OPERATORS DEFINED BY CONDITIONAL EXPECTATIONS AND RANDOM MEASURES

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Abstract

This study revolves around operators defined by conditional expectations and operators generated by random measures.

Studies of operators in function spaces defined by conditional expectations first appeared in the mid 1950's by S-T.C. Moy [22] and S. Sidak [26]. N. Kalton studied them in the setting of L_p-spaces $0 < p < 1$ in [15, 13] and in L_1-spaces, [14], while W. Arveson [5] studied them in L_2-spaces. Their averaging properties were studied by P.G. Dodds and C.B. Huijsmans and B. de Pagter in [7] and C.B. Huijsmans and B. de Pagter in [10]. A. Lambert [17] studied their relationship with multiplication operators in C^*-modules. It was shown by J.J. Grobler and B. de Pagter [8] that partial integral operators that were studied A.S. Kalitvin et al in [2, 4, 3, 11, 12] and the special cases of kernel operators that were, inter alia, studied by A.R. Schep in [25] were special cases of conditional expectation operators.

On the other hand, operators generated by random measures or pseudo-integral operators were studied by A. Sourour [28, 27] and L.W. Weis [29, 30], building on the studies of W. Arveson [5] and N. Kalton [14, 15], in the late 1970's and early 1980's.

In this thesis we extend the work of J.J. Grobler and B. de Pagter [8] on Multiplication Conditional Expectation-representable (MCE-representable) operators. We also generalize the result of A. Sourour [27] and show that order continuous linear maps between ideals of almost everywhere finite measurable functions on σ-finite measure spaces are MCE-representable. This fact enables us to easily deduce that sums and compositions of MCE-representable operators are again MCE-representable operators. We also show that operators generated by random measures are MCE-representable.

The first chapter gathers the definitions and introduces notions and concepts that are used throughout. In particular, we introduce Riesz spaces and operators therein, Riesz and Boolean homomorphisms, conditional expectation operators, kernel and absolute τ-kernel operators.

In Chapter 2 we look at MCE-operators where we give a definition different from that given by J.J. Grobler and B. de Pagter in [8], but which we show to be equivalent.

Chapter 3 involves random measures and operators generated by random measures. We solve the problem (positively) that was posed by A. Sourour in [28] about the relationship of the lattice properties of operators generated by random measures and the lattice properties of their generating random measures. We show that the total variation of a random signed measure representing an order bounded operator T, it being the difference of two random measures, is again a random measure and represents $|T|$.

We also show that the set of all operators generated by a random measure is a band in the Riesz space of all order bounded operators.

In Chapter 4 we investigate the relationship between operators generated by random measures and MCE-representable operators. It was shown by A. Sourour in [28, 27] that every order bounded order continuous linear operator acting between ideals of almost everywhere measurable functions is generated by a random measure, provided that the measure spaces involved are standard measure spaces. We prove an analogue of this theorem for the general case where the underlying measure spaces are σ-finite. We also, in this general setting, prove that every order continuous linear operator is MCE-representable. This rather surprising result enables us to easily show that sums, products and compositions of MCE-representable operator are again MCE-representable.

Key words: Riesz spaces, conditional expectations, multiplication conditional expectation-representable operators, random measures.

Opsomming

In hierdie studie word gekyk na operatore gedefinieer deur voorwaardelike verwagtings en operatore voortgebring deur stogastiese mate.

Studies van operatore wat deur voorwaardelike verwagtings in funksieruimtes gedefinieer is, het hul verskyning in die middel vyftiger jare gemaak in publikasies deur S-T.C. Moy [22] en S. Sidak [26]. N. Kalton, [15, 13], het 'n studie daarvan gemaak in L_p-ruimtes, $(0 < p < 1)$ en in L_1-ruimtes, [14], terwyl W. Arveson, [5], dit in L_2-ruimtes beskou het. Die vergemiddelding eienskappe daarvan is deur P.G. Dodds, C.B. Huijsmans en B. de Pagter in [7], en deur C.B. Huijsmans en B. de Pagter in [10] ondersoek. A. Lambert, [17], het die verband van die operatore met vermenigvuldigingsoperatore in C^*-modules ondersoek. J.J. Grobler en B. de Pagter, [8], het bewys dat parsiële integraal operatore soos bestudeer deur A.S. Kalitvin et al. in [2, 4, 3, 11, 12] en die spesiale geval van kern-operatore, soos onder andere bestudeer in [25] deur A.R. Schep, spesiale gevalle is van produkte van voorwaardelike verwagtings en vermenigvuldigings operatore.

Andersyds is operatore voortgebring deur stogastiese mate (ook genoem pseudo-integraal operatore) deur A. Sourour [28, 27] and L.W. Weis [29, 30], bestudeer voortbouend op studies onderneem deur W. Arveson [5] en N. Kalton, [14, 15], in die laat sewentiger en vroeg tagtiger jare.

In hierdie proefskrif brei ons die werk wat J.J. Grobler en B. de Pagter [8] oor vermenigvuldigings voorwaardelike verwagtings representeerbare (vvv-representeerbare) operatore gedoen het uit. Ons veralgemeen ook 'n resultaat van Sourour, [27], en toon aan dat orde kontinue lineêre afbeeldings tussen ideale van byna-oral eindige meetbare funksies op σ-eindige maatruimtes vvv-representeerbaar is. Hierdie feit stel ons in staat om maklik te sien dat somme en samestellings van vvv-representeerbare operatore weer vvv-representeerbaar is. Ons toon ook aan dat operatore voortgebring deur stogastiese mate, vvv-representeerbaar is.

Die eerste hoofstuk dien as inleiding waarin ons die definisies en begrippe wat verder gebruik word saamvat. In die besonder gee ons aandag aan die teorie van Rieszruimtes en die operatore wat daarin 'n rol speel, naamlik Riesz- en Boole-homomorfismes, voorwaardelike verwagtingsoperatore, kernoperatore and absolute τ-kern operatore.

In Hoofstuk 2 bestudeer ons vvv-representeerbare operatore en ons gee 'n definisie daarvan wat verskil van die een wat deur Grobler en de Pagter in [8] gebruik word. Ons toon egter aan dat die twee definisies ekwivalent is.

Hoofstuk 3 bevat die teorie van stogastiese mate en die operatore voortgebring deur stogasiese mate. Ons los 'n probleem gestel deur Sourour in [28] positief op deur aan te toon dat die totale variasie van 'n betekende stogastiese maat wat 'n orde begrensde operator T voortbring en wat die verskil van twee stogastiese mate is, weer 'n stogastiese maat is en dat dit die operator $|T|$ voortbring.

Ons toon ook aan dat die versameling van alle operatore voortgebring deur 'n stogastiese maat 'n band is in die Rieszruimte van alle orde begrensde operatore.

In Hoofstuk 4 ondersoek ons die verband tussen operatore voortgebring deur stogastiese mate en vvv-representeerbare operatore. Sourour het in [28, 27] bewys dat elke orde begrensde ordekontinue lineêre operator wat ideale van byna-oral eindige meetbare funksies afbeeld in ideale van soortgelyke funksies deur stogastiese mate voorgebring word mits die onderliggende maatruimtes standaard maatruimtes is. Ons bewys 'n analoog van hierdie stelling vir die algemener geval waar die onderliggende maatruimtes σ-eindig is. In hierdie algemene geval toon ons ook aan dat elke ordekontinue lineêre operator vvv-representeerbaar is. Hierdie verrassende resultaat stel ons in staat om sonder moeite te wys dat somme, produkte en samestellings van vvv-representeerbare operatore weer vvv-representeerbaar is.

Sleutelterme: Rieszruimtes, voorwaardelike verwagtings, vermenigvuldigings voorwaardelike verwagtings representeerbare-operatore, stogastiese mate.

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Contents

Chapter 1

Preliminaries

1.1 Introduction

Operators in function spaces defined by conditional expectations were first studied by, among others, S - T.C. Moy [22], Z. Sidak [26] and H.D. Brunk in the setting of L^p spaces. Conditional expectation operators on various function spaces exhibit a number of remarkable properties related to the underlying structure of the given function space or to the metric structure when the function space is equipped with the norm. P.G. Dodds, C.B. Huijsmans and B. de Pagter [7] linked these operators to averaging operators defined on abstract spaces earlier by J.L. Kelley [16], while A. Lambert [17] studied their link to classes of multiplication operators which form Hilbert C^*-modules. J.J. Grobler and B. de Pagter [8] showed that the classes of partial integral operators which were studied by A.S. Kalitvin and others (see [12], [2], [3] and [4]) were a special case of conditional expectation operators.

