\) with \(\Phi\)-kernel \(m := w_2^{-1}k\).

(3) Let \((X, \Sigma, \mu)\), \((Y, \Lambda, \nu)\) and \((Z, \Gamma, \lambda)\) be \(\sigma\)-finite measure spaces and suppose that \(L \subseteq L^0(Y, \Lambda, \nu)\) and \(M \subseteq L^0(X, \Sigma, \mu)\) are ideals with carriers
Y and X respectively. Suppose that \( \tau : X \times Z \to Y \) is a \((\Sigma \otimes \Gamma, \Lambda)\)-measurable null-preserving mapping. We want to show that every absolute \( \tau \)-kernel operator \( T \in L_{\tau k}(L, M) \) is MCE-representable via a fixed representation triple \( \Phi \). We first introduce the following sub-\( \sigma \)-algebras of \( \Sigma \otimes \Gamma \):

(a) \( \Lambda_0 := \{\tau^{-1}(A) : A \in \Lambda\} = \{\tau_*(A) : A \in \Lambda\} \);

(b) \( \Sigma_0 := \{A \times Z : A \in \Sigma\} \).

Let \( 0 \leq w_1 \in L^1(X, \Sigma, \mu) \) be such that \( w_1(x) > 0 \ \mu\text{-a.e. and} \int_X w_1 \, d\mu = 1 \) and let \( 0 \leq w_2 \in L^1(Z, \Gamma, \lambda) \) be such that \( w_2(z) > 0 \ \lambda\text{-a.e. and} \int_Z w_2 \, d\lambda = 1 \). Define \( \Omega := X \times Z, \mathcal{F} = \Sigma \otimes \Gamma \) and \( \mathbb{P} = (w_1 \mu) \otimes (w_2 \lambda) \).

Then \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and \( \mathbb{P} \) is equivalent to the product measure \( \mu \otimes \lambda \). Define \( \phi_L := \phi_\tau : L^0(Y, \Lambda, \nu) \to L^0(\Omega, \mathcal{F}, \mathbb{P}) \).

Then \( \phi_L \) is a Riesz homomorphism satisfying \( \phi_L(1) = 1 \) and \( \mathcal{F}_L = \Lambda_0 \) in the notation defined above. Let \( \mathcal{F}_M = \Sigma_0 \) and note that \( f \in L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \) if and only \( f = g1_Z \) with \( g \in L^0(X, \Sigma, \mu) \). Define \( \psi_M : L^0(\Omega, \mathcal{F}_M, \mathbb{P}) \to L^0(X, \Sigma, \mu) \) by \( \psi_M(f) = \psi_M(g1_Z) = g \). Then \( \psi_M \) is a Riesz isomorphism which is interval preserving (in fact it is a surjection). We will use \( \Phi = (\Omega, \mathcal{F}, \mathbb{P}, \phi_L, \psi_M) \) as a representation triple of a \( \tau \)-kernel operator.

Since \( L_\Omega = \{f \circ \tau : f \in L\} \), it follows from the result in Example 1.5.5
on Page 30 that $m \in \mathfrak{M}(L_\Omega, M_\Omega)$ if and only if

$$E(|mf| \mid \Sigma_0)(x, z) = \int |m(x, z)f(\tau(x, z))w_2(z)|d\lambda(z) < \infty$$

$\mu$-a.e. on $X$, and

$$E(|mf| \mid \Sigma_0)(\cdot, z) = \int |m(\cdot, z)f(\tau(\cdot, z))w_2(z)|d\lambda(z) \in M \forall f \in L.$$

Consequently, $m \in \mathfrak{M}(L_\Omega, M_\Omega)$ if and only if $m \cdot w_2 \in \mathcal{H}(L, M)$. Moreover, if $m \in \mathfrak{M}(L_\Omega, M_\Omega)$ then

$$\psi_M S_m \phi_L f(x) = \int m(x, z)f(\tau(x, z))w_2(z)\,d\lambda(z),$$

which shows that $\psi_M S_m \phi_L \in \mathcal{L}_{rk}(L, M)$ with kernel $k(x, z)$ given by

$m(x, z)w_2(z)$. 

Chapter 3

Operators defined by random measures

In this chapter we look at operators generated by a class of measures known as random measure. We start by introducing random measures as studied by Sourour [28, 27] and Weis [29, 30]. We then look at operators that these random measures generate which are sometimes called pseudo-integral operators. In addition to Sourour and Weis who studied these operators in the setting of ideals of measurable functions over standard spaces, they were also studied in $L_2$ by Arveson [5] and in $L_p$ ($0 < p < 1$) by Kalton in [14, 15, 13].

Sourour in [27] showed that the lattice properties of operators generated by random measures are closely related to the properties of the generating (signed) measures in standard measure spaces, where a Borel space $X$ is called standard if $X$ is Borel isomorphic to a Borel subset of a separable complete metric space (see [20] or [5]). He assumed that the total variation of a random measure that generates an order bounded operator is again a random measure and showed that it generates the absolute value of that
operator. We show that for an order bounded operator that can be expressed as a difference of two positive operators there exists a signed random measure which is a difference of two measures and that the total variation of that random measure generates the absolute value of that operator in more general measure spaces. In order to do this we make use of the Luxemburg-Schep Radon-Nikodym theorem [18] and the Kantorovich extension theorem [1].

We also develop the concept of random measure-representable operators analogous to MCE-representable operators that were introduced by Grobler and de Pagter in [8].

3.1 Random measures

Sourour in [27] studies operators generated by random measures. He called random measures kernels and the operators that they generate pseudo-integral operators. Weis in [29] called them operators represented by random measures. We will call them operators generated by random measures. Both Sourour and Weis use the same implicit assumption that \( \nu(x, \cdot) \) and \( |\nu|(x, \cdot) \) are Borel functions, see [28, Definition 1.1 (ii)] and [29, Remark 2.2 (i)]. Our results give a more general setting than those of Sourour and Weis as we do not make use of their assumption but actually show that the mapping \( x \mapsto |\nu|(x, B) \) is a measurable function for each \( B \in \Lambda \).

We consider measure spaces \( (Y, \Lambda, \lambda) \) and \( (X, \Sigma, \mu) \) with \( (X, \Sigma, \mu) \) complete \( \sigma \)-finite, i.e., subsets of \( \mu \)-a.e. zero sets in \( \Sigma \) are also \( \mu \)-a.e. zero sets.

**Definition 3.1.1** A class of measures \( \nu(x, \cdot) \) defined on \( \Lambda \) for each \( x \in X \)
such that $\nu(x, B)$ is $\Sigma$-measurable as a function of $x$ for each fixed $B \in \Lambda$, is
called a random measure on $Y$.

**Definition 3.1.2** A random measure $\nu(x, \cdot)$ is called uniformly $\sigma$-finite on
$Y$ if $Y = \bigcup Y_n$ with $Y_n \in \Lambda$ and $\nu(x, Y_n) \leq k_n < \infty$ for all $x \in X$, where $k_n$
is a sequence of positive real constants.

For a uniformly $\sigma$-finite random measure $\nu(x, \cdot)$ on $Y$, there exists (see [6]
Theorem 2.6.2) a unique measure $\nu$ on $\Sigma \otimes \Lambda$ such that

$$\nu(A \otimes B) = \int_A \nu(x, B)d\mu(x) \text{ for all } A \in \Sigma \text{ and } B \in \Lambda. \quad (3.1.1)$$

This measure is given by

$$\nu(E) = \int_X \nu(x, E(x))d\mu(x) \text{ for } E \in \Sigma \otimes \Lambda.$$ 

This measure is $\sigma$-finite on $\Sigma \otimes \Lambda$ and is a probability measure if $\mu$ and $\nu(x, \cdot)$
are probability measures.

We will make use of the following weaker condition.

**Definition 3.1.3** A random measure $\nu(x, \cdot)$ is said to be $\sigma$-finite on $Y$ if
$Y = \bigcup Y_n$ and $\nu(x, Y_n) < \infty \mu$-a.e. on $X$.

We can use $\sigma$-finite random measure $\nu(x, \cdot)$ to construct a unique $\sigma$-finite
measure $\nu$ as in (3.1.1).

We will use the notation $F = \Sigma \otimes \Lambda$.

**Lemma 3.1.4** Let $\nu(x, \cdot)$ be a $\sigma$-finite random measure. Then the function
$x \mapsto \nu(x, C(x))$ belongs to $M^+(X, \Sigma, \mu)$ for every $C \in F$. 
Proof Let $X_0$ be such that $\mu(X - X_0) = 0$ and $\nu(x, Y_n) < \infty$ for every $x \in X_0$ and $n \in \mathbb{N}$. Put $C = A \times B$ with $A \in \Sigma$ and $B \in \Lambda$. Then

$$\nu(x, C(x)) = 1_A(x)\nu(x, B),$$

which, by definition, is in $M^+(X, \Sigma, \mu)$.

Let $Z_n = X \times Y_n$, $C_n = Z_n \cap C$ for every $n \in \mathbb{N}$ and $C \subset X \times Y$. Put

$$D_n = \{ C \in \mathcal{F} | \nu(x, C_n(x)) \in M^+(X, \Sigma, \mu) \}.$$

Then $D_n$ contains the measurable rectangles $A \times B \in \mathcal{F}$. Also, if $B$ and $C$ are in $D_n$ with $B \subset C$, we have, from the fact that $\nu(x, \cdot)$ is finite on subsets of $Y_n$, for every $x \in X_0$, that

$$\nu(x, (Z_n \cap (C - B))(x)) = \nu(x, (C_n(x) - B_n(x)))$$

$$= \nu(x, C_n(x)) - \nu(x, B_n(x)) \text{ a.e.}$$

Hence $\nu(x, (Z_n \cap (C - B))(x))$ is equal to a measurable function except possibly on a subset of a $\mu$–null set. By the completeness of $\mu$, $\nu(x, (Z_n \cap (C - B))(x))$ is in $M^+(X, \Sigma, \mu)$. Thus $C - B \in D_n$.

If $C_k \uparrow C$ with $C_k \in D_n$ then $\nu(x, C_k(x)) \uparrow \nu(x, C_n(x))$, and so $C \in D_n$. Thus, by the Dynkin principle, $D_n$ contains the $\sigma$-algebra $\mathcal{F}$. From the fact that $\nu(x, C_n(x)) \uparrow \nu(x, C(x))$, we then have that $\nu(x, C(x)) \in M^+(X, \Sigma, \mu)$.

\[ \square \]

Theorem 3.1.5 Let $\nu(x, \cdot)$ be a $\sigma$-finite random measure on $(Y, \Lambda)$ and let $\mu$ be a $\sigma$-finite measure on $X$. For every $C \in \mathcal{F}$ define

$$\nu(C) = \int_X \nu(x, C(x))d\mu.$$
Then $\nu$ is a $\sigma$-finite measure on $(X \times Y, \mathcal{F})$, moreover, it is a unique $\sigma$-finite measure that satisfies

$$\nu(A \times B) = \int_A \nu(x, B)d\mu$$

for all $A \in \Sigma$ and $B \in \Lambda$.