In this thesis we are going to discuss the notion of multiplication conditional expectation operators and extend the work by J.J. Grobler and B. de

Pagter [8] to operators that can be represented by multiplication conditional expectation operators.

We also investigate results obtained by, among others, Sourour [28], [27] and Weis [29] and [30]. They worked on operators in ideals of almost everywhere finite measurable functions on standard measures spaces and showed that these can be generated by random measures. We generalize their results and show that order continuous linear maps between ideals of almost everywhere finite measurable functions on σ-finite measure spaces are Multiplication Conditional Expectation-representable (MCE-representable).

The first chapter gives the preliminaries and the background where definitions, basic concepts and notions that are needed in the sequel are stated. Chapter 2 focuses on characterizing conditional expectation operators as order continuous functions. It also identifies those operators that can be represented by conditional expectation operators. Chapter 3 looks at pseudo-integral operators or those operators that can be generated by random measures. In this chapter we also look at operators that are random measure representable. In Chapter 4 we investigate the relationship between random measure-representable operators and MCE-representable operators. Here we present our main result.

1.2 Riesz spaces

In this section we give a short introduction to the theory of Riesz spaces. We will concentrate only on definitions, notions and results that will be useful later. For deeper results we refer to W.A.J, Luxemburg and A.C. Zaanen [19], H.H. Schaefer [24], P. Meyer - Nieberg [21] and A.C. Zaanen [34].

Definition 1.2.1 A binary relation that is reflexive, anti-symmetric and transitive is called a *partial order.* We will denote a partial order by $\leq$. A set in which a partial order has been defined is called a *partially ordered set.* A partially ordered set X with a partial order $\leq$ will be denoted by $(X, \leq)$.

If, however, the partial order $\leq$ is obvious from the context, or if there is no fear of confusion, we will write X for a partially ordered set $(X, \leq)$.

For elements x and y in a partially ordered set X we will sometimes write $y \geq x$ if $x \leq y$ and $x < y$ to express the fact that $x \leq y$ and $x \neq y$, similarly for $y > x$.

Definition 1.2.2 Let X be a partially ordered set. A set $Y \subset X$ is said to be *bounded from above* if there is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. The element x is called an *upper bound of* Y. An upper bound u of Y is called *the least upper bound* or *supremum of* Y if $u \leq x$ for every upper bound x of Y. The supremum of Y will be denoted by $\sup Y$. The notions of *bounded from below, lower bound* and *the greatest lower bound* or *infimum* of the set $Y \subset X$ are defined similarly with the inequalities reversed. The infimum of Y will be denoted by $\inf Y$.

Definition 1.2.3 Let X be a partially ordered set. For $x, y \in X$ with $x \leq y$, the *order interval* $[x, y]$ is defined as the subset

$$[x, y] = \{z \in X : x \leq z \leq y\}.$$

A subset $A \subset X$ is said to be *order bounded* if A is contained in an order interval.

From the definition of boundedness it is clear that every order bounded set is bounded from above and from below.

Definition 1.2.4 A partially ordered set X is called a *lattice* whenever $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$ exist for all $x, y \in X$.

Definition 1.2.5 A real vector space E which is partially ordered is called an *ordered vector space* if

$$\begin{aligned} x &\leq y \text{ implies that } x + y \leq y + z \text{ for all } x, y, z \in E \\ x &\leq y \text{ implies that } \lambda x \leq \lambda y \text{ for all } x, y \in E \text{ and } 0 < \lambda \in \mathbb{R}. \end{aligned}$$

An ordered vector space is called a *Riesz space* or *vector lattice* if it is a lattice.

We list some examples of Riesz spaces which we will use further on. For more details on these we refer the reader to [8, 34]

Example 1.2.6 (1) Let (Ω, Σ, μ) be a measure space and consider $L^0 = L^0(\Omega, \Sigma, \mu)$, the set of all real μ-a.e. finite measurable functions on Ω. We identify functions which differ only on a set of measure zero, i.e., elements of L^0 are equivalence classes of functions, two functions being in the same equivalence class if and only if they differ only on a set of measure zero. If we define for $f, g \in L^0$, $f \leq g$ if $f(x) \leq g(x)$ $\mu-$ a.e. on Ω then L^0 is a Riesz space.

For $1 \leq p < \infty$ define $L^p = L^p(\Omega, \Sigma, \mu)$ as the subset of L^0 consisting of all $f \in L^0$ such that $\int_\Omega |f|^p \, d\mu < \infty$. If we define the order in L^p as that defined for L^0 then L^p is a Riesz space.

Define the set of all essentially bounded $f \in L^0$ by $L^\infty = L^\infty(\Omega, \Sigma, \mu)$; here $f \in L^0$ is said to be essentially bounded if there exists a non - negative finite number M such that $|f(x)| < M$ for μ-almost every $x \in \Omega$. In other words $f \in L^\infty$ if $f \in L^0$ and $\operatorname{ess\,sup} |f(x)| < \infty$. Again, if we define the order in L^∞ as that in L^0 then L^∞ is a Riesz space.

(2) Let X be a compact Hausdorff space and $C(X)$ be a vector space of all real continuous functions on X. Define the order $\leq$ in $C(X)$ by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$, this makes $C(X)$ a Riesz space.

(3) Let $\mathfrak{F}$ be an algebra (σ-algebra) of subsets of a non-empty set X. Let μ be a bounded finitely additive signed measure on $\mathfrak{F}$, i.e., for $A, B \in \mathfrak{F}$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A is disjoint with B, $|\mu(A)| \leq K$ for some constant $K > 0$ and

$$||\mu|| = \sup\{|\mu(A)| : A \in \mathfrak{F}\} < \infty.$$

Put $\mathfrak{C}$ to be the set of all bounded finitely additive signed measures on $\mathfrak{F}$, if we define the order on $\mathfrak{C}$ by saying, for $\mu, \nu \in \mathfrak{C}$, that $\mu \leq \nu$ whenever $\mu(A) \leq \nu(A)$ for all $A \in \mathfrak{F}$ then $\mathfrak{C}$ is a Riesz space.

Here for μ_1 and μ_2 in $\mathfrak{C}$ we have that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are given by

$$\mu_1 \vee \mu_2(A) = \inf_{A \supseteq B} \{\mu_1(B) + \mu_2(A - B)\}$$

$$\mu_1 \wedge \mu_2(A) = \sup_{A \supseteq B} \{\mu_1(B) + \mu_2(A - B)\}$$

where $A, B \in \mathfrak{F}$, see [19, 24, 34].

Definition 1.2.7 The subset $E^+ = \{x \in E : x \geq 0\}$ is called the *positive cone* of the ordered vector space E, $x \in E$ is said to be a *positive element* of E.

From this definition, it follows that the positive cone E^+ exhibits the following properties:

If $x, y \in E^+$ then $x + y \in E^+$.

If $x \in E^+$ and $0 \leq \lambda \in \mathbb{R}$ then $\lambda x \in E^+$.

If $x \in E^+$ and $-x \in E^+$ then $x = 0$.

Definition 1.2.8 Let E be a Riesz space. For every $x \in E$ we define *the positive part* of x by $x^+ = x \vee 0$, the *negative part* of x by $x^- = (-x) \vee 0$ and the *absolute value* of x by $|x| = x \vee (-x)$.

It is immediately clear that x^+ and x^- are in E^+ and that $|-x| = |x|$. Also $(-x)^- = -(-x) \vee 0 = x \vee 0 = x^+$ and $(-x)^+ = (-x) \vee 0 = x^-$.

The proof of the following properties can be found in Meyer-Nieberg [21].

Proposition 1.2.9 *Let E be a Riesz space. Then the following hold for every $x \in E$*

(i) $x = x^+ - x^-, x^+ \wedge x^- = 0.$

(ii) $0 \leq x^+ \leq |x|.$

(iii) $-x^- \leq x \leq x^+.$

(iv) $(\lambda x)^+ = \lambda x^+$ *and* $(\lambda x)^- = \lambda x^-$ *for* $\lambda \geq 0.$

$(\lambda x)^+ = -\lambda x^-$ *and* $(\lambda x)^- = -\lambda x^+$ *for* $\lambda \geq 0.$

$|\lambda x| = |\lambda|\,|x|$ *for* $\lambda \in \mathbb{R}.$

(v) $x \leq y$ *if and only if* $x^+ \leq y^+$ *and* $x^- \geq y^-.$

Consider the space $\mathfrak{C}$ in Example 1.2.6 (3). We have that $\mu^+ = \mu \vee 0$, $\mu^- = (-\mu) \vee 0$ and $|\mu| = (-\mu) \vee \mu$. Also $|\mu| = \mu^+ + \mu^-$

The above properties can be used to prove the following properties:

Proposition 1.2.10 *If E is a Riesz space then*

$$x \vee y = \tfrac{1}{2}(x + y + |x - y|) \quad \text{and} \quad x \wedge y = \tfrac{1}{2}(x + y - |x - y|)$$

and

$$|x| \vee |y| = \tfrac{1}{2}(|x + y| + |x - y|) \quad \text{and} \quad |x| \wedge |y| = \tfrac{1}{2}(|x + y| - |x - y|).$$

We state a property of Riesz spaces whose importance is apparent when looking at the space of linear functionals on a Riesz space. It is known as the Riesz Decomposition Property, its proof can be found in Zaanen [32].

Proposition 1.2.11 *Let E be a Riesz space and let $x, y \in E^+$*

(i) *If* $0 \leq z \leq x + y$ *there exist elements* $0 \leq u \leq x$ *and* $0 \leq v \leq y$ *such that* $z = u + v$.

(ii) *For all* $x, y \in E^{+}$ *we have that* $[0, x + y] = [0, x] + [0, y]$.

Definition 1.2.12 If E is a Riesz space, x and y in E are said to be *disjoint* whenever $|x| \wedge |y| = 0$ and we write $x \perp y$.

Two subsets A and B in E are called *disjoint* whenever $a \perp b$ for every $a \in A$ and $b \in B$.

The set $A^{d} = \{x \in E : x \perp a \text{ for all } a \in A\}$ is called the *disjoint complement* of $A \subset E$.

A subset $S \subset E$ is called a *disjoint system* whenever $0 \in S$ and $x \perp y$ for every $x, y \in S$.

We present the following result, the proof of which can be found in Meyer-Nieberg [21].

Proposition 1.2.13 *If* x, y *are elements of a Riesz space* E *then the following are equivalent*

(a) $x \perp y$.

(b) $|x| \vee |y| = |x + y| = |x - y|$.

(c) $|x| \vee |y| = |x| + |y|$.

The following definitions and notions are adapted from the book of Zaanen [34].

Let Δ be a non-empty set and E be a Riesz space. Assume that for each $\alpha \in \Delta$ there exists an element f_α in E, i.e., there is some mapping $\alpha \mapsto f_\alpha$ from Δ into E. Put $A = \{f_\alpha : \alpha \in \Delta\}$. Here Δ is an indexing set for A.

Definition 1.2.14 With the notation above, the set A is said to be *upwards directed* if for any α and β in Δ there exists a δ in Δ such that $f_\delta \geq f_\alpha \vee f_\beta$ and it is said to be *downwards directed* if there is a $\delta \in \Delta$ such that $f_\delta \leq f_\alpha \wedge f_\beta$. We denote an upwards directed set A by $A \uparrow$ and if A is downwards directed we denote this by $A \downarrow$. If $A \uparrow$ and $\sup A = a$ we write $A \uparrow a$. Similarly, we write $A \downarrow a$ if $A \downarrow$ and $\inf A = a$.

We give, as a definition, a more specific case of a directed set.

Definition 1.2.15 A sequence (x_n) in a Riesz space E is said to be *increasing* if $x_1 \leq x_2 \leq \cdots$ and it is said to be *decreasing* if $x_1 \geq x_2 \geq \cdots$. We denote an increasing sequence (x_n) by $x_n \uparrow$ and a decreasing sequence (x_n) by $x_n \downarrow$. If $x_n \uparrow$ and $x = \sup x_n$ exists in E we write $x_n \uparrow x$ and $x_n \downarrow x$ whenever $x_n \downarrow$ and $x = \inf x_n$ exists in E. In the latter case we sometimes say that (x_n) *converges monotonely* to x (as n tends to infinity).

We introduce a more general notion of convergence, which is convergence associated with the order structure of E. This kind of convergence is known as order convergence.

Definition 1.2.16 A sequence (x_n) in a Riesz space E is said to *converge in order* to x if there exists a sequence $(y_n) \downarrow 0$ such that $|x_n - x| \leq y_n$ for all

n. We shall denote that (x_n) converges in order to x (or is *order convergent to* x) by $x_n \to (o)x$ or by $x_n \to x$ in order. In this case x is called the *order limit* of the sequence (x_n).

We give a remark from [34] that the notion of a downwards directed sequence and that of a decreasing sequence are different. For instance, in the Riesz space $\mathbb{R}$ consider the sequence

$$(x_n) = \begin{cases} \frac{1}{n} & \text{if } n = 1, 3, 5, \cdots \\ 1 + \frac{1}{n} & \text{if } n = 2, 4, 6, \cdots . \end{cases}$$

We have that (x_n) is not monotone but $x_n \downarrow$, in fact $x_n \downarrow 0$.

Example 1.2.17 Let (X, Σ, μ) be a measure space and let $E = M(X, \mu)$, the space of all μ-a.e. finite functions on X. Then a sequence $f_n \downarrow 0$ if and only if $f_n(t) \downarrow 0$ for almost every $t \in X$. Therefore, $f_n \to f$ in order if and only if there exists a positive $g \in E$ such that $|f_n| \leq g$ and $(f_n(t))$ converges to $f(t)$ for almost every $t \in X$.

We also put in a few non-topological properties of vector lattices. In particular we look at some subsets of Riesz spaces that will be encountered as we progress.

Definition 1.2.18 Let E be a Riesz space

(i) E is called *Archimedean* if, for all $x, y \in E$, it follows from $nx \leq y$ for all $n \in \mathbb{N}$ that $x \leq 0$.

(ii) E is said to be *laterally complete* if every set of pairwise disjoint elements has a supremum.

(iii) E is said to be *Dedekind complete* or *order complete* if every non-empty subset of E that is bounded from above has a supremum, or equivalently, if every non-empty subset of E that is bounded from below has an infimum.

(iv) E is called *$\sigma-$Dedekind complete* if every non-empty finite or countable subset of E that is bounded from above (bounded from below) has a supremum (infimum).

(v) E is called *order separable* if every non-empty subset D of E that has a supremum contains a subset that is at most countable with the same supremum as D.

(vi) E is called *super Dedekind complete* if it is Dedekind complete and order separable.

For σ-finite measure spaces (X, Σ, μ), the spaces $L^0(X, \Sigma, \mu), M(X, \Sigma, \mu)$ and $L^p(X, \Sigma, \mu)$, with $0 \leq p < \infty$, are examples of super Dedekind complete spaces. (The space $\mathfrak{C}$ is Dedekind complete but in general not super Dedekind complete, see [19, Example 23.3]. $L^0(X, \Sigma, \mu)$ is an example of a laterally complete Riesz space.

Definition 1.2.19 A linear subspace $A \subset E$ is called a *Riesz subspace* or a *sublattice* of the Riesz space E if $x \vee y$ and $x \wedge y$ belong to A for all $x, y \in A$.

(i) A subset A is called *order dense* in E whenever for each $0 < x \in E$ there exists some $y \in A$ that satisfies $0 < y \leq x$.

(ii) A subset A is called *solid* if $|x| < |y|$ and $y \in A$ implies that $x \in A$.