**Proof** The proof that $\nu$ is a measure on $\mathcal{F}$ is exactly as for the case of a uniformly $\sigma$-finite measure (see [6]). We have that $\nu(A \times B) = \int_A \nu(x, B)d\mu$ for all $A \in \Sigma$ and $B \in \Lambda$. We now show that it is a $\sigma$-finite measure. Let $X_0 = \{x \in X : \nu(x, Y_n) < \infty \text{ for all } n\}$. We therefore have, by assumption, that $\mu(X - X_0) = 0$. Let $X_n \uparrow X$ and $Y_n \uparrow Y$ be such that $\mu(X_n) < \infty$ and $\mu(Y_n) < \infty \mu$-a.e. on $X$. Let $X_{n,k} = \{x \in X_n \cap X_0 : \nu(x, Y_n) \leq k\}$ for $n, k = 1, \ldots$. Then for each $n$ we have that $X_{n,k} \uparrow_k X_n \cap X_0$, and so,

$$\bigcup_{n,k} (X_{n,k} \times Y_n) \cup (X_0^c \times Y) = (X_0 \times Y) \cup (X_0^c \times Y) = X \times Y.$$

Furthermore,

$$\nu(X_{n,k} \times Y_n) = \int_{X_{n,k}} \nu(x, Y_n)d\mu \leq k\mu(X_n) \leq \infty$$

and $\nu(X_0^c \times Y) = \int_{X_0^c} \nu(x, Y)d\mu$. The uniqueness of $\nu$ follows from the fact that it is a $\sigma$-finite measure. \qed
Corollary 3.1.6 For every \( p \in M^+(X \times Y, \mathcal{F}) \) the integral
\[
\int_Y p(x, y)\nu(x, dy)
\]
exists and is in \( M^+(X, \Sigma, \mu) \). Moreover,
\[
\int_{X \times Y} p(x, y)\nu(x, y) = \int_X \left( \int_Y p(x, y)\nu(x, dy) \right) d\mu(x)
\]
For a proof the reader may consult the relevant part of the proof of Fubini's theorem in [6].

Definition 3.1.7 The set \( B \subset Y \) is a null set with respect to the random measure \( \nu(x, \cdot) \) if \( \nu(x, B) = 0 \) \( \mu \)-a.e. on \( X \). This is equivalent to saying that \( \int_A \nu(x, B)d\mu(x) = 0 \) for all \( A \in \Sigma \).

A subset of a null set is also a null set. This follows from the fact that if \( A \subset B \) then \( \nu(x, A) \leq \nu(x, B) \). The union of a countable number of null sets is again a null set.

3.2 Operators generated by random measures

We now take a closer look at operators that are generated by random measures.

Definition 3.2.1 Let \( L \subset L^0(Y, \Lambda, \lambda) \) and \( M \subset L^0(X, \Sigma, \mu) \) be order dense ideals of measurable functions. We say the positive operator \( T \in \mathcal{L}_b(L, M) \) is generated by the random measure \( \nu(x, \cdot) \) if
\[
Tf(x) = \int_Y f(y)\nu(x, dy) \quad \text{for almost every} \quad x \in X.
\]
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The space of the differences $T_1 - T_2$, where $T_1$ and $T_2$ are positive operators from $L$ to $M$ with $T_1$ generated by the random measure $\nu_1(x, \cdot)$ and $T_2$ generated by the random measure $\nu_2(x, \cdot)$, will be denoted by $\mathcal{L}_{rm}(L, M)$. It is obvious that this is a subspace of order bounded linear operators from $L$ to $M$, i.e., $\mathcal{L}_{rm}(L, M) \subseteq \mathcal{L}_b(L, M)$.

We give examples of operators generated by random measures (taken from [31]).

Example 3.2.2 (1) Let $(Y, \Lambda, \lambda)$ and $(X, \Sigma, \mu)$ be $\sigma$-finite measure spaces and $L \subseteq L^0(Y, \Lambda, \lambda)$ and $M \subseteq L^0(X, \Sigma, \mu)$ be ideals with carriers $Y$ and $X$, respectively. Let $T : L \to M$ be a positive, or, more generally, a regular operator. If $T$ is a kernel operator defined by $Tf(x) = \int_Y f(y)t(x, y)d\lambda(y)$ for all $f \in Y$ with kernel $t(x, y)$ then if we put $\nu_T(x, \cdot) = t\lambda(\cdot)$, i.e., $\nu_T(x, B) = \int_B t(x, y)d\lambda(y)$ for all $B \in \Lambda$. Then $T$ is an operator generated by the random measure $\nu_T(x, \cdot)$.

(2) Let $K_1$ and $K_2$ be locally compact Hausdorff spaces and $T : C(K_2) \to C(K_1)$ be a lattice homomorphism of the form $Tf(x) = g(x)f(\sigma(x))$, with $\sigma : K_1 \to K_2$ a continuous map and $g \in C(K_1)$, see [21][Theorem 3.2.10]. Then $T$ is an operator generated by the random measure $\nu_T(x, \cdot) = g(x)\delta_{\sigma(x)}$, where, for $B \in K_2$, we have

$$\nu_T(x, B) = \begin{cases} g(x) & \text{if } x \in \sigma^{-1}(B) \\ 0 & \text{if } x \notin \sigma^{-1}(B), \end{cases}$$

i.e., $\nu_T(x, B)$ is a point measure on $K_2$. 
(3) Let $\eta$ be a measure on a locally compact group $G$ and $T$ be the convolution by the measure $\eta$ given by $Tf(x) = \int_G f(x-y)d\eta(y)$. Then $T$ is an operator generated by the random measure $\nu_T(x,A) = \eta(A-x)$ for $A$ a Borel subset of $G$.

(4) If $T_t$ is a semi group generated by the transition probabilities $P_t$ of a Markov process, with $T_tf(x) = \int_Y f(y)dP_t(x,dy)$, then $T_t$ is an operator generated by the random measure $\nu_{T_t}(x,\cdot) = P_t(x,\cdot)$.

We give an example of an operator that is not generated by random measures (see [27])

**Example 3.2.3** Let $\Gamma$ be the unit circle with normalized Lebesgue measure, $L^2 = L^2(\Gamma)$ and $H^2$ the usual Hardy spaces. The projection $P$ of $L^2$ onto $(H^2)^\perp$ is not a operator generated by random measure. Suppose, to the contrary, that $(Pf)(w) = \int f(z)\mu(w,dz)$ almost everywhere. Let $e_n(z) = z^n$, and remove a set $\Gamma_0$ of measure zero such that

$$\int e_n(z)\mu(w,dz) = \begin{cases} e_n(w) & \text{if } n \leq 0 \\ 0 & \text{if } n > 0, \end{cases}$$

for every $w \in \Gamma - \Gamma_0$. Choose one $w \not\in \Gamma_0$ and let $\sigma(dz) = \mu(w,dz)$. Therefore the Fourier transform $\hat{\sigma}$ of $\sigma$ is given by

$$\hat{\sigma} = \begin{cases} w^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

The theorem of F. and M. Riesz in [9, Page 47] implies that $\hat{\sigma}$ is absolutely continuous and the Riemann-Lebesgue lemma (see [13, Page 13]) $\hat{\sigma}(n) \to 0$, i.e. $1 = |w^n| \to 0$, which is a contradiction.
Proposition 3.2.4 Let \(L \subset L^0(Y, \Lambda, \lambda)\) and \(M \subset L^0(X, \Sigma, \mu)\) be order dense ideals of measurable functions, let \(\lambda\) be a \(\sigma\)-finite measure on \(\Lambda\) and let \(T: L \rightarrow M\) be a linear operator that is generated by a random measure \(\nu(x, \cdot)\).

\(T\) is well defined if and only if \(\lambda(B) = 0\) implies that \(\nu(x, B) = 0\) \(\mu\)-a.e. on \(X\).

Proof Let \(T\) be well defined and let \((Y_n)\) be a sequence in \(\Lambda\) with \(Y_n \uparrow Y\) and \(1_{Y_n} \in L\). Let \(B\) be a \(\lambda\)-null set. Then \(B_n = B \cap Y_n\) is also a \(\lambda\)-null set. \(T\) is well defined and so \(0 = T1_{B_n}(x) = \nu(x, B_n)\) since \(1_{B_n} = 0\) for almost every \(x\). Thus \(B_n\) is a \(\nu(x, \cdot)\)-null set for every \(n\). Now \(B_n \uparrow B\) and so \(B\) is also a \(\nu(x, \cdot)\)-null set.

On the other hand, suppose that for \(B \in \Lambda\), \(\lambda(B) = 0\) implies that \(\nu(x, B) = 0\) almost everywhere on \(X\). It then follows that if \(f\) is any positive \(\Lambda\)-measurable function then \(\int_B f(y)\nu(x, dy) = 0\), \(\mu\)-a.e. on \(X\). Let \(f, g \in L\) be positive functions with \(f(y) = g(y)\) \(\lambda\)-a.e. on \(Y\). Then for the set \(B = \{y \in Y : f(y) \neq g(y)\}\) we have that

\[ Tf(x) = \int_{Y-B} f(y)\nu(x, dy) + \int_B f(y)\nu(x, dy) \]

and

\[ Tg(x) = \int_{Y-B} g(y)\nu(x, dy) + \int_B g(y)\nu(x, dy). \]

Therefore \(Tf = Tg\), and so \(T\) is well defined.

Lemma 3.2.5 If two random measures \(\nu_1(x, \cdot)\) and \(\nu_2(x, \cdot)\) generate the same positive operator \(T\), then \(\nu_1(x, \cdot) = \nu_2(x, \cdot)\) for \(\mu\) almost every \(x \in X\).
Proof Let $Y_n$ be a sequence in $\Lambda$ with $Y_n \uparrow Y$ and $1_{Y_n} \in L$. Let $B \in \Lambda$ and put $B_n = B \cap Y_n$. We have that $B_n \uparrow B$ and so $\nu_1(x, B_n) \uparrow \nu_1(x, B)$ and $\nu_2(x, B_n) \uparrow \nu_2(x, B)$ for every $x \in X$. But $\nu_1(x, B_n) = T1_{B_n}(x) = \nu_2(x, B_n)$ for almost every $x$. Hence $\nu_1(x, B) = \nu_2(x, B)$ for almost every $x$.

Let $T : L \to M$ be a positive operator generated by the random measure $\nu(x, \cdot)$. If $Y_0 \subset Y$ is such that $1_{Y_0} \in L$ and if $\Lambda_0$ is the $\sigma$-algebra induced by $\Lambda$ on $Y_0$, then the random measure defines a vector-valued measure $\tilde{\nu}$ on $(Y_0, \Lambda_0)$ by

$$\tilde{\nu}(B) = \nu(x, B) \in M \text{ for all } B \in \Lambda_0.$$ 

In particular, if $L^\infty(Y, \Lambda) \subset L$ then the random measure defines an $M$-valued measure in $(Y, \Lambda)$.

Let $T_1$ be generated by the random measure $\nu_1(x, \cdot)$ and $T_2$ be generated by the random measure $\nu_2(x, \cdot)$. Then the operator $T_1 + T_2$ is generated by the random measure $\tau := \nu_1(x, \cdot) + \nu_2(x, \cdot)$. We have that $T_1 + T_2$ maps $L$ into $M$, and since $L$ is an order dense ideal, there is a sequence $Y_n \uparrow Y$ such that $1_{Y_n} \in L$ and so $\tau(x, Y_n)$ is finite $\mu$-a.e. on $X$. Let $X_0$ be a subset of $X$ such that $\tau(X - X_0) = 0$ and $\tau(x, Y_n) < \infty$ for all $n$ and all $x \in X_0$. Let $\Lambda_0$ be the ideal in $\Lambda$ consisting of all $B \in \Lambda$ with $B \subset Y_n$ for some $n$. For $B \in \Lambda_0$, define

$$\nu(x, B) = \begin{cases} \nu_1(x, B) - \nu_2(x, B) & \forall x \in X_0 \\ 0 & \forall x \in X - X_0. \end{cases}$$
Then $\nu(x, \cdot)$ is a signed random measure defined on $\Lambda_0$ and is $\sigma$-additive on $\Lambda_0$. Now define

$$\int_Y f(y)\nu(x, dy) = \begin{cases} \int_Y f(y)\nu_1(x, dy) - \int_Y f(y)\nu_2(x, dy) & \forall x \in X_0 \\ 0 & \forall x \in X - X_0. \end{cases}$$

for all $f \in L$. Then $Tf(x) = \int_Y f(y)\nu(x, dy)$ and we say that $T$ is generated by the signed measure $\nu(x, \cdot)$.

**Proposition 3.2.6** If $L \subseteq L^0(Y, \Lambda, \lambda)$ and $M \subseteq L^0(X, \Sigma, \mu)$ are ideals of measurable functions and $T \in \mathcal{L}_{rm}(L, M)$. Then

(i) The operator $T$ is order continuous.

(ii) If $0 \leq S \leq T \in \mathcal{L}_{rm}(L, M)$ then $S \in \mathcal{L}_{rm}(L, M)$.