(iii) A solid linear subspace of the Riesz space E is called an *ideal.*

(iv) An ideal B is a *band* if for every subset $A \subset E$ we have $\sup A \in B$ whenever $\sup A$ exists in E. An ideal generated by a singleton set is called a *principal ideal.*

(v) A band B is called a *projection band* if there exists a linear projection $P : E \to B$ satisfying $0 \leq Px \leq x$ for all $x \in E^+$. P is then called a *band projection.*

From the definition it follows that the ideal B is a band in E if $0 \leq f_\alpha \uparrow f \in E$ with $f_\alpha \in B$ implies that $f \in B$ and that an ideal A is order dense in E if the band generated by A is the whole E.

Example 1.2.20 (1) $L^p(\Omega, \Sigma, \mu)$ is an ideal in $L^0(\Omega, \Sigma, \mu)$ (both in Example 1.2.6).

(2) Again $L^p(\Omega, \Sigma, \mu)$ in Example 1.2.6 is an ideal in the space $M(\Omega, \Sigma, \mu)$, the space of all μ-a.e. finite functions on Ω, but it is not a band. $B \subset M(\Omega, \Sigma, \mu)$ is a band if and only if there exists a measurable set $E_B \subset \Omega$ such that

$$B = \{f \in M(\Omega, \Sigma, \mu) : f(x) = 0 \text{ for almost all } x \notin E_B\}.$$

We will, to a great extent, be concerned with ideals of measurable functions, i.e., ideals L in the Riesz space $L^0(Y, \Lambda, \nu)$. The collection of all Λ-measurable functions into $[0, \infty]$ will be denoted by $M^+(Y, \Lambda, \nu)$.

If L is an ideal in $L^0(Y,\Lambda,\nu)$, the set $Z \in \Lambda$ is called an L-zero set if every $f \in L$ vanishes ν-a.e. on Z. There exists a maximal L-zero set $Z_1 \in \Lambda$ and the set $Y_1 = Y \backslash Z_1$ is called the carrier of the ideal L. Also, there exists a sequence $A_n \uparrow Y_1$ in Λ such that $\nu(A_n) < \infty$ and $\mathbf{1}_{A_n} \in L$ for all $n \in \mathbb{N}$. For proofs of these see [32].

We note that if the carrier of the ideal L in $L^0(Y,\Lambda,\nu)$ is the whole set Y, i.e., $Y = Y_1$ then L is order dense in $L^0(Y,\Lambda,\nu)$.

1.3 Operators in Riesz spaces

We now look at the basic theory of operators in Riesz spaces. We consider two Riesz spaces E and F. We will denote the set of all linear operators from E to F by $\mathcal{L}(E,F)$. $\mathcal{L}(E,F)$ is an ordered space if, for $T, S \in \mathcal{L}(E,F)$, we define $S \leq T$ to mean $T - S \geq 0$.

Definition 1.3.1 Let E and F be Riesz spaces and let $T \in \mathcal{L}(E,F)$.

(i) T is called *order bounded* if it maps order bounded subsets into order bounded subsets.

(ii) T is called *positive* whenever $Tx \geq 0$ for all $x \geq 0$. If T is positive we write $T \geq 0$.

(iii) T is called a *Riesz homomorphism* or *lattice homomorphism* whenever $T(x \vee y) = Tx \vee Ty$. A bijective Riesz homomorphism is called a *Riesz isomorphism*.

(iv) T is said to be *order continuous* whenever $\inf\{|Tx| : x \in D\} = 0$ in F for every set D such that $D \downarrow 0$ in E.

(v) T is said to be *σ-order continuous*, if, for any monotone sequence $x_n \downarrow 0$ we have that $\inf\{|Tx_n|\} = 0$.

(vi) T is said to be *regular* if it can be expressed as a difference of two positive linear operators from E into F, i.e., if $T = T_1 - T_2$ where T_1 and T_2 are positive linear operators from E into F.

Note that for a positive T, we have that T is order continuous if and only if it follows that $T(D) \downarrow 0$ in F for all sets $D \downarrow 0$ in E. In the case of σ-order continuity, we have that a positive T is σ-order continuous if $x_n \downarrow 0$ implies that $Tx_n \downarrow 0$.

The set of order bounded linear operators from E to F will be denoted by $\mathcal{L}_b(E,F)$. The bounded order dual of E, which is $\mathcal{L}_b(E,\mathbb{R})$, will be denoted by $E^\sim$. Note that if F is Dedekind complete then $\mathcal{L}_b(E,F)$ is Dedekind complete. We will denote the set of order continuous linear operators in $\mathcal{L}_b(E,F)$ by $\mathcal{L}_n(E,F)$ and in the case where $F = \mathbb{R}$ by $E_n^\sim$. The set of all $\sigma-$order continuous linear operators will be denoted by $\mathcal{L}_c(E,F)$ and for the case where $F = \mathbb{R}$ by $E_c^\sim$.

We also note that a positive operator is regular and that if F is Dedekind complete then every operator $T \in \mathcal{L}(E,F)$ is regular if and only if it is order bounded, i.e., the set of regular operators and $\mathcal{L}_b(E,F)$ coincide if F is Dedekind complete, see [32].

We state a few characteristics of positive operators whose proofs can be found in [24] and in [34].

Lemma 1.3.2 *Let $T \in \mathcal{L}_b(E,F)$. Then $T \in \mathcal{L}_n(E,F)$ if and only if $|T| \in \mathcal{L}_n(E,F)$, i.e., if $T^+, T^- \in \mathcal{L}_n(E,F)$. Similarly $T \in \mathcal{L}_c(E,F)$ if and only if $|T| \in \mathcal{L}_c(E,F)$.*

Proposition 1.3.3 *(i) $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are bands in $\mathcal{L}_b(E,F)$.*

(ii) $E_n^\sim$ and $E_c^\sim$ are bands in $E^\sim$.

Proposition 1.3.4 *Let $T \in \mathcal{L}(E,E)$, then*

(i) T is positive if and only if $|Tx| \leq T|x|$ for all $x \in E$.

(ii) $(Tf)^+ \leq Tf^+$

(iii) Every Riesz homomorphism is positive.

(iv) T is a Riesz isomorphism if and only if T and T^{-1} are positive.

We will denote the range of $T \in \mathcal{L}(E,F)$ by $\operatorname{ran}(T)$ and the kernel of T by $\ker(T)$. Recall that $\ker(T) = \{f \in E : Tf = 0\}$.

Definition 1.3.5 For an operator $T \in \mathcal{L}_b(E,F)$ the set

$$N_T = \{x \in E : |T|\,|x| = 0\}$$

is called the *null ideal* of T or the *absolute kernel* of T.

We have that the absolute kernel of a positive operator is an ideal and that $N_T \subset T^{-1}(0)$.

Definition 1.3.6 The disjoint complement of the absolute kernel of $T \in \mathcal{L}_b(E,F)$ is called the *carrier* of T and will be denoted by C_T.

Note that if $T \in \mathcal{L}_n(E,F)$ then N_T is a band and, furthermore, by the Riesz decomposition of Dedekind complete Riesz spaces we have that

$$E = N_T \oplus C_T.$$

The following result implies that every regular operator is norm bounded, its proof can be found in [34].

Proposition 1.3.7 *Let E be a Banach lattice and F a normed Riesz space. Then every positive linear operator from E into F is continuous.*

1.4 Riesz and Boolean homomorphisms

In this section we insert some remarks concerning Riesz homomorphisms that will be used later on. We start by stating some results about order convergence and σ-order convergence.

Lemma 1.4.1 *If (f_n) is a sequence in $L^0(X,\Sigma,\mu)$ such that for μ-a.e. x, $f_n \to 0$ then $f(x) = \sup_n |f_n(x)|$ exists in $L^0(X,\Sigma,\mu)$.*

Proof Each $f_n(x)$ is a.e. μ-measurable and so $f(x)$ is a.e. μ-measurable.

We show that it is μ-a.e. finite valued. We have that for μ-a.e. x,

$f_n \to 0$. Let x_0 be one of these points, then there exists a natural number $N_{(x_0)}$ such that $|f_n(x_0)| \leq 1$ whenever $n > N_{(x_0)}$. Put $M = \sup\{1, |f_1(x_0)|, |f_2(x_0)|, \cdots, |f_{N_{(x_0)}}(x_0)|\}$. Then

$$f(x_0) = \sup_n |f_n(x_0)| \leq M.$$

Hence $f(x_0)$ is finite. Thus $f(x)$ is μ-a.e. finite valued on X. □

If (f_n) is a sequence in $L^0(X, \Sigma, \mu)$ such that $f_n(x) \downarrow 0$ μ−a.e., then there exists a sequence of positive real numbers $a_n \uparrow \infty$ such that $a_n f_n(x) \to 0$, see [19, Theorem 71.4]. Putting $f_0(x) = \sup_n a_n f_n(x)$, and using Lemma 1.4.1, we have that $f_0 \in L^0(X, \Sigma, \mu)$. But then $0 \leq f_n(x) \leq \frac{1}{a_n} f_0(x)$ and so (f_n) converges f_0-uniformly to 0.

The above observation can be stated as

Theorem 1.4.2 *Order convergent sequences in $L^0(X, \Sigma, \mu)$ are relatively uniformly convergent.*

We use this in the proof of the following

Proposition 1.4.3 *Let (Y, Λ, ν) and (X, Σ, μ) be σ-finite measure spaces. Suppose that ϕ is a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$. Then*

(i) ϕ is order continuous;

(ii) if ϕ is interval preserving, $\operatorname{ran}(\phi)$ *is a band in $L^0(X, \Sigma, \mu)$;*

(iii) if ϕ is surjective, then it is interval preserving and in particular, if $L \subset (Y, \Lambda, \nu)$ is an ideal, then $\phi(L)$ is an ideal in $L^0(X, \Sigma, \mu)$.

(iv) If $L \subset L^0(Y, \Lambda, \nu)$ and $M \subset L^0(X, \Sigma, \mu)$ are order dense ideals and if $\phi : L \to M$ is an order continuous Riesz homomorphism, then it can be extended uniquely to a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$.

Proof (i) By Theorem 1.4.2 we have that an order convergent sequence in $L^0(X, \Sigma, \mu)$ is relatively uniformly convergent. Since ϕ is positive it is σ-order continuous. But $L^0(X, \Sigma, \mu)$ is super Dedekind complete, and therefore order separable and so ϕ is order continuous.

(ii) Assume that ϕ is interval preserving. If $G = \text{ran}(\phi)$, then G is an ideal in $L^0(X, \Sigma, \mu)$. Since $L^0(Y, \Lambda, \nu)$ is laterally complete and ϕ is order continuous, G is laterally complete. We have that for every $0 \le w \in G$, $B_w = \{w\}^{dd} \subset G$. Indeed, if we let $0 \le f \in B_w$ and put

$$\begin{aligned} f_n &= f\mathbf{1}_{E_n} \text{ where} \\ E_n &= \{t \in X \mid nw(t) < f(t) \le (n+1)w(t)\}, \end{aligned}$$

and $n = 1, 2, \cdots$, then f_n is a disjoint system in G and so $\sup_n f_n$ belongs to G. Since this supremum is equal to $f \in L^0(X, \Sigma, \mu)$ we have that $f \in G$.

Now, let $\{w_\alpha\}$ be a maximal disjoint system in G, then

$$w = \sup w_\alpha \in G$$

and the band generated by w is contained in G.

On the other hand, if $0 \leq v \in G$, we write $v = v_1 + v_2$ with $v_1 \in B_w$ and $v_2 \in B_w^d$. Since v_2 is disjoint to every w_α and since w_α is a maximal disjoint system in G, we have that $v_2 = 0$. Thus $G \subset B_w$ and so $G = B_w$ is a band in $L^0(X, \Sigma, \mu)$.

(iii) Let ϕ be a surjective Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ to $L^0(X, \Sigma, \mu)$ and let $0 \leq g \leq \phi(u)$ for some $0 < u \in L^0(Y, \Lambda, \nu)$. Let $w \in L^0(Y, \Lambda, \nu)$ be such that $\phi(w) = g$ and consider $w^+ \wedge u$. Then $0 \leq w^+ \wedge u \leq u$ and $0 \leq \phi(w^+ \wedge u) \leq \phi(u)$. This gives us that $0 \leq \phi(w^+) \wedge \phi(u) \leq u$ and so $\phi(w^+ \wedge u) = g$ since $\phi(w^+) = g^+ = g$.

(iv) Since $L^0(X, \Sigma, \mu)$ is laterally complete and $L \subset L^0(Y, \Lambda, \lambda)$ is Dedekind complete we have by Theorem 7.20 in [1], that ϕ can be extended into a Riesz homomorphism ϕ' from $L^0(Y, \Lambda, \mu)$ into $L^0(X, \Sigma, \mu)$ that satisfy the equation

$$\phi'(x) = \sup\{\phi(y) : y \in L \text{ and } 0 \leq y \leq x\},$$

for all $0 < x \in L^0(Y, \Lambda, \mu)$. Since every extension satisfies this formula it is unique. □

Let $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$, be a Riesz homomorphism, we denote its null-ideal and carrier by N_ϕ and C_ϕ, respectively, i.e.,

$$N_\phi = \{f \in L^0(Y, \Lambda, \nu) : \phi(f) = 0\} \text{ and } C_\phi = N_\phi^d.$$

Since ϕ is order continuous, N_ϕ is a band and so

$$L^0(Y, \Lambda, \nu) = C_\phi \oplus N_\phi.$$

If $Y_1 \in \Lambda$ is the carrier of C_ϕ, then

$$\begin{aligned} C_\phi &= \{f\mathbf{1}_{Y_1} : f \in L^0(Y, \Lambda, \nu)\} \\ &= L^0(Y_1, \Lambda_{Y_1}, \nu), \end{aligned}$$

where $\Lambda_{Y_1} = \{A \cap Y_1 : A \in \Lambda\}$.

Note that the restriction of ϕ to $L^0(Y_1, \Lambda_{Y_1}, \nu)$ is a Riesz isomorphism into $L^0(X, \Sigma, \mu)$.

We will denote by Λ_ν the measure algebra of (Y, Λ, ν) and by Σ_μ the measure algebra of (X, Σ, μ). $\dot{C}$ will denote the equivalence class in either of the algebras to which the measurable set C belongs.

Let $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$ be a Riesz homomorphism such that $\phi(\mathbf{1}_Y)$ is an a.e. strict positive function on X. If $A \in \Lambda$ then $\phi(\mathbf{1}_A) = \phi(\mathbf{1}_Y)\mathbf{1}_B$, for some $B \in \Sigma$ which is uniquely determined modulo μ−null sets by A. If we put $\widehat{\phi}(\dot{A}) = \dot{B}$ it follows that

$$\widehat{\phi} : \Lambda_\nu \to \Sigma_\mu$$

is an order continuous Boolean homomorphism that satisfies $\widehat{\phi}(\dot{Y}) = \dot{X}$. Put

$$\Sigma_\phi = \{B \in \Sigma : \dot{B} \in \widehat{\phi}(\Lambda_\nu)\}$$

Then Σ_ϕ is a sub-σ-algebra of Σ and we write $\Sigma_\phi = \widehat{\phi}(\Lambda)$. (In this case we do not distinguish between the σ-algebra Λ and the measure algebra Λ_ν).

We show that $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma_\phi, \mu)$ is interval preserving. To that end let $0 \le u \in L^0(Y, \Lambda, \nu)$ and let $v \in L^0(X, \Sigma_\phi, \mu)$ be such that

$0 \leq v \leq \phi(u)$. For every step function $f(y) = \sum_{i=1}^{n} a_i \mathbf{1}_{B_i}(y)$ with $B_i = \widehat{\phi}(A_i) \in \Sigma_\phi$ we have

$$\begin{aligned}\phi\left(\frac{1}{\phi(\mathbf{1}_Y)}\sum_{i=1}^{n} a_i \mathbf{1}_{A_i}\right) &= \frac{1}{\phi(\mathbf{1}_Y)}\sum_{i=1}^{n} a_i \phi(\mathbf{1}_Y)\mathbf{1}_{B_i} \\ &= \sum_{i=1}^{n} a_i \mathbf{1}_{B_i}.\end{aligned}$$

Let (v_n) be a sequence of step functions such that $0 \leq v_n \uparrow v$ in $L^0(X, \Sigma_\phi, \mu)$. By the preceding argument there exists a sequence (u_n) of step functions in $L^0(Y, \Lambda, \nu)$ such that $\phi(u_n) = v_n$. Since ϕ is a Riesz homomorphism we may assume, without loss of generality, that $0 \leq u_n \uparrow\leq u$ [else we can successively replace u_n firstly by u_n^+ (to get $\phi(u_n^+) = \phi(u_n \vee 0) = \phi(u_n) \vee 0 = v_n^+ = v_n$), secondly by $u_n \vee u_{n-1}$ (to get $\phi(u_n \vee u_{n-1}) = v_n \vee v_{n-1} = v_n$) and lastly by $u_n \wedge u$ (to get $\phi(u_n \wedge u) = v_n \wedge \phi(u) = v_n$)]. Let $0 \leq u_n \uparrow w$. Then

$$\phi(w) = \sup \phi(u_n) = \sup(v_n) = v,$$

which shows that ϕ is interval preserving.