(iii) $\mathcal{L}_{rm}(L, M)$ is a Riesz subspace of $\mathcal{L}_b(L, M)$.

(iv) $\mathcal{L}_{rm}(L, M)$ is an ideal in $\mathcal{L}_b(L, M)$.

**Proof** (i) Let $T \in \mathcal{L}_{rm}(L, M)$ with $T = T_1 - T_2$. In order to show that $T$ is order continuous we have to show that $T_1$ and $T_2$ are order continuous. We may therefore assume that $T$ is positive and is generated by the random measure $\nu(x, \cdot)$. Since $L$ is super Dedekind complete it is sufficient to show that $T$ is $\sigma$-order continuous. Let $\{f_n\}$ be a sequence in $L$ such that $f_n \downarrow 0$ $\nu$-a.e. There exists a set $Y_0$ such that $\lambda(Y - Y_0) = 0$ and $f_n(y) \downarrow 0$ for all $y \in Y_0$. But then $Y - Y_0$ is, by Proposition 3.2.4, a $\nu(x, \cdot)$-null set, and so, for almost every $x$, $f_n(y) \downarrow 0$ $\nu(x, \cdot)$-a.e. on $Y$ and $\int_Y f_1(y)\nu(x, dy) < \infty$ $\mu$-a.e. By the Lebesgue theorem $\int_Y f_n(y)\nu(x, dy) \downarrow 0$ for almost
every $x$, i.e., $Tf_n(x) \downarrow 0$ for almost every $x$. Thus $Tf_n \downarrow 0$ and so $T$ is order continuous.

(ii) Let $T$ be generated by the random measure $\nu(x, \cdot)$ and let $\nu$ be the $\sigma$-finite measure on $\mathcal{F}$ generated by $\nu(x, \cdot)$ and $\mu$ as constructed in Theorem 3.1.5. Consider the map $f(y) \rightarrow \overline{f}(x, y) = 1_x(x)f(y)$, $f \in L$. This map is well defined, for, if $f(y) = 0$ $\lambda$-a.e., then, if $C = \{(x, y) : 1_x(x)f(y) \neq 0\}$ we have that $C = X \times N$, with $N = \{y : f(y) \neq 0\}$. Hence $C(x) = N$ for all $x$. Since $\lambda(N) = 0$, we have, by assumption, that $\nu(x, N) = 0$ $\mu$-a.e. on $X$ and so

$$\nu(C) = \int_X \nu(x, N)d\mu = 0.$$ 

Taking $C = \{(x, y) : 1_x(x)f(y) = \infty\}$, a similar argument shows that this map maps $L$ into $L^0(X \times Y, \Sigma \otimes \Lambda, \nu)$. It is also clear that this map is a Riesz homomorphism. Let $\overline{L}$ be the image of $L$ in $L^0(X \times Y, \Sigma \otimes \Lambda, \nu)$ under this Riesz homomorphism. Put $I(\overline{L})$ to be the ideal generated in $L^0(X \times Y, \Sigma \otimes \Lambda, \nu)$ by $\overline{L}$. We extend the operator $T$ to $\overline{T}$ on $I(\overline{L})$ by

$$\overline{T}h(x) = \int_Y h(x, y)\nu(x, dy) \text{ for } h \in I(\overline{L}).$$

Now $\overline{T}$ is well defined, since if $h(x, y) = 0$ $\nu$-a.e. and if $C = \{(x, y) : h(x, y) \neq 0\}$, it follows from $0 = \nu(C) = \int_Y \nu(x, C(x))d\mu$, that $\nu(x, C(x)) = 0$ $\mu$-a.e. on $X$. Hence, for $\mu$-almost every $x$, the set $C(x)$ is a $\nu(x, \cdot)$-null set. It then follows that $h(x, y) = 0$
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\[ \nu(x, \cdot) - a.e. \text{ on } Y \text{ for } \mu\text{-almost every } x. \] This then gives that \( \int_Y h(x, y) \nu(x, dy) = 0 \), i.e., \( \bar{T}h(x) = 0 \) for almost every \( x \) in \( X \).

We can use a similar argument to show that for \( g, h \in I(\overline{L}) \) with \( g(x, y) \leq h(x, y) \) \( \nu \)-a.e. we have

\[
\int_Y g(x, y) \nu(x, dy) \leq \int_Y h(x, y) \nu(x, dy)
\]

holds \( \mu \)-a.e. on \( X \). In particular, if \( h \in I(\overline{L}) \), then \( |h(x, y)| \leq 1_x f(y) \) for some \( f \in L \). Therefore

\[
0 \leq |\bar{T}h(x)| = \left| \int_Y h(x, y) \nu(x, dy) \right|
\leq \int_Y |h(x, y)| \nu(x, dy)
\leq \int_Y f(y) \nu(x, dy) \in M,
\]

which shows that \( \bar{T} : I(\overline{L}) \to M \)

Now, if \( h_n(x, y) \downarrow 0 \) \( \nu \)-a.e., we have that

\[
\int_X \int_Y h_n(x, y) \nu(x, dy) d\mu = \int_{X \times Y} h_n(x, y) d\nu \downarrow 0.
\]

This implies that \( \int_Y h_n(x, y) \nu(x, dy) d\mu \downarrow 0 \) for \( \mu \)-almost every \( x \), which shows that \( \bar{T} \) is order continuous.

Let \( S \) be an operator such that \( 0 \leq S \leq T \) and for \( \bar{f}(x, y) \in \overline{L} \) define the operator \( \overline{S} \bar{f}(x) = Sf(x) \in M \). Let \( \bar{f}(x, y) = 0 \) \( \nu \)-a.e., then \( |\overline{S} \bar{f}(x)| = |Sf(x)| \leq |f(x)| \leq T|f|(x) = \bar{T}|\bar{f}|(x) = 0 \), thus \( \overline{S} \) is well defined. We show that \( 0 \leq \overline{S} \leq \overline{T} \). Let \( \bar{f}(x, y) \geq 0 \) \( \nu \)-a.e.
and let \( C = \{(x, y) : f(x, y) < 0 \} \). We again have that \( C = X \times N \)
with \( C(x) = \{ y : f(y) < 0 \} = N \) and \( \nu(x, N) = 0 \) for almost every \( x \in X \). We thus have that \( T(1_N|f|)(x) = \int_N |f|(y)\nu(x, dy) = 0 \)
for almost every \( x \). From \( 0 \leq S \leq T \) we get that \( S(1_N|f|)(x) = 0 \)
for almost every \( x \). Therefore \( S(1_Nf)(x) = 0 \) for almost every \( x \).

We have that \( 1_{N^c}f(x) \) is positive for all \( y \) and so it is a positive
function in \( L \) and \( 0 \leq S(1_{N^c}f)(x) \leq T(1_{N^c}f)(x) \) for almost every \( x \). Putting these together we see that \( 0 \leq \overline{S}f(x) \leq \overline{T}f(x) \) \( \mu \)-a.e.
on \( X \).

We use the theorem of Kantorovich to extend \( \overline{S} \) as a mapping on \( \overline{L} \) to a mapping \( \overline{S} : I(\overline{L}) \rightarrow M \) such that \( 0 \leq \overline{S} \leq \overline{T} \). Now \( \overline{T} \)
is interval preserving, i.e., it has the Maharam property, for, if \( 0 \leq g \leq \overline{T}h \) put

\[
\sigma(x) = \begin{cases} 
g/\overline{T}h(x) & \text{if } (\overline{T}h) \neq 0 \\
0 & \text{otherwise.} \end{cases}
\]

We have \( 0 \leq \sigma(x) \leq 1 \) and it is \( \Sigma \)-measurable, i.e., \( \sigma(x)h(x, y) \in I(\overline{L}) \), and

\[
\overline{T}(\sigma h)(x) = \int_Y \sigma(x)h(x, y)\nu(x, dy) \\
= \sigma(x)\overline{T}h \\
= g(x).
\]

By the Luxemburg-Schep Radon-Nikodym theorem we have that \( \overline{S} = \overline{T}\pi \) for some orthomorphism \( 0 \leq \pi \leq I(\overline{L}) \). But it is known
that $\pi h = p(x,y)h(x,y)$ for some $\nu$-measurable function $p(x,y)$ satisfying $0 \leq p(x,y) \leq 1$ $\nu$-a.e. So,

$$\mathcal{F}(h(x)) = \mathcal{T}(p(x,y)v(x,dy))$$

for all $h \in I(L)$. Therefore,

$$(Sf)(x) = \mathcal{F}(f(x)) = \int_Y f(y)p(x,y)v(x,dy)$$

Now define $\nu_S(x,B) = \int_B p(x,y)v(x,dy)$. This is a random measure which generates $\mathcal{F}$ and so it also generates $S$.

(iii) Let $T_1, T_2 \in \mathcal{L}_{rm}(L,M)$. For $T = T_1 - T_2$ we have that $|T| \leq T_1 + T_2$. But $T_1 + T_2 \in \mathcal{L}_{rm}(L,M)$. Thus by (ii) above $|T| \in \mathcal{L}_{rm}(L,M)$. Hence $\mathcal{L}_{rm}(L,M)$ is a Riesz subspace of $\mathcal{L}_b(L,M)$.

(iv) Let $0 \leq |S| \leq T \in \mathcal{L}_{rm}(L,M) \subset \mathcal{L}_b(L,M)$. Then by (ii), $|S| \in \mathcal{L}_{rm}(L,M)$. Again from (ii) and the fact that $S^+ \leq |S| \in \mathcal{L}_{rm}(L,M)$ and $S^- \leq |S| \in \mathcal{L}_{rm}(L,M)$, we get that $S^+, S^- \in \mathcal{L}_{rm}(L,M)$. Therefore $S = S^+ - S^- \in \mathcal{L}_{rm}(L,M)$.

From the proof of the proceeding result we obtain the following:

**Corollary 3.2.7** Let $0 \leq S \leq T$ and let $T$ be generated by the random measure $\nu_T(x,\cdot)$. Then there exists a measurable function $0 \leq p(x,y) \leq 1$ such that $S$ is generated by the random measure $\nu_S(x,\cdot) = p(x,y)\nu_T(x,\cdot)$, i.e.,

$$(Sf)(x) = \int_Y f(y)p(x,y)\nu_T(x,dy)$$

for almost every $x \in X$. 
In particular it follows that there exists a generating measure $\nu_S(x, \cdot)$ for $S$ satisfying $\nu_S(x, B) \leq \nu_T(x, B)$ for all $B \in \Lambda$ and all $x \in X$.

The following result characterizes the random measures which generate the operators $T^+, T^-$ and $|T|$ in terms of the random measure which generates $T$. We first look at the following case: If $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ are ideals of measurable functions and if $T$ is generated by $\nu(x, \cdot)$, then we show that for every $x \in X$ the total variation $|\nu|(x, \cdot)$ is also a random measure which generates the operator $|T|$. This result was proved by Sourour in the special case where the measure spaces in question are standard measure spaces and in which it is assumed, a priori, that $|\nu|(x, B)$ is a measurable function of $x$ for every $B \in \Lambda$. (see [28, Remark 1.3 (ii)]). Without this assumption one would almost be tempted to use the following reasoning:

Let $B \subset Y$ be such that $1_B \in L$. For a signed measure $\nu$ we have that

$$|\nu|(B) = \sup \left\{ \sum_{i=1}^n |\nu(B_i)| : B_i \cap B_j = \emptyset, \bigcup_{i=1}^n B_i = B \right\}.$$ 

Thus for a fixed $x$ we have

$$|\nu|(x, B) = \sup \left\{ \sum_{i=1}^n |\nu(x, B_i)| : B_i \cap B_j = \emptyset, \bigcup_{i=1}^n B_i = B \right\}$$

$$= \sup \left\{ \sum_{i=1}^n (|T1_{B_i}|)(x) : 1_{B_i} \wedge 1_{B_j} = 0, \sum_{i=1}^n 1_{B_i} = 1_B \right\}$$

for $\mu$-a.e. $x \in X$.

$$|T|1_B = \sup \left\{ |Tg_i| : g_i \wedge g_j = 0, \sum_{i=1}^n g_i = 1_B \right\}$$

$$= \sup \left\{ \sum_{i=1}^n |T(1_{B_i})| : 1_{B_i} \wedge 1_{B_j} = 0, \sum_{i=1}^n 1_{B_i} = 1_B \right\}$$
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The problem with this is that the first supremum is a pointwise supremum which holds $\lambda$–a.e. and the second supremum is a supremum taken in Riesz space $L^0(X, \Sigma, \mu)$ and, in general, for uncountable families these two suprema need not be equal as the following well known example shows: For $\alpha \in [0, 1]$ let

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x = \alpha \\ 0 & \text{if } x \neq \alpha. \end{cases}$$

We have that $f_\alpha = 0$ $\lambda$–a.e. on $[0, 1]$ for a Lebesgue measure $\lambda$ and so $\sup_\alpha f_\alpha = 0$ in $L^0([0, 1], \lambda)$. On the other hand, the pointwise supremum of $\{f_\alpha\}$ is 1.