By Proposition 1.4.3 (ii) we get that the ran(ϕ) is a band in $L^0(X, \Sigma_\phi, \mu)$ containing the weak order unit $\phi(\mathbf{1}_Y)$. This band is the whole of $L^0(X, \Sigma_\phi, \mu)$ and so $\phi(L^0(Y, \Lambda, \nu)) = L^0(X, \Sigma_\phi, \mu)$.

On the other hand, if $\sigma : \Lambda_\nu \to \Sigma_\mu$ is an order continuous Boolean homomorphism with $\sigma(\dot{Y}) = \dot{X}$, then there exists a unique Riesz homomorphism $\phi : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$ with $\phi(\mathbf{1}) = \mathbf{1}$ such that $\widehat{\phi} = \sigma$.

Now assume that $\tau : X \to Y$ is a (Σ, Λ)-measurable mapping which is null preserving (i.e., if $B \in \Lambda$ and $\nu(B) = 0$ then $\mu(\tau^{-1}(B)) = 0$). The mapping $B \mapsto \tau^{-1}(B)$ defines an order continuous Boolean homomorphism

$$\tau_* : \Lambda_\nu \to \Sigma_\mu$$

with

$$\tau_*(\dot{Y}) = \dot{X}.$$

The associated Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(X, \Sigma, \mu)$ will be denoted by ϕ_τ, i.e., $\widehat{\phi}_\tau = \tau_*$ and $\phi_\tau(\mathbf{1}) = \mathbf{1}$. It is easily verified that

$$(\phi_\tau f)(x) = f(\tau x) \ \mu\text{-a.e. on } X$$

for all $f \in L^0(Y, \Lambda, \nu)$.

Let $L \subset L^0(Y, \Lambda, \nu)$ be an order dense ideal with order continuous dual $L_n^\sim$. As usual we identify $L_n^\sim$ with an ideal L' of functions in $L^0(Y, \Lambda, \nu)$ and we will assume that L' is again an order dense ideal (which is always the case if L is a Banach function space; (see [33], Theorem 112.1 or [21], Theorem 2.6.4). Equivalent to this assumption is that $L_n^\sim$ separates the points of L. The duality relation between L and L' is given by

$$\langle f, g \rangle = \int_Y fg d\nu \text{ for } f \in L \text{ and } g \in L',$$

(see [33] Section 86). Let $T \in \mathcal{L}_n(L, M)$ with L and M ideals of functions in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$ respectively. We define its order continuous adjoint $T' : M' \to L'$ by $\langle g, T'f \rangle = \langle Tg, f \rangle$ for all $f \in M'$ and $g \in L$ (see [33], Section 97). Then $T' \in \mathcal{L}_n(M', L')$.

The next result is from [8]. In it we gather some results relating to the adjoints of homomorphisms.

Lemma 1.4.4 *Let (X, Σ, μ) and (Y, Λ, ν) be $\sigma-$finite measure spaces and let $L \subseteq L^0(Y, \Lambda, \nu)$ and $M \subseteq L^0(X, \Sigma, \mu)$ be order dense ideals for which L' and M' are order dense ideals as well. Let $\phi : L \to M$ be an order continuous interval preserving Riesz homomorphism. Then,*

(i) the adjoint $\phi' : M' \to L'$ is an order continuous interval preserving Riesz homomorphism as well and ϕ' extends uniquely to an order continuous interval preserving Riesz homomorphism $\phi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)$;

(ii) if $\phi(L)$ is order dense in $L^0(X, \Sigma, \mu)$, then ϕ' is injective;

(iii) if ϕ is injective, then $\phi'(\mathbf{1}_X)$ is strictly positive and $\phi'(M')$ is order dense in L';

(iv) if ϕ is injective and $\phi(L)$ is a band in M then $\phi' : M' \to L'$ is surjective.

Proof (i) It follows from [1], Theorem 7.7 that ϕ' is a Riesz homomorphism and from [1] Theorem 7.8 that ϕ' is an interval preserving. By Proposition 1.4.3, ϕ' can be extended to an order continuous Riesz homomorphism $\phi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)$. We show that ϕ' is interval preserving: Let $0 \leq u \in L^0(X, \Sigma, \mu) \supset M$ and $v \in L^0(Y, \Lambda, \nu) \supset L$ be such that $0 \leq v \leq \phi'(u)$. Since M' is order dense in $L^0(X, \Sigma, \mu)$ there is a sequence $0 \leq u_n \uparrow u$ in M' and so $0 \leq \phi'(u_n) \uparrow \phi'(u)$. We have that $\phi' : M' \to L'$ is interval preserving. Since $0 \leq v \wedge \phi'(u_n) \leq \phi'(u_n)$ there exists, for each $n = 1, 2, \cdots$, an element $w_n \in M'$ such that $\phi'(w_n) = v \wedge \phi'(u_n)$.

Again, since ϕ' is a Riesz homomorphism, we may assume without loss of generality that $0 \le w_n \uparrow \le u$. Let $w_n \uparrow w$, then

$$\phi'(w) = \sup \phi'(w_n) = \sup(v \wedge \phi'(u_n)) = v \wedge \phi'(u) = v.$$

This proves the assertion.

(ii) Now assume that $\phi(L)$ is order dense in $L^0(X, \Sigma, \mu)$. Let $0 \le g \in M'$ be such that $\phi'(g) = 0$. Then

$$\langle \phi(u), g \rangle = \langle u, \phi'(g) \rangle = 0 \quad \forall\, 0 \le u \in L,$$

and so $g = 0$ by order denseness of $\phi(L)$. Hence ϕ' is injective.

(iii) Now assume that ϕ is injective. To show that $\phi'(\mathbf{1}_X)$ is strictly positive, take $0 \le g \in L$ such that $g \wedge \phi'(\mathbf{1}_X) = 0$. Take $X_n \in \Sigma$ such that $X_n \uparrow X$ and $\mathbf{1}_{X_n} \in M'$ for all n. Then

$$\begin{aligned} \int_{X_n} \phi(g)\, d\mu &= \langle \phi(g), \mathbf{1}_{X_n} \rangle \\ &= \langle g, \phi'(\mathbf{1}_{X_n}) \rangle \\ &= 0 \end{aligned}$$

for all n, and so $\int_X \phi(g)\, d\mu = 0$. This implies that $\phi(g) = 0$ μ-a.e. on X. Hence $g = 0$, which shows that $\phi'(\mathbf{1}_X)$ is strictly positive, as L is order dense in $L^0(Y, \Lambda, \nu)$.

In order to see that $\phi'(M')$ is order dense in L', let $0 < h \in L'$. Since $\phi'(\mathbf{1}_X)$ is strictly positive, $0 < \phi'(\mathbf{1}_X) \wedge h \le \phi'(\mathbf{1}_X)$ and as ϕ' is interval preserving, there exists $0 < f \in L^0(X, \Sigma, \mu)$ such that $0 < f \leq \mathbf{1}_X$ and $\phi'(f) = \phi'(\mathbf{1}_X) \wedge h$. But M' is order dense in $L^0(X, \Sigma, \mu)$ and so for some $g \in M'$, we have $0 < g \leq f$. By injectivity, $0 < \phi'(g) \leq \phi'(f) \leq h$ and it follows that $\phi'(M')$ is order dense in L'.