We present a different approach to show, among others, that if $\nu(x, \cdot)$ generates an operator $T$ then $|\nu|(x, \cdot)$ generates the operator $|T|$.

Theorem 3.2.8 Let $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ be ideals of measurable functions and let $T \in \mathcal{L}_{r\nu}(L, M)$. Then there exists a random signed measure $\nu_T(x, \cdot)$ that generates $T$, such that $|T|, T^+$ and $T^-$ are generated by the random measures $|\nu_T|(x, \cdot)$, $\nu_T^+(x, \cdot)$ and $\nu_T^-(x, \cdot)$ respectively. Moreover, $S \in \mathcal{L}_{r\nu}(L, M)$ has a generating random signed measure $\nu_S(x, \cdot)$ such that $(\nu_S \vee \nu_T)(x, \cdot)$ and $(\nu_S \wedge \nu_T)(x, \cdot)$ are random signed measures that generate $S \vee T$ and $S \wedge T$ respectively.

Proof Assume that $T = T_1 - T_2 \in \mathcal{L}_{r\nu}(L, M)$ with $\nu_1(x, \cdot)$ and $\nu_2(x, \cdot)$ random measures which generate $T_1$ and $T_2$ respectively. Let $\nu(x, \cdot) = \nu_1(x, \cdot) + \nu_2(x, \cdot)$. This is a random measure which generates $T_1 + T_2$. We have that $T^+ \leq T_1 + T_2$ and $T^- \leq T_1 + T_2$, therefore, from
Corollary 3.2.7 there exist functions

\[ 0 \leq p_{T^+}(x, y) \leq 1 \text{ such that } p_{T^+}(x, y)\nu(x, \cdot) \text{ generates } T^+ \]

and

\[ 0 \leq p_{T^-}(x, y) \leq 1 \text{ such that } p_{T^-}(x, y)\nu(x, \cdot) \text{ generates } T^-. \]

Define \( p_T(x, y) = p_{T^+}(x, y) - p_{T^-}(x, y) \). Then \( \nu_T(x, \cdot) = p_T(x, y)\nu(x, \cdot) \) generates \( T \), for

\[
(T f)(x) = (T^+ f)(x) - (T^- f)(x) \\
= \int_Y f(y)p_{T^+}(x, y)\nu(x, dy) - \int_Y f(y)p_{T^-}(x, y)\nu(x, dy) \\
= \int_Y f(y)p_T(x, y)\nu(x, dy)
\]

holds \( \mu \)-a.e. on \( X \). Let \( C = \{(x, y) : p_T(x, y) < 0\} \). Then \( C \) is \( \Sigma \times \Lambda \) measurable and so is \( C(x) \) for almost every \( x \in X \). For all \( B \in \Lambda \) we have that

\[
\nu_T(x, B \cap C(x)) = \int_{B \cap C(x)} p_T(x, y)\nu(x, dy) \leq 0
\]

and

\[
\nu_T(x, B \cap C(x)^c) = \int_{B \cap (Y - C(x))} p_T(x, y)\nu(x, dy) \geq 0.
\]

Therefore

\[
\nu_T^-(x, B) = \int_{B \cap C(x)} p_T(x, y)\nu(x, dy) = \int_B p_T^-(x, y)\nu(x, dy)
\]

and

\[
\nu_T^+(x, B) = \int_{B \cap (Y - C(x))} p_T(x, y)\nu(x, dy) = \int_B p_T^+(x, y)\nu(x, dy).
\]
This shows that $\nu_T^-$ and $\nu_T^+$ are random measures. We now show that they generate $T^-$ and $T^+$ respectively. Suppose $p_T^+(x,y)\nu(x,\cdot)$ generates $S_1$ and $p_T^-(x,y)\nu(x,\cdot)$ generates $S_2$. We then have, since $p_T = p_T^+(x,y) - p_T^-(x,y)$ and consequently $p_T^+ \leq p_T^+$ and that $p_T^- \leq p_T^-$, that
\[
T^+ f(x) = \int_Y f(y)p_T^+(x,y)\nu(x,dy) \\
\geq \int_Y p_T^+(x,y)\nu(x,dy) \\
= S_1 f(x).
\]
for all $f \in L^+$. Hence $T^+ \geq S_1$. Similarly, $T^- \geq S_2$. Since $T = S_1 - S_2$ we have that $T^+ \leq S_1$ and $T^- \leq S_2$. Therefore $T^+ = S_1$ and $T^- = S_2$. By putting
\[
|\nu|(x,\cdot) = \nu^+(x,\cdot) + \nu^-(x,\cdot) \\
= p_T^+(x,y)\nu(x,\cdot) + p_T^-(x,y)\nu(x,\cdot) \\
= |p_T(x,y)|\nu(x,\cdot)
\]
we see that $|\nu|(x,B)$ is measurable for all $B \in \Lambda$, therefore it is a random measure and, moreover, it generates $|T|$.

From the preceding proof we see that we have a useful “functional calculus” for the generating random measures: Let $\{T_i\}$ be a finite set of elements of $\mathcal{L}_{rm}(L,M)$ and let $\nu(x,\cdot)$ be a generating measure for $\sup_i |T_i|$. Then, for every $T$ in the ideal generated by $\{T_i\}$ in $\mathcal{L}_{rm}(L,M)$ there exists a $\nu$-measurable function $h_T(x,y)$ such that $h_T(x,y)\nu(x,\cdot)$ is a random signed measure which generates $T$ and the map $T \mapsto h_T$ is a Riesz homomorphism.
We can apply this to derive the following result:

**Proposition 3.2.9** Let $0 \leq T \in \mathcal{L}_{rm}(L, M)$ be generated by the random measure $\nu_T(x, \cdot)$ and let $0 \leq S \in \mathcal{L}_{rm}(L, M)$. The following statements are equivalent:

(i) $S \in \{T\}^{dd}$;

(ii) there exists a $\Sigma \otimes \Lambda$-measurable function $0 \leq p : X \times Y \to \mathbb{R}$ such that for any generating measure $\nu_S(x, \cdot)$ for $S$, $p(x, B) = p\nu_T(x, B)$ for almost every $x \in X$ for all $B \in \Lambda$.

(iii) $S$ has a generating measure $\nu_S(x, \cdot)$ satisfying $\nu_S(x, \cdot) \ll \nu_T(x, \cdot)$ for $\mu$-almost every $x \in X$;

(iv) $S$ has a generating measure $\nu_S(x, \cdot)$ satisfying $\nu_S \ll \nu_T$ on $\Sigma \otimes \Lambda$;

**Proof** (i) $\Rightarrow$ (ii): Let $S \in \{T\}^{dd}$ and define $S_n := S \wedge nT$; then $S_n \uparrow S$ and $S_n \leq nT$, the latter operator generated by the random measure $n\nu_T$. It follows from Corollary 3.2.7 that there exists a $\Sigma \times \Lambda$-measurable function $q_n$ such that $0 \leq q_n \leq 1$ such that $S_n$ is generated by the random measure $q_n(x, y)n\nu_T(x, \cdot) := p_n(x, y)n\nu_T(x, \cdot)$, with $0 \leq p_n(x, y) \leq n\nu_T$-a.e on $X \times Y$. Since $S_n \leq S_{n+1}$ we may assume that $p_n(x, y) \leq p_{n+1}(x, y)$ holds $\nu$-almost everywhere on $X \times Y$. Define $p(x, y) := \sup_n p_n(x, y)$ in
the space $M^+(X \times Y, \Sigma \otimes \Lambda, \nu_T)$. We then have for every $B \in \Lambda$ that
\[
\int_A \nu_S(x, B) d\mu(x) = \int_A S1_B(x) d\mu(x) \\
= \sup_n \int_A S_n1_B d\mu(x) \\
= \sup_n \int_B \int_A p_n(x, y) \nu_T(x, dy) d\mu(x) \\
= \sup_n \int_{A \times B} p_n(x, y) d\nu_T \\
= \int_{A \times B} f(y)p(x, y) d\nu_T \\
= \int_A \int_B p(x, y) \nu_T(x, dy) d\mu(x).
\]
This holds for every $A \in \Sigma$ and hence, $\nu_S(x, B) = \int_B p(x, y) \nu_T(x, dy)$ for $\mu$-almost every $x \in X$. This proves (ii).

(ii) $\Rightarrow$ (iii): If $\nu_T(x, B) = 0$ for some $B \in \Lambda$, then, $\int_B p(x, y) \nu_T(x, dy) = 0$. It follows that the measure $\nu_S(x, \cdot) := p\nu_T(x, \cdot)$, is a generating measure for $S$ for which we have $\nu_S(x, \cdot) \ll \nu_T(x, \cdot)$ for every $x \in X$ and so (iii) holds.

(iii) $\Rightarrow$ (iv): Let $N \in \Sigma$ be such that $\mu(N) = 0$ and $\nu_S(x, \cdot) \ll \nu_T(x, \cdot)$ for all $x \in X - N$. Take $C \in \Sigma \otimes \Lambda$ such that $\nu_T(C) = 0$. Then there exists a set $M \in \Sigma$ such that $\mu(M) = 0$ and $\nu_T(x, C(x)) \ll \nu_T(x, \cdot)$ for all $x \in X - M$. Hence, if $x \in X - (N \cup M)$, then $\nu_S(x, C(x)) = 0$, so $\nu_S(x, C(x)) = 0$ $\mu$-almost everywhere on $X$. This implies that $\nu_S(C) = \int_X \nu_S(x, C(x)) d\mu = 0$.

(iv) $\Rightarrow$ (ii): Applying the Radon-Nikodym theorem there exists some
0 ≤ \( p(x, y) \) such that \( \nu_S = \mu_T \). In particular, for every \( A \in \Sigma \) we have

\[
\int_A \nu_S(x, B) = \nu_S(A \times B) = \int_{X \times Y} 1_{A \times B} p(x, y) d\nu_T = \int_A \int_B p(x, y) \nu_T(x, dy),
\]

from which it follows that \( \nu_S(x, B) = \int_B p(x, y) \nu_T(x, dy) \) for \( \mu \)-almost every \( x \in X \) and we are done.

(ii) \( \Rightarrow \) (i):

Let \( p_n(x, y) := n \wedge p(x, y) \) and set \( S_n f(x) := \int_Y f(y) p_n(x, y) \nu_T(x, dy) \). It then follows that \( 0 ≤ S_n ≤ nT \) and so each \( S_n \) is in the ideal generated by \( T \) in \( \mathcal{L}_{rm}(L, M) \). But, since \( p_n(x, y) \uparrow p(x, y) \) we have for every \( 0 ≤ f \in L \) that

\[
S_n f(x) = \int_Y f(y) p_n(x, y) \nu_T(x, dy) \uparrow \int_Y f(y) p(x, y) \nu_T(x, dy) = S f(x)
\]

holds \( \mu \)-almost everywhere on \( X \). Thus \( S_n \uparrow S \) which proves that \( S \in \{T\}^{dd} \).

\( \square \)

Note that the equivalence of (ii), (iii) and (iv) in the above theorem still holds if we replace \( \nu_T(x, \cdot) \) by \( \lambda \), although \( \lambda \) may not generate an operator from \( L \) into \( M \). Therefore, taking \( \nu_T(x, \cdot) = \lambda \), we have the following result in which condition (ii) translates into the assertion that \( S \) is a kernel operator.

**Proposition 3.2.10** Let \( 0 ≤ S : L \to M \) be generated by the random measure \( \nu_S(x, \cdot) \). Then the following are equivalent.
(i) $S$ is a kernel operator.

(ii) $\nu_S(x, \cdot) \ll \lambda$ for almost every $x \in X$.

(iii) $\nu_S \ll \mu \otimes \lambda$ on $\Sigma \otimes \Lambda$.

**Proof (i) ⇒ (ii):** Let $S$ be a kernel operator with kernel $0 \leq p(x, y)$. So $Sf(x) = \int_Y f(y)p(x, y)d\lambda(y)$ for all $f \in L$ showing that $S$ has a generating measure $\nu_S(x, \cdot) = p\lambda(\cdot)$. If $\lambda(B) = 0$ for some $B \in \Lambda$ then $\int_B p(x, y)d\lambda(y) = 0$ for almost every $x \in X$. It follows that $\nu_S(x, \cdot) \ll \lambda$ for almost every $x \in X$.

(ii) ⇒ (iii): Let $N \in \Sigma$ be such that $\mu(N) = 0$ and $\nu_S(x, \cdot) \ll \lambda$ for all $x \in X - N$. Take $C \in \Sigma \otimes \Lambda$ such that $\mu \otimes \lambda(C) = 0$. Then, by Fubini, there exists a set $M \in \Sigma$ such that $\mu(M) = 0$ and $\lambda(C(x)) = 0$ for all $x \in X - M$ with $C(x)$ the section of $C$ at $x$. Hence if $x \in X - (N \cup M)$, then $\lambda(C(x)) = 0$ and so $\nu_S(x, C(x)) = 0$ $\mu$-a.e. on $X$. This implies that $\nu_S(C) = \int_X \nu_S(x, C(x))d\mu(x) = 0$ and therefore $\nu_S \ll \mu \otimes \lambda$ in $\Sigma \otimes \Lambda$ for every $x \in X$.

(iii) ⇒ (i): By the Radon–Nikodym theorem there exists a function $0 \leq p(x, y)$ such that $\nu_S = p(\mu \otimes \lambda)$. Thus, for $A \in \Sigma$ we have

$$\int_A \nu_S(x, B)d\mu(x) = \nu_S(A \times B) = \int_{X \times Y} 1_{A \times B}p(x, y)d(\mu \otimes \lambda) = \int_A \int_B p(x, y)d\lambda(y)d\mu(x),$$
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from which it follows that \( \nu_S(x, B) = \int_B p(x, y) d\lambda(y) \) for \( \mu \)-almost every \( x \in X \), i.e., \( \nu_S(x, \cdot) = p\lambda(\cdot) \). Since \( S \) is generated by the random measure \( \nu_S(x, \cdot) \) we have

\[
Sf(x) = \int_Y f(y) \nu_S(x, dy) = \int_Y f(y) p(x, y) d\lambda(y)
\]

for almost every \( x \in X \) and for all \( f \in L \). So \( S \) is a kernel operator with kernel \( p(x, y) \).

For the proof of the following result we will make use of the following order separable condition:

**Lemma 3.2.11** Let \( L \) and \( M \) be two Archimedean Riesz spaces with \( M \) order separable. Let \((u_n)\) be a sequence in \( L^+ \) such that \( \{u_n : n = 1, 2, \ldots\}^{dd} = L \). Then the space \( L_n(L, M) \) is order separable.

**Proof** Let \((T_\alpha)\) be an upward directed net in \( L_n(L, M) \) such that \( 0 \leq T_\alpha \uparrow T \in L_n(L, M) \). We have that \( 0 \leq T_\alpha u_n \uparrow T u_n \). Also, from the fact that \( M \) is order separable there exists a sequence \((\alpha_n, k)\) such that \( \sup_k T_{\alpha_n, k} u_n = T u_n \). Put \( S = \sup_{n, k} T_{\alpha_n, k} \) in \( L_n(L, M) \). We then have that \( 0 \leq T_{\alpha_n, k} \leq S \leq T \) for all \( n, k \). In particular, for any \( n \) we have \( 0 \leq T_{\alpha_n, k} u_n \leq S u_n \leq T u_n \) for all \( k \) and so \( S u_n = T u_n \). Now \( 0 \leq T - S \) and so \( T - S = 0 \) on the ideal generated by \( \{u_n : n = 1, 2, \ldots\} \).

Since \( T - S \) is order continuous we get that \( T - S = 0 \) on \( L \). Thus \( T = \sup_{n, k} T_{\alpha_n, k} \). \( \square \)
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Theorem 3.2.12 $\mathcal{L}_{rm}(L, M)$ is a band in $\mathcal{L}_b(L, M)$.

Proof The space $\mathcal{L}_n(L, M)$ is order separable since the underlying measure spaces are assumed to be $\sigma$-finite. It is therefore sufficient to show that if $(T_n)$ is a sequence in $\mathcal{L}_{rm}(L, M)$ such that $0 \leq T_n \uparrow T$ in $\mathcal{L}_n(L, M)$, then it follows that $T \in \mathcal{L}_{rm}(L, M)$. Put $T_0 = 0$ and let $S_n = T_n - T_{n-1}$ for all $n = 1, 2, \ldots$. We have that $0 \leq S_n \leq T_n$ and since $\mathcal{L}_{rm}(L, M)$ is an ideal in $\mathcal{L}_n(L, M)$, it then follows that $0 \leq S_n \in \mathcal{L}_{rm}(L, M)$ and so $S_n$ is generated by some random measure $\tau_n(x, \cdot)$.

Let $\nu_n(x, \cdot) = \sum_{k=1}^n \tau_k(x, \cdot)$, then $\nu_n(x, \cdot)$ is a generating measure for $T_n$ for all $n = 1, 2, \ldots$. By definition, $\nu_n(x, B) \uparrow_n$ for all $B \in \Lambda$ and all $x \in X$. Define, for all $B \in \Lambda$ and all $x \in X$

$$\nu(x, B) = \sup_n \nu_n(x, B).$$

The function $x \mapsto \nu(x, B)$ is $\Sigma$-measurable on $X$ for every $B \in \Lambda$ and $\nu(x, \cdot)$ is a random measure on $X$.

Lastly we show that $\nu(x, \cdot)$ is a generating measure for $T$.

Let $B \in \lambda$ be such that $f = 1_B \in L$. Then

$$Tf(x) = \sup_n T_n f(x)$$

$$= \sup_n \nu_n(x, B)$$

$$= \nu(x, B)$$

$$= \int_Y f(y) \nu(x, dy) \mu \text{ a.e. on } X.$$
This then implies that $Tf(x) = \int_Y f(y)\nu(x, dy)$ holds $\mu$-a.e. on $X$ for the step function $f$ given by

$$f = \sum_{j=1}^{k} \alpha_j 1_{B_j}$$

(3.2.1)

where $0 \leq \alpha_j \in \mathbb{R}$ and $B_j \in \Lambda$ for $j = 1, 2, \ldots, k$. Let $0 \leq f \in L$ be arbitrary, then there exists a sequence $(f_n)$ in $L$ such that each $f_n$ is of the form (3.2.1) and $0 \leq f_n \uparrow f$ holds $\lambda$-a.e. on $Y$. Since $T$ is order continuous, we also have that $Tf_n \uparrow Tf$ $\mu$-a.e. on $X$. Using the monotone convergence theorem we then get that

$$\int_Y f_n(y)\nu(x, dy) \uparrow \int_Y f(y)\nu(x, dy)\nu$ a.e. on $X$.

Thus for each $n$ we have that $Tf_n(x) = \int_Y f_n(y)\nu(x, dy)\mu$-a.e. on $X$. We therefore conclude that $Tf(x) = \int_Y f(y)\nu(x, dy)$ holds $\mu$-a.e. on $X$ and the proof is complete.

\[\square\]

### 3.3 Random measure-representable operators

Here we develop a theory of random measure-representable operators analogous to the theory of MCE-representable operators. Our main theorem will be that every order continuous operator is random measure-representable. This gives us an extension of the theorem by Sourour [28] to a more general setting.
Consider measure spaces \((Y, \Lambda, \lambda)\) and \((X, \Sigma, \mu)\) with \((X, \Sigma, \mu)\) complete \(\sigma\)-finite and let \(L \subseteq L^0(Y, \Lambda, \lambda)\) and \(M \subseteq L^0(X, \Sigma, \mu)\) be order dense ideals of measurable functions.

**Definition 3.3.1** We say that a linear operator \(T : L \rightarrow M\) is random measure-representable if there exist \(\sigma\)-finite measure spaces \((\Omega_1, \mathcal{F}_1, \pi_1)\) and \((\Omega_2, \mathcal{F}_2, \pi_2)\), order dense ideals \(L_{\Omega_2} \subseteq L^0(\Omega_2, \mathcal{F}_2, \pi_2)\) and \(M_{\Omega_1} \subseteq L^0(\Omega_1, \mathcal{F}_1, \pi_1)\) and order continuous Riesz homomorphisms \(\phi_L : L \rightarrow L_{\Omega_2}\) and \(\psi_M : M_{\Omega_1} \rightarrow M\) with \(\phi_L\) surjective and \(\psi_M\) a Riesz isomorphism onto an ideal in \(M\); such that

\[
T = \psi_M \circ T_\nu \circ \phi_L,
\]

with \(T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})\), i.e., such that the following diagram commutes

\[
\begin{array}{ccc}
L & \xrightarrow{T} & M \\
\phi_L \downarrow & & \downarrow \psi_M \\
L_{\Omega_2} & \xrightarrow{T_\nu} & M_{\Omega_1}
\end{array}
\]

As in the case of MCE-representable operators the following also holds.

**Proposition 3.3.2** Let \((X, \Sigma, \mu)\) and \((Y, \Lambda, \nu)\) be \(\sigma\)-finite measure spaces and let \(L \subseteq L^0(Y, \Lambda, \nu)\) and \(M \subseteq L^0(X, \Sigma, \mu)\) be ideals with carriers \(Y\) and \(X\) respectively. Then the linear operator \(T : L \rightarrow M\) is random measure-representable if and only if there exist \(\sigma\)-finite measure spaces \((\Omega_1, \mathcal{F}_1, \pi_1)\) and \((\Omega_2, \mathcal{F}_2, \pi_2)\), Riesz homomorphisms \(\phi_L : L^0(Y, \Lambda, \nu) \rightarrow L^0(\Omega_2, \mathcal{F}_2, \pi_2)\) and \(\psi_M : L^0(\Omega_1, \mathcal{F}_1, \pi_1) \rightarrow L^0(X, \Sigma, \mu)\) with \(\phi_L(1)\) strictly positive and \(\psi_M 1 - 1\) and interval preserving and \(T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})\) with \(L_{\Omega_2} = \phi(L)\) and \(M_{\Omega_1} = \psi_M M_{\Omega_1}\).
\( \psi_M^{-1}(M) \) order dense ideals such that

\[
T = \psi_M \circ T_\nu \circ \phi_L,
\]

i.e., such that the following diagram commutes

\[
\begin{array}{c}
L^0(Y; \Lambda, \nu) \supset L \xrightarrow{T} M \subset L^0(X; \Sigma, \mu) \\
\phi_L \downarrow \quad \quad \quad \quad \quad \quad \psi_M \uparrow \\
L^0(\Omega_2; \mathcal{F}_2, \pi_2) \supset L_{\Omega_2} \xrightarrow{T_\nu} M_{\Omega_2} \subset L^0(\Omega_1; \mathcal{F}_1, \pi_1)
\end{array}
\]

**Proof** The proof of the corresponding result for MCE-representable operators (Proposition 2.2.4 and Proposition 2.2.5) involve only properties of the ideals and Riesz homomorphisms that are used there, which are similar to those in this theorem. We can, therefore, apply the same argument replacing \( S_m \) with \( T_\nu \).