(iv) Assume now that ϕ is injective and that $\phi(L)$ is a band in M. By (ii), $\phi'(\mathbf{1}_X)$ is strictly positive and since ϕ' is interval preserving, it follows that $\phi' : L^0(X, \Sigma, \mu) \to L^0(Y, \Lambda, \nu)$ is surjective. Let $X_1 \in \Sigma$ be the carrier of $\phi(L)$. By hypothesis, $\phi(L) = \{f\mathbf{1}_{X_1} : f \in M\}$. Furthermore it is easy to see that $\phi'(g) = 0$ for all $g \in L^0(X, \Sigma, \mu)$ such that $g = 0$ on X_1.

Now let $0 < h \in L'$ be given. Then $h = \phi'(g)$ for some $0 \leq g \in L^0(X, \Sigma, \mu)$, and we may assume that $g = 0$ on $X \backslash X_1$. It remains to show that $0 \leq g \in M'$, i.e., that $\int_X gf \, d\mu < \infty$ for all $0 \leq f \in M$. To this end, take $0 \leq f \in M$. Then $f\mathbf{1}_{X_1} \in \phi(L)$, so $f\mathbf{1}_{X_1} = \phi(u)$ for some $0 \leq u \in L$. Let $0 \leq g_n \in M'$ be such that $0 \leq g_n \uparrow g$. We find that

$$\begin{aligned} \int_X gf \, d\mu &= \int_X g(f\mathbf{1}_{X_1}) \, d\mu \\ &= \int_X g\phi(u) \, d\mu \\ &= \lim_{n\to\infty} \int_X g_n\phi(u) \, d\mu. \end{aligned}$$

Thus

$$\begin{aligned}\int_X gf\ d\mu &= \lim_{n\to\infty}\int_Y \phi'(g_n)u\ d\nu \\ &= \int_Y \phi'(g)u\ d\nu \\ &= \int_Y hu\ d\nu \\ &< \infty,\end{aligned}$$

which shows that $0 \leq g \in M'$. □

1.5 Conditional Expectations

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, i.e., $\mathfrak{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ a countably additive measure on $\mathfrak{F}$ with $\mathbb{P}(\Omega) = 1$.

Let $\mathfrak{G}$ be a σ-algebra (sub-σ-algebra) of subsets of Ω with $\mathfrak{G} \subset \mathfrak{F}$. For every $A \in \mathfrak{G}$ the equation

$$\phi(A) = \int_A f d\mathbb{P},$$

where $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$, defines a countably additive function ϕ on $\mathfrak{G}$. By the Randon-Nikodym theorem there is an extended real valued function g defined on Ω which is measurable with respect to $\mathfrak{G}$ such that

$$\phi(A) = \int_A g d\mathbb{P}$$

for every $A \in \mathfrak{G}$. If we denote g by $\mathbb{E}(f\,|\mathfrak{G})$ we get the following

Definition 1.5.1 Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let $\mathfrak{G}$ be a sub-σ-algebra of $\mathfrak{F}$. For $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ we denote by $\mathbb{E}(f\,|\mathfrak{G})$ the $\mathbb{P}$-a.e. unique

$\mathfrak{G}$−measurable function with the property that

$$\int_A \mathbb{E}(f\,|\mathfrak{G})\,d\mathbb{P} = \int_A f d\mathbb{P}$$

for all $A \in \mathfrak{G}$. The function $\mathbb{E}(f\,|\mathfrak{G})$ is called the *conditional expectation of f with respect to (or given) $\mathfrak{G}$.*

Proposition 1.5.2 *Let $\mathbb{E}(\cdot\,|\mathfrak{G})$ be a conditional expectation. Then $\mathbb{E}(\cdot\,|\mathfrak{G})$ can be extended from a mapping of $L^1(\Omega, \mathfrak{F}, \mathbb{P})$ into itself to a mapping from $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ into itself.*

Proof If $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$, then, (see [23] Corollary I-2-10), we can, for a sequence (f_n) such that $0 \leq f_n \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ satisfying $0 \leq f_n \uparrow f$ $\mathbb{P}$-a.e., define $\mathbb{E}(f|\mathfrak{G}) \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ by

$$\mathbb{E}(f|\mathfrak{G}) = \sup_n \mathbb{E}(f_n|\mathfrak{G}).$$

Next we show that this definition is independent of the choice of the sequence (f_n). To that end let (g_m) be a sequence in $L^1(\Omega, \mathfrak{F}, \mathbb{P})$ such that $0 \leq g_m \uparrow f$ $\mathbb{P}$−a.e. Then $g_m = \sup_n (f_n \wedge g_m)$ and so

$$\mathbb{E}(g_m|\mathfrak{G}) = \sup_n \mathbb{E}(f_n \wedge g_m|\mathfrak{G}) \leq \sup_n \mathbb{E}(f_n|\mathfrak{G}).$$

Thus

$$\sup_m \mathbb{E}(g_m|\mathfrak{G}) \leq \sup_n \mathbb{E}(f_n|\mathfrak{G}). \tag{1.5.1}$$

On the other hand $f_n = \sup_m (f_n \wedge g_m)$ and

$$\mathbb{E}(f_n|\mathfrak{G}) = \sup_m \mathbb{E}(f_n \wedge g_m|\mathfrak{G}) \leq \sup_m \mathbb{E}(g_m|\mathfrak{G}).$$

Thus

$$\sup_n \mathbb{E}(f_n|\mathfrak{G}) \leq \sup_m \mathbb{E}(g_m|\mathfrak{G}), \tag{1.5.2}$$

Equation 1.5.1 and Equation 1.5.2 together give that

$$\sup_n \mathbb{E}(f_n|\mathfrak{G}) = \sup_m \mathbb{E}(g_m|\mathfrak{G}).$$

We can use these suprema and write

$$\begin{aligned}\sup_n \mathbb{E}(f_n|\mathfrak{G}) &= \sup_m \mathbb{E}(g_m|\mathfrak{G}) \\ &= \mathbb{E}(f|\mathfrak{G}).\end{aligned}$$

□

We list some properties of $\mathbb{E}(\cdot\,|\mathfrak{G})$ and refer the reader to [23].

Proposition 1.5.3

(i) *If* $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ *and* $g \in L^\infty(\Omega, \mathfrak{G}, \mathbb{P})$, *then* $\mathbb{E}(gf|\mathfrak{G}) = g\mathbb{E}(f|\mathfrak{G})$.

(ii) $\mathbb{E}(\alpha f + \beta g|\mathfrak{G}) = \alpha\mathbb{E}(f|\mathfrak{G}) + \beta\mathbb{E}(g|\mathfrak{G})$ *for all* $f, g \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *and all* $0 \leq \alpha, \beta \in \mathbb{R}$.

(iii) $0 \leq f \leq g$ *in* $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *implies that* $0 \leq \mathbb{E}(f\,|\mathfrak{G}) \leq \mathbb{E}(g\,|\mathfrak{G})$.

(iv) $0 \leq f_n \uparrow f$ $\mathbb{P}-a.e.$ *implies that* $0 \leq \mathbb{E}(f_n\,|\mathfrak{G}) \uparrow \mathbb{E}(f\,|\mathfrak{G})\,\mathbb{P}-a.e.$

(v) *For all* $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *and all* $g \in M^+(\Omega, \mathfrak{G}, \mathbb{P})$ *we have* $\mathbb{E}(gf\,|\mathfrak{G}) = g\mathbb{E}(f\,|\mathfrak{G})$

(vi) *If* $g \in M^+(\Omega, \mathfrak{G}, \mathbb{P})$ *and* $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *then* $\int_A g d\mathbb{P} = \int_A f d\mathbb{P}$ *for all* $A \in \mathfrak{G}$ *if and only if* $g = \mathbb{E}(f\,|\mathfrak{G})\,\mathbb{P}-a.e.$

(vii) *If $\mathfrak{G}$ and $\mathfrak{H}$ are sub-σ-algeras of $\mathfrak{F}$, such that $\mathfrak{G} \subset \mathfrak{H}$, then* $\mathbb{E}(f\,|\mathfrak{G}) = \mathbb{E}(\mathbb{E}(f\,|\mathfrak{H})\,|\mathfrak{G})$ *for all* $0 \leq f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$.