As in the case of MCE-representable operators, putting \( \Omega = \Omega_1 \times \Omega_2 \), \( \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \), and \( \pi = \pi_1 \otimes \pi_2 \) We call \( \Phi = (\Omega, \mathcal{F}, \pi, \phi_L, \psi_M) \) a representing triple for the operator \( T \). We will denote the set of all linear operators \( T : L \to M \) which are random measure-representable via the triple \( \Phi \) by \( \mathcal{L}_\Phi(L, M) \) and the operator \( T \in \mathcal{L}_\Phi(L, M) \) will be called \( \Phi \)-representable.

The following are examples that show that operators generated by random measures and kernel operators are amongst the class of operators that are random measure-representable.

**Example 3.3.3** (1) Let \( (X, \Sigma, \mu) \) and \( (Y, \Lambda, \lambda) \) be \( \sigma \)-finite measure spaces and let \( L \subset L^0(Y, \Lambda, \lambda) \) and \( M \subset L^0(X, \Sigma, \mu) \) be ideals with carriers \( Y \).
and \(X\) respectively. Let \(T \in \mathcal{L}_{rm}(L, M)\). If we take \(\phi_L\) and \(\psi_M\) to be identity operators then \(T\) is Random measure-representable.

(2) Consider \(\sigma\)-finite measure spaces \((X, \Sigma, \mu)\) and \((Y, \Lambda, \lambda)\) and let \(L \subset L^0(Y, \Lambda, \lambda)\) and \(M \subset L^0(X, \Sigma, \mu)\) be ideals with carriers \(Y\) and \(X\) respectively. Let \(K : L \to M\) be the absolute kernel operator from \(L\) into \(M\) defined by

\[
K f(\omega) = \int_Y k(\omega, x) f(x) d\lambda(x) \text{ for all } f \in L,
\]

with kernel \(k\). We have that the \(k(\omega, \cdot)\lambda(\cdot)\) is a random measure that generates \(K\) (see Example 3.2.2 (1) on Page 68). By (1) above \(K\) is random measure-representable.

**Proposition 3.3.4** \(\mathcal{L}_\phi(L, M)\) is a band in \(\mathcal{L}_n(L, M)\).

**Proof** Let the carrier of \(\phi_L\) be denoted by \(C_{\phi_L}\) and put \(L_1 := L \cap C_{\phi_L}\). We have that \(C_{\phi_L}\) is a band in \(L^0(Y, \Lambda, \lambda)\) and so \(L_1\) is a band in \(L\). Let \(\phi_1\) be the restriction of \(\phi_L\) to \(L_1\), then \(\phi_1\) is a Riesz isomorphism from \(L_1\) to \(L_{\Omega_1}\). By Proposition 1.4.3 (ii) we get that \(\text{ran}(\psi_M)\) is a band in \(L^0(X, \Sigma, \mu)\). If we put \(M_1 := M \cap \text{ran}(\psi_M)\) then \(M_1\) is a band in \(M\). Let the restriction of \(\psi_M\) to \(M_1\) be denoted by \(\psi_1\). Then \(\psi_1\) is a Riesz isomorphism from \(M_{\Omega_1}\) onto \(M_1\). Using these Riesz isomorphisms the space \(\mathcal{L}_n(L, M)\) can be identified with the band in \(\mathcal{L}_n(L_1, M_1)\) consisting of all operators \(T \in \mathcal{L}_n(L, M)\) for which the carrier of \(T\), denoted by \(C_T\), is in \(L_1\) and \(\text{ran}(T)\) is contained in \(M_1\). From Proposition 3.3.2
we get that $L_\phi(L,M) \subset L_n(L_1,M_1)$. It will therefore suffice to show that $L_\phi(L,M)$ is a band in $L_n(L_1,M_1)$. To that end we define

$$\tau : L_n(L_{\Omega_1}, M_{\Omega_2}) \rightarrow L_n(L_1, M_1)$$

by

$$\tau(S) = \psi_1 \circ S \circ \phi_1$$

for all $S \in L_n(L_{\Omega_1}, M_{\Omega_2})$. Note that $\tau$ is bipositive and therefore a Riesz isomorphism from $L_n(L_{\Omega_1}, M_{\Omega_2})$ onto $L_n(L_1, M_1)$. We can now express the relation $T = \psi_\nu \circ T_\nu \circ \phi_L$ in Proposition 3.3.2 by $T = \tau(S_\nu)$ and so

$$L_\phi(L,M) = \tau(L_{rm}(L_{\Omega_1}, M_{\Omega_2})).$$

By Theorem 3.2.12 $L_{rm}(L_{\Omega_1}, M_{\Omega_2})$ is a band in $L_n(L_{\Omega_1}, M_{\Omega_2})$. It therefore follows that $L_\phi(L,M)$ is a band in $L_n(L_1, M_1)$, and hence a band in $L_n(L,M)$. \qed

The main result of this section is that every order continuous operator is random measure-representable. As in the last section we consider order dense ideals $L$ and $M$ of almost everywhere finite measurable functions on measure spaces $(Y, \Lambda, \lambda)$ and $(X, \Sigma, \mu)$ respectively. We note firstly that for an operator from $L$ into $M$ we may assume without loss of generality that its range is in $L^0(X, \Sigma, \mu)$, for if it is a random measure representable operator from $L$ into the latter space, then it is also representable as an operator from $L$ into $M$. For the same reason we can consider $T$ defined on its natural domain, which is defined to be $\{ f \in L^0(Y, \Lambda, \lambda) : T|f| \in L^0(X, \Sigma, \mu) \}$. From
CHAPTER 3. OPERATORS DEFINED BY RANDOM MEASURES

this observation we infer that we may assume without loss of generality that
$L$ has a weak order unit. This is a consequence of the next lemma which was
communicated to us by A.R. Schep.

**Lemma 3.3.5** Let $0 \leq T : L \to M$ and let $0 \leq f_n$ be disjoint in $L$.
Then there exists $0 \leq g_n \leq f_n$ such that $\{g_n\}_{n=1}^{\infty}$ and such that
$\sum_{n=1}^{\infty} Tg_n(x) < \infty \mu$-a.e. on $X$.

**Proof** Fix $n$. Then $(1/k)Tf_n(x) \to 0$ $\mu$-a.e. as $k \to \infty$. It follows that
there exists some $k_n$ such that $\mu\{x : (1/k_n)Tf_n(x) > 2^{-n}\} < 2^{-n}$.
Let $g_n := f_n/k_n$. Let $\varepsilon > 0$ and choose $N$ such that $2^{-N} < \varepsilon$. Then
$\mu\{x : Tg_n(x) > 2^{-n}\} < 2^{-n}$ for all $n \geq N + 1$ implies

$$
\mu\{x : \sum_{n=N+1}^{\infty} Tg_n(x) > 1\} \leq \mu\left( \bigcup_{n=N+1}^{\infty} \{x : Tg_n(x) > 2^{-n}\} \right)
\leq \sum_{n=N+1}^{\infty} 2^{-n} = 2^{-N} < \varepsilon.
$$

Since $Tg_1, \ldots, Tg_N$ are finite $\mu$-a.e., we get that

$$
\mu\{x : \sum_{n=1}^{\infty} Tg_n(x) = \infty\} < \varepsilon.
$$

Since this holds for arbitrary $\varepsilon > 0$, we have that $\sum_{n=1}^{\infty} Tg_n(x) < \infty$
holds $\mu$-a.e. on $X$. \qed

In our case we apply the lemma to the disjoint functions $1_{Y_n+1 \setminus Y_n}$ with
$Y_n \uparrow Y$ as sequence such that $1_{Y_n} \in L$. The function $g := \sum_n g_n$ constructed
in the lemma is then strictly positive and belongs to the natural domain of $T$.

Let $0 \leq u \in L$ be a weak order unit for $L$. The principal ideal generated by $u$, denoted by $L_u$, normed by the gauge function of $[-u,u]$ is an AM-space with unit $u$. By the Kakutani-Krein theorem (see [21, Theorem 2.1.3] or [24, Theorem II.7.4, Corollary 1]), $L_u$ is Riesz and isometric isomorphic to a space $C(\Omega)$ with $\Omega$ a compact Hausdorff space. If $\phi$ denotes this Riesz isomorphism, then $\phi(u) = 1_{\Omega}$. Since $L_u$ is Dedekind complete, so also is $C(\Omega)$ and this implies that $\Omega$ is an extremally disconnected topological space (i.e., the closure of each open set is open).

Since our results are measure theoretic a few facts on the $\sigma$-algebra of subsets of $\Omega$ inherited from $\Lambda$ are recalled. Since $u$ is a strictly positive function, its Boolean algebra of components are in one-to-one correspondence with the elements of the measure algebra associated with $\Lambda$ (we will not distinguish between the elements of $\Lambda$ and the elements of the measure algebra). On the other hand $\phi$ establishes a one-to-one correspondence between the components of $u$ and those of $1_{\Omega}$. The latter components are of the form $1_C$ and we have that such a $C$ is both open and closed (clopen). Conversely, for each clopen set $C \subset \Omega$, $1_C$ is a continuous function and hence a component of $1_{\Omega}$ in $C(\Omega)$. It follows that there is a one-to-one correspondence between the measure algebra associated with $\Lambda$, and the set of clopen subsets of $\Omega$.

From this it follows already that in our case the set $C$ of clopen subsets of $\Omega$ is a $\sigma$-algebra (a fact which can also be proven directly using the fact
that $C(\Omega)$ is $\sigma$-Dedekind complete in this case. For every clopen set $C$, we have $C = \overline{C}$ and so $C$ is trivially an open $F_\sigma$ set. Since the $\sigma$-algebra $\mathcal{A}(\Omega)$ of Baire subsets of $\Omega$ is generated by the open $F_\sigma$ sets, we have that $C \subset \mathcal{A}(\Omega)$. Conversely, since every positive $f \in L_u$ can be approximated $u$-uniformly by linear combinations of components of $u$ (Freudenthal), and since $\phi$ is order continuous, every positive element of $C(\Omega)$ can be uniformly approximated by $C$-step functions and so $f$ is $C$-measurable. However, the smallest $\sigma$-algebra of subsets of $\Omega$ with respect to which all continuous functions on $\Omega$ are measurable is $\mathcal{A}(\Omega)$. Therefore, $\mathcal{A}(\Omega) \subset C$. Thus, the image of $\Lambda$ is the $\sigma$-algebra $\mathcal{A}(\Omega)$ of Baire subsets of $\Omega$.

It follows from the characterization of $\mathcal{A}(\Omega)$ as $C$, that every $\mathcal{A}(\Omega)$ step function is continuous and so every positive, bounded Baire-measurable function is continuous. Denoting the image of the measure $\lambda$ on $\mathcal{A}(\Omega)$ by $\hat{\lambda}$, it follows from this that $L^\infty(\Omega, \mathcal{A}(\Omega), \hat{\lambda}) = C(\Omega)$.

**Theorem 3.3.6** Let $(Y, \Lambda, \lambda)$ and $(X, \Sigma, \mu)$ be $\sigma$--finite measure spaces and let $L$ and $M$ be order dense ideals in $L^0(Y, \Lambda, \lambda)$ and $L^0(X, \Sigma, \mu)$ respectively. If $T : L \to M$ is an order continuous linear operator then $T$ is random measure-representable.

**Proof** We assume, without loss of generality that $L$ has a weak order unit, i.e., assume that there exists a strictly positive function $u$ on $Y$ such that $u \in L$. Furthermore it is sufficient to prove the theorem for a positive $T$. Therefore, let $0 \leq T \in \mathcal{L}_+(L, M)$, i.e., $T$ is an order continuous operator from $L$ to $M$, and let $v = Tu \in M$. Denote the principal
ideals generated by \( u \) and \( v \) in \( L \) and \( M \) respectively by \( L_u \) and \( M_v \). Then \( T : L_u \to M_v \) is also order continuous.

Let \( \Omega_u \) and \( \Omega_v \) be compact Hausdorff spaces such that \( L_u \) and \( M_v \) are Riesz isometric isomorphic to \( C(\Omega_u) \) and \( C(\Omega_v) \), respectively, via the Riesz isomorphisms \( \phi_u : L_u \to C(\Omega_u) \) and \( \phi_v : M_v \to C(\Omega_v) \). The Riesz isomorphisms \( \phi_u \) and \( \phi_v \) are surjections. It follows that they are order continuous Riesz isomorphisms. Indeed, for \( \phi_u \), if we let \( f_n \uparrow f \) in \( L \) we have that \( \phi_u(f_n) \uparrow \phi_u(f) \). Since \( L_u \) is Dedekind complete, so is \( \phi_u(L_u) \) and hence there exists a \( g \in C(\Omega_u) \) such that \( \phi_u(f_n) \uparrow g \) in \( C(\Omega_u) \), and \( g \leq \phi_u(f) \). This implies that \( \phi_u^{-1}(\phi_u(f_n)) \uparrow \phi_u^{-1}(g) \). Hence \( f_n \leq \phi_u^{-1}(g) \) and so \( f \leq \phi_u^{-1}(g) \).

Thus \( \phi_u(f) \leq g \). Therefore \( \phi_u(f) = g \), i.e., \( \phi_u(f_n) \uparrow \phi_u(f) \) and this proves our assertion. (This actually holds for any Riesz isomorphism which is onto, and so for \( \phi_u^{-1} \) as well). Note that \( \phi_u(u) = 1_{\Omega_u} \) and \( \phi_v(v) = 1_{\Omega_v} \) and that \( \mathcal{A}(\Omega_u) \) and \( \mathcal{A}(\Omega_v) \) are images of \( \Lambda \) and \( \Sigma \) under \( \phi_u \) and \( \phi_v \), respectively. We also have that \( L^\infty(\Omega_u, \mathcal{A}(\Omega_u), \lambda) = C(\Omega_u) \).

We define the positive linear map \( \overline{T} : C(\Omega_u) \to C(\Omega_v) \) by \( \overline{T}g = \phi_v \circ T \circ \phi_u^{-1}(g) \) for \( g \in C(\Omega_u) \). Since \( \phi_v \) and \( \phi_u^{-1} \) are Riesz isomorphisms onto, it then follows that \( \overline{T} \) is order continuous. Furthermore, \( \overline{T}1_{\Omega_u} = 1_{\Omega_v} \), so \( \overline{T} \) is a continuous map from the Banach space \( C(\Omega_u) \) into \( C(\Omega_v) \) and \( ||\overline{T}|| = 1 \).

Let \( \overline{T}' : C(\Omega_v)' \to C(\Omega_u)' \) be the adjoint of \( \overline{T} \) mapping the dual of \( C(\Omega_v) \) into \( C(\Omega_u)' \), the dual of \( C(\Omega_u) \). For each \( x \in \Omega_v \) let \( F_x \in C(\Omega_v)' \)
be defined by \( F_x = T^* \delta_x \) where \( \delta_x(g) = g(x) \) is a continuous linear functional in \( C(\Omega_u) \). Then

\[
F_x(f) = (T^* \delta_x)(f) = \delta_x(T f) = (T f)(x) \quad \text{and} \quad F_x(1_{\omega_u}) = 1.
\]

By the Riesz Representation theorem (see [6, Theorem 4.3.9]) \( F_x \) can be represented by a unique probability measure \( \nu(x, \cdot) \) defined on \( A(\Omega_u) \) such that

\[
\int_{\Omega_u} f(y) \nu(x, dy) = F_x(f) = (T f)(x) \tag{3.3.1}
\]

for all \( f \in C(\Omega_u) = L^\infty(\Omega_u, A(\Omega_u), \hat{\lambda}) \). Thus, for each \( f \in Lu \) we have that

\[
\overline{T} (\phi_u f)(x) = \int_{\Omega_u} \phi_u f(y) \nu(x, dy),
\]

and since \( Tf = \phi_u^{-1}(\overline{T} \phi_u f) \) we have that

\[
[\phi_u(Tf)](x) = \int_{\Omega_u} (\phi_u f)(y) \nu(x, dy).
\]

From Equation (3.3.1) it follows that \( \nu(x, C) \) is \( A(\Omega_u) \)-measurable for every \( C \in A(\Omega_u) \). Therefore \( \nu(x, \cdot) \) is a random measure on \( A(\Omega_u) \).

Thus \( \overline{T} : L^\infty(\Omega_u, A(\Omega_u), \hat{\lambda}) \rightarrow L^\infty(\Omega_u, A(\Omega_u), \hat{\mu}) \) is generated by the random measure \( \nu(x, \cdot) \), i.e. the following diagram commutes

\[
\begin{array}{ccc}
Lu & \xrightarrow{T} & M_\nu \\
\phi_u \downarrow & & \phi_\nu \\
C(\Omega_u) & \xrightarrow{\overline{T}} & C(\Omega_u)
\end{array}
\]

Note that the Riesz homomorphism \( \phi_u \) can be extended to an order continuous Riesz homomorphism \( \phi_L \) from \( L^0(Y, \Lambda, \lambda) \) into \( L^0(\Omega_2, \mathcal{F}_2, \pi_2) \).
and similarly $\phi_v^{-1}$ can be extended to an order continuous Riesz homomorphism $\psi_M$ from $L^0(\Omega_1, \mathcal{F}_1, \pi_1)$ into $L^0(X, \Sigma, \mu)$. Since $L_u$ is order dense in $L$ we have, for $0 \leq g \in L$, that there exists a sequence $0 \leq (g_n)$ in $L_u$ such that $g_n \uparrow g$. So $\phi_L(g_n) \uparrow \phi_L(g)$. Let $f_n = \phi_L(g_n)$. We have that $f_n \uparrow \phi_L(g) := f$, so that $\int_{\Omega_1} f_n(y)\nu(x, dy) \uparrow \int_{\Omega_1} f(y)\nu(x, dy)$. But, $\psi_L \int_{\Omega_1} f_n(y)\nu(x, dy) = Tg_n$ and since $T$ is order continuous we have that $Tg_n \uparrow Tg$, so that

$$
Tg = \psi_L \int_{\Omega_1} f(y)\nu(x, dy) = \psi_L \int_{\Omega_1} \phi_L(g(y))\nu(x, dy).
$$

This holds for all $0 \leq g \in L$ and so for all $g \in L$. Thus $T = \psi_L \circ T \circ \phi_L$. This completes the proof of the theorem. \qed
Chapter 4

Random measures and MCE-Operators

There seems to be a strong connection between conditional expectation operators and operators generated by random measures. By investigating the relationship between these operators, we prove the main theorem of this thesis, which is that every order continuous operator is MCE-representable.

We also forge a link amongst order continuous operators, MCE-representable operators and random measure-representable operators.

This solves many open problems involving MCE-representable operators. For instance, it becomes easy, almost trivial, to show that the sum, composition, product, etc of MCE-representable operators are again MCE-representable operators. Direct proofs of these results are not trivial.

Note that if we put $\mathcal{R}(L, M)$ to be the set of all random measures $\nu(x, \cdot)$ such that $\int_Y f(y)\nu(x, dy) \in M$ for all $f \in L$ then $\mathcal{L}_{rm}(L, M)$ is Riesz isomorphic to $\mathcal{R}(L, M)$. Thus $\mathcal{R}(L, M)$ is a Riesz space since $\mathcal{L}_{rm}(L, M)$ is a Riesz space.
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4.1 Random measures and MCE-Operators

Before we embark on a proof of the main result we first show that operators generated by random measures are MCE-representable, i.e., we want to show that if \( L \subset L^0(Y, \Lambda, \lambda) \) and \( M \subset L^0(X, \Sigma, \mu) \) are ideals of measurable functions and \( T : L \to M \) a positive operator generated by the random measure \( \nu(x, \cdot) \), then \( T \) is MCE-representable.

The case where \( \nu(x, \cdot) \) is a probability measure for each \( x \in X \) and \( \mu \) is also a probability measure on \( X \) is trivial.

To show this let \( \Omega = X \times Y \). Since \( \nu(x, \cdot) \) and \( \mu \) are probability measures, there exists, by the Product of Measures theorem see [6, Theorem 2.6.2], a unique probability measure \( P \) on \( \Sigma \otimes \Lambda \) defined by

\[
P(C) = \int_X \nu(x, C(x)) d\mu(x).
\]

The operator \( T \) is given by \( Tf(x) = \int_Y f(z) \nu(x, dz) \) for almost every \( x \in X \).

For \( f \in L \) define the homomorphism \( \phi_L \) by \( \phi_L(f) = f(z) 1_X(x) \) and for \( g \in M \) define the homomorphism \( \psi_M \) by \( \psi_M^{-1}(g) = g(x) 1_Y(y) \). Let \( f \in L \), put \( \mathcal{F}_M = \{ B \times Y, \text{ such that } B \in \Sigma \} = \Sigma \otimes Y \). We have that

\[
\int_{B \times Y} (Tf)(x) 1_Y(y) dP = \int_B \int_Y Tf(x) 1_Y(y) \nu(x, dy) d\mu(x)
= \int_B \int_Y f(z) \nu(x, dz) \nu(x, dy) d\mu(x),
\]

which by the theorem of Fubini (see [6, Theorem 2.6.6]) we get

\[
\int_{B \times Y} (Tf)(x) 1_Y(y) dP = \int_B \int_Y f(z) \nu(x, dz) \int_Y \nu(x, dy) d\mu(x).
\]
Since $\int_Y \nu(x, dy) = 1$ we then have that
\[
\int_{B \times Y} (Tf)(x) 1_Y(y) d\mathbb{P} = \int_B \int_Y f(z) \nu(x, dz) d\mu(x) = \int_{B \times Y} f(z) 1_X(x) d\mathbb{P}.
\]

Since $(Tf)(x) 1_Y(y)$ is $\mathcal{F}_M$-measurable, this shows that
\[
Tf(x) 1_Y(y) = \mathbb{E}(f 1_X(x) \mid \mathcal{F}_M).
\]

Thus $\psi^{-1}_M(Tf) = \mathbb{E}(\phi_L(f) \mid \mathcal{F}_M)$, and so $\psi_M \mathbb{E}(\phi_L(f) \mid \mathcal{F}_M) = Tf$.

We present a formal proof for the general case. We will use the following convention:

**Convention(A)** If $h : \Omega \rightarrow \mathbb{R}$ is a measurable function we define
\[
\frac{1}{h} = \begin{cases} \frac{1}{h(\omega)}, & h(\omega) \neq 0 \\ 0, & h(\omega) = 0 \end{cases}
\]
The function $\frac{1}{h}$ is thus again a measurable function.

**Theorem 4.1.1** If $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ are ideals of measurable functions and $T : L \rightarrow M$ a positive operator generated by the random measure $\nu(x, \cdot)$, then $T$ is MCE-representable.

**Proof** Since $L$ is order dense, there exists a sequence $Y_n \uparrow Y$ such that $1_{Y_n} \in L$ and $T1_{Y_n} = \nu(x, Y_n) \in M$ for each $n$. For $B \in \Lambda$ put $\nu_n(x, B) = \nu(x, B \cap Y_n)$. Following convention (A), we define
\[
\overline{\nu}(x, \cdot) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu_n(x, Y_n)} \nu_n(x, \cdot).
\]
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Let \( \tilde{X} = \{ x \in X : \nu(x, Y) \neq 0 \} \). Since \( \nu(x, B) = 0 \) for all \( x \in \tilde{X}^c \) we have that \( \tilde{\nu}(x, \cdot) = 0 \) for all \( x \in \tilde{X}^c \). Also, since \( \nu(x, Y_n) \uparrow \nu(x, Y) > 0 \), we have that \( \tilde{\nu}(x, \cdot) \geq 0 \) for all \( x \in \tilde{X} \). Therefore

\[
\tilde{\nu}(x, Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu_n(x, Y_n)} \nu_n(x, Y)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(x, Y \cap Y_n)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_n(x, Y_n)
\]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{2^n}
\]

\[
= 1.
\]

This holds for every \( x \in \tilde{X} \). Put, using convention (A),

\[
\frac{\tilde{\nu}(x, \cdot)}{\tilde{\nu}(x, Y)} = \frac{\nu(x, \cdot)}{\nu(x, Y)}
\]

Note that \( \frac{\tilde{\nu}(x, \cdot)}{\tilde{\nu}(x, Y)} \) is non-zero for all \( x \in \tilde{X} \). Also, \( \tilde{\nu}(x, Y) = 1 \) for all \( x \in \tilde{X} \), i.e., it is a probability measure on \( Y \) for all \( x \in \tilde{X} \) and so \( \tilde{\nu}(x, Y) = 1_{\tilde{X}}(x) \). It follows that \( \tilde{\nu}(x, \cdot) \) is uniformly \( \sigma \)-finite (in fact uniformly finite).

We have that \( \nu \) restricted to \( \tilde{X} \) is a \( \sigma \)-finite measure and it can be normalized to be a probability measure on \( \tilde{X} \) as we did with the measure \( \nu \). We denote this normalized measure by \( \tilde{\mu} \). Hence there exists a unique product measure \( \mathbb{P} \) on \( \Sigma \times \Lambda \) such that \( \mathbb{P}(A \times B) = \int_{\tilde{X}} \tilde{\nu}(x, B) \tilde{\mu}(A) \)

for
all $A \in \Sigma$ and $B \in \Lambda$. Thus

$$\mathbb{P}(X \times Y) = \int_X \bar{\mu}(x, Y) d\bar{\mu}(x)$$

$$= \int_X 1_X d\bar{\mu}$$

$$= \bar{\mu}(\tilde{X})$$

$$= 1.$$  

It follows that $\mathbb{P}$ is a probability measure on $\Omega = X \times Y$. Let $\mathcal{F} = \Sigma \otimes \Lambda$ and $g$ be a step function given by $g = \sum_{j=1}^k \alpha_j 1_{E_j}$, where $E_j \in \Lambda$. Then

$$\int_Y g(z) \nu_n(x, dz) = \sum_{j=1}^k \alpha_j \int_{E_j} \nu_n(x, dz)$$

$$= \sum_{j=1}^k \alpha_j \nu_n(x, E_j)$$

$$= \sum_{j=1}^k \alpha_j \nu(E_j \cap Y_n)$$

$$= \sum_{j=1}^k \alpha_j \int_{E_j \cap Y_n} \nu(x, dz)$$

$$= \int_Y 1_{Y_n}(z) g(z) \nu(x, dz)$$
so that $\int_Y g(z)\nu_n(x, dz) = \int_Y 1_{Y_n}(z)g(z)\nu(x, dz)$. Thus

$$
\int_Y g(z)\bar{\nu}(x, dz) = \int_Y g(z)\frac{\nu(x, dz)}{\bar{\nu}(x, Y)}
= \int_Y g(z)\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu(x, Y_n)} \nu_n(x, dz) \frac{1}{\bar{\nu}(x, Y)}
= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu(x, Y_n)} \int_Y \frac{1}{\bar{\nu}(x, Y)} g(z)\nu_n(x, dz)
= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu(x, Y_n)} \int_Y \frac{1}{\bar{\nu}(x, Y)} 1_{Y_n}(z)g(z)\nu(x, dz)
= \int_Y \frac{1}{\bar{\nu}(x, Y)} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu(x, Y_n)} 1_{Y_n}(z)g(z)\nu(x, dz).
$$

Put, using convention (A),

$$
h(x, z) = \frac{1}{\bar{\nu}(x, Y)} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu(x, Y_n)} 1_{Y_n}(z).
$$

It follows that

$$
\int_Y g(z)\bar{\nu}(x, dz) = \int_Y h(x, z)g(z)\nu(x, dz) \quad (4.1.1)
$$

for every step function $g$ and so it holds for arbitrary positive functions $g \in L^0(Y, \Lambda, \lambda)$.

For an arbitrary $A$ in $\mathcal{F}_M = \{A \times Y : A \in \Sigma\}$ and $0 < f \in L$ we have,
that

\[
\int_{A \times Y} (Tf)(x)1_Y(y) \, d\mathbb{P} = \int_A \int_Y Tf(x)1_Y(y) \, \bar{\nu}(x, dy) \, d\bar{\mu}(x)
\]

\[
= \int_A \int_Y f(z) \nu(x, dz) \, \bar{\nu}(x, dy) \, d\bar{\mu}(x)
\]

\[
= \int_A \int_Y f(z) \nu(x, dz) \, \bar{\nu}(x) \, d\bar{\mu}(x)
\]

\[
= \int_A \int_Y f(z) \nu(x, dz)1_{\bar{\mathcal{R}}}(x) \, d\bar{\mu}(x)
\]

\[
= \int_A \int_Y f(z) \nu(x, dz) \, d\bar{\mu}(x)
\]

\[
+ \int_A \int_{\mathcal{R}} \int_Y f(z) \nu(x, dz) \, d\bar{\mu}(x)
\]

\[
= \int_A \int_Y f(z) \nu(x, dz) \, d\bar{\mu}(x)
\]

\[
= \int_A \int_Y \frac{1}{h(x, z)} h(x, z) f(z) \nu(x, dz) \, d\bar{\mu}(x),
\]

which, by using Equation (4.1.1) gives

\[
\int_{A \times Y} (Tf)(x)1_Y(y) \, d\mathbb{P} = \int_A \int_Y \frac{f(z)}{h(x, z)} \bar{\nu}(x, dz) \, d\bar{\mu}(x)
\]

\[
= \int_{A \times Y} \frac{1}{h(x, z)} f(z) \, d\mathbb{P}.
\]

We thus have shown that \(E(\frac{1}{h(x, z)} f(z)(x) | \mathcal{F}_M) = Tf(x)1_Y(y)\). Putting \(m(x, \cdot) = \frac{1}{h(x, z)}\), we get that \(\psi_M E(m \phi_L(f) | \mathcal{F}_M) = Tf\). This holds for positive \(f \in L\) and consequently for all \(f \in L\). This completes the proof.

\[\square\]

### 4.2 Main result

We are now in a position to prove the main result of this thesis.
Theorem 4.2.1 Let \((Y, \Lambda, \nu)\) and \((X, \Sigma, \mu)\) be \(\sigma\)-finite measure spaces and let \(L\) and \(M\) be order dense ideals in \(L^0(Y, \Lambda, \nu)\) and \(L^0(X, \Sigma, \mu)\) respectively. If \(T : L \rightarrow M\) is an order continuous linear operator then \(T\) is MCE representable.

Proof From Theorem 3.3.6 we get that the operator \(T\) is random measure-representable. Therefore there exist \(\sigma\)-finite measure spaces \((\Omega_1, \mathcal{F}_1, \pi_1)\) and \((\Omega_2, \mathcal{F}_2, \pi_2)\), order dense ideals \(L_{\Omega_2} \subset L^0(\Omega_2, \mathcal{F}_2, \pi_2)\) and \(M_{\Omega_1} \subset L^0(\Omega_1, \mathcal{F}_1, \pi_1)\) and order continuous Riesz homomorphisms \(\phi'_L : L \rightarrow L_{\Omega_2}\) and \(\psi'_M : M_{\Omega_1} \rightarrow M\) with \(\phi'_L\) surjective and \(\psi'_M\) a Riesz isomorphism onto an ideal in \(M\); such that

\[
T = \psi'_M \circ T_\nu \circ \phi'_L,
\]

with \(T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})\), i.e., such that the following diagram commutes

\[
\begin{array}{ccc}
L & \xrightarrow{T} & M \\
\phi'_L \downarrow & & \uparrow \psi'_M \\
L_{\Omega_2} & \xrightarrow{T_\nu} & M_{\Omega_1}
\end{array}
\]

From Theorem 4.1.1 we get that the operator \(T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})\) is MCE-representable. Thus there exist a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), order dense ideals \(L_\Omega \subset L^0(\Omega, \mathcal{F}_L, \mathbb{P})\) and \(M_\Omega \subset L^0(\Omega, \mathcal{F}_M, \mathbb{P})\), order continuous Riesz homomorphisms \(\phi''_L : L_{\Omega_2} \rightarrow L_\Omega\) and \(\psi''_M : M_\Omega \rightarrow M_{\Omega_1}\), with \(\phi''_L\) surjective and \(\psi''_M\) a Riesz isomorphism onto an ideal in \(M_{\Omega_1}\), and a measurable function \(m \in L^0(\Omega, \mathcal{F}, \mathbb{P})\) such that

\[
T_\nu = \psi''_M \circ S_m \circ \phi''_L,
\]
i.e., such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{L}_\Omega & \xrightarrow{T} & \mathcal{M}_{\Omega_1} \\
\phi_L' \downarrow & & \uparrow \psi_M' \\
\mathcal{L}_\Omega & \xrightarrow{s_m} & \mathcal{M}_\Omega \\
\end{array}
\]

If we take \( \phi_L := \phi_L' \circ \phi_L'' \) and \( \psi_M := \psi_M' \circ \psi_M'' \) then \( \phi_L \) and \( \psi_M \) are order continuous Riesz homomorphisms with \( \phi_L \) surjective and \( \psi_M \) a Riesz isomorphism onto an ideal in \( \mathcal{M} \) and we have that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{T} & \mathcal{M} \\
\phi_L \downarrow & & \uparrow \psi_M \\
\mathcal{L}_\Omega & \xrightarrow{s_m} & \mathcal{M}_\Omega \\
\end{array}
\]

and the proof is complete.

\[\square\]

The following result now follows from the fact that the sum and the composition of order continuous operators are also order continuous.

**Corollary 4.2.2** Let \((Y, \Lambda, \nu), (X, \Sigma, \mu)\) and \((Z, \Gamma, \gamma)\) be \(\sigma\)-finite measure spaces and let \(L, M\) and \(N\) be order dense ideals in \(L^0(Y, \Lambda, \nu), L^0(X, \Sigma, \mu)\) and \((Z, \Gamma, \gamma)\), respectively.

(a) If \(T_1\) and \(T_2\) are order continuous linear operators that map \(L\) into \(M\) then \(T = T_1 + T_2\) is MCE-representable.

(b) If \(T_1 : M \to N\) and \(T_2 : L \to M\) are order continuous linear operators then \(T = T_2 \circ T_1\) is MCE-representable.
Theorem 4.2.3 Let \((Y, \Lambda, \nu)\) and \((X, \Sigma, \mu)\) be \(\sigma\)-finite measure spaces and let \(L\) and \(M\) be order dense ideals in \(L^0(Y, \Lambda, \nu)\) and \(L^0(X, \Sigma, \mu)\) respectively. For a linear operator \(T : L \to M\) the following are equivalent

(i) \(T\) is order continuous;

(ii) \(T\) is random measure-representable;

(iii) \(T\) is MCE-representable.

Proof (i) \(\Rightarrow\) (ii)

See Theorem 3.3.6 on Page 94.

(ii) \(\Rightarrow\) (iii)

See Theorem 4.1.1 on Page 100.

(iii) \(\Rightarrow\) (i)

If \(T\) is MCE-representable then by Definition 2.2.1 on Page 49 there is a representation triple \(\Phi = ((\Omega, \mathcal{F}, \mathcal{P}), \phi_L, \psi_M)\) for the operator \(T\) and a \(\Phi\)-kernel \(m\) of \(T\) such that

\[
T = \psi_M \circ S_m \circ \phi_L.
\]

Since \(\phi_L, S_m\) and \(\psi_M\) are all order continuous we have that \(T\) is order continuous.

We remark that if \(T\) is generated by a random measure then \(T\) is random measure-representable (and therefore MCE-representable). The converse is true if the measure spaces \((Y, \Lambda, \nu)\) and \((X, \Sigma, \mu)\) are standard Borel measure spaces see [27].
Bibliography


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