(viii) *If* $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ *is such that* $\mathbb{E}(f\,|\mathfrak{G}) \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ *then we also have that* $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$.

Proof We prove (viii) only

If $\mathbb{E}(\cdot\,|\mathfrak{G}) \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ then there exists a sequence $(\Omega_n) \subset \mathfrak{G}$ such that $\Omega_n \uparrow \Omega$ and $\int_{\Omega_n} \mathbb{E}(f|\mathfrak{G})\, d\mathbb{P} < \infty$. This implies that $\int_{\Omega_n} f d\mathbb{P} < \infty$, so that $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$. □

We remark that the converse of the above does not hold in general. For instance, let $\Omega = [0, 1]$ and $\mathbb{P}$ be a Lebesgue measure. Put

$$\mathfrak{G} = \left\{\emptyset, \left[0, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, 1\right], [0, 1]\right\}$$

and $f(x) = \frac{1}{x}$, where $0 \leq x \leq 1$. Then $\mathbb{E}(f|\mathfrak{G}) = \infty$ on $\left[0, \frac{1}{2}\right]$. We therefore need the following:

Definition 1.5.4 The *domain* of $\mathbb{E}(\cdot\,|\mathfrak{G})$ is the set $\mathrm{dom}\mathbb{E}(\cdot\,|\mathfrak{G})$ given by

$$\mathrm{dom}\mathbb{E}(\cdot\,|\mathfrak{G}) = \left\{f \in L^0(\Omega, \mathfrak{F}, \mathbb{P}) : \mathbb{E}(|f|\,|\mathfrak{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P})\right\}.$$

It is clear that $\mathrm{dom}\mathbb{E}(\cdot\,|\mathfrak{G})$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ which contains $L^1(\Omega, \mathfrak{F}, \mathbb{P})$ and therefore it is order dense in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$. For an element f of $\mathrm{dom}\mathbb{E}(\cdot\,|\mathfrak{G})$ we define

$$\mathbb{E}(f\,|\mathfrak{G}) = \mathbb{E}\left(f^+\,|\mathfrak{G}\right) - \mathbb{E}\left(f^-\,|\mathfrak{G}\right).$$

This defines a positive linear operator

$$\mathbb{E}\left(\cdot\,|\mathfrak{G}\right):\mathrm{dom}\mathbb{E}\left(\cdot\,|\mathfrak{G}\right)\to L^{0}\left(\Omega,\mathfrak{G},\mathbb{P}\right)\subset L^{0}\left(\Omega,\mathfrak{F},\mathbb{P}\right).$$

Example 1.5.5 Let $(\Omega_1, \mathfrak{F}', \mathbb{P}_1)$ and $(\Omega_2, \mathfrak{F}'', \mathbb{P}_2)$be two propability spaces and $\Omega = \Omega_1 \times \Omega_2$, $\mathfrak{F} = \mathfrak{F}' \otimes \mathfrak{F}''$ and $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$. Put $\mathfrak{G} = \{A \times \Omega_2 : A \in \mathfrak{F}'\}$. Then $\mathfrak{G}$ is a sub-σ-algebra of $\mathfrak{F}$. An $\mathfrak{F}$-measurable function $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is $\mathfrak{F}'$-measurable if and only if $g(x_1, x_2) = g(x_1)$, i.e., if and only if g is independent of x_2. For $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ we now have that $\mathbb{E}(f|\mathfrak{G})(x_1, x_2) = \int_{\Omega_2} f(x_1, y)\, d\mathbb{P}_2(y)$ $\mathbb{P}$-a.e. on Ω. From the way in which $\mathbb{E}(\cdot|\mathfrak{G})$ is extended to $M^+(\Omega, \mathfrak{F}, \mathbb{P})$ it then follows that for $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ we have $f \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$ if and only if $\int_{\Omega_2} |f(x_1, y)|\, d\mathbb{P}_2(y) < \infty$ $\mathbb{P}$-a.e. on Ω. In this case we have that $\mathbb{E}(f|\mathfrak{G})(x_1, x_2) = \int_{\Omega_2} f(x_1, y)\, d\mathbb{P}_2(y)$ $\mathbb{P}$-a.e. on Ω. We also note that $\int_{\Omega_2} |f(x_1, y)|\, d\mathbb{P}_2(y) < \infty$ $\mathbb{P}$-a.e. on Ω is equivalent to $\int_{\Omega_2} |f(x_1, y)|\, d\mathbb{P}_2(y) < \infty$ $\mathbb{P}_1-$a.e. on Ω_1.

We give the following characterization of conditional expectation:

Proposition 1.5.6

(i) If $f \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$ *and* $g \in L^0(\Omega, \mathfrak{G}, \mathbb{P})$, *then* $gf \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$ *and* $\mathbb{E}(gf|\mathfrak{G}) = g\mathbb{E}(f|\mathfrak{G})$

(ii) If $f \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$, *then* $f \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$ *if and only if there is a sequence* $\{A_n\}$ *in* $\mathfrak{G}$ *such that* $A_n \uparrow \Omega$ *and*

$$\int_{A_n} |f|\, d\mathbb{P} < \infty \textit{ for } n = 1, 2, \cdots.$$

Moreover, if $f \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$*, then*

$$\int_A \mathbb{E}(f|\mathfrak{G})\, d\mathbb{P} = \int_A f d\mathbb{P}$$

for all $A \in \mathfrak{G}$ *with* $\int_A |f|\, d\mathbb{P} < \infty$.

Proof (i) Since $\mathbb{E}(gf|\mathfrak{G}) = g\mathbb{E}(f|\mathfrak{G})$ for all $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ and $g \in M^+(\Omega, \mathfrak{G}, \mathbb{P})$ we get that

$$\mathbb{E}(|gf|\,|\mathfrak{G}) = |g|\,\mathbb{E}(|f|\,|\mathfrak{G}) \in L^0(\Omega, \mathfrak{G}, \mathbb{P}).$$

This implies that $gf \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$. If we take $g = g^+ - g^-$ and $f = f^+ - f^-$ we have that $gf = (g^+ - g^-)(f^+ - f^-)$ and it then follows that

$$\mathbb{E}(gf|\mathfrak{G}) = g\mathbb{E}(f|\mathfrak{G}).$$

(ii) We first assume that there exists a sequence $A_n \subset \mathfrak{G}$ such that $A_n \uparrow \Omega$ and $\int_{A_n} |f|\, d\mathbb{P} < \infty$ for $n = 1, 2, \cdots$. We have that if $f \in M^+(\Omega, \mathfrak{F}, \mathbb{P})$ then $\int_A \mathbb{E}(f|\mathfrak{G})\, d\mathbb{P} = \int_A f d\mathbb{P}$ for all $A \in \mathfrak{G}$. It then follows that

$$\int_{A_n} \mathbb{E}(|f|\,|\mathfrak{G})\, d\mathbb{P} = \int_{A_n} |f|\, d\mathbb{P} < \infty \text{ for } n = 1, 2, \cdots.$$

Hence $\mathbb{E}(|f|\,|\mathfrak{G}) < \infty$ $\mathbb{P}-$ a.e. on A_n for all $n = 1, 2, \cdots$. Since $A_n \uparrow \Omega$, this implies that $\mathbb{E}(|f|\,|\mathfrak{G}) \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$, i.e., $f \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$.

Conversely, assume that $f \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{G})$. Then $\mathbb{E}(|f|\,|\mathfrak{G})$ is in $L^0(\Omega, \mathfrak{G}, \mathbb{P})$ and so there is a sequence $A_n \in \mathfrak{G}$ with $A_n \uparrow \Omega$ and

$$\int_{A_n} |f|\, d\mathbb{P} = \int_{A_n} \mathbb{E}(|f|\,|\mathfrak{G})\, d\mathbb{P} < \infty \text{ for } n = 1, 2, \cdots.$$

Since $|\mathbb{E}(|f|\mathfrak{G})|, \mathbb{E}(f^+|\mathfrak{G})$ and $\mathbb{E}(f^-|\mathfrak{G})$ are all less than or equal to $\mathbb{E}(|f|\,|\mathfrak{G})$ it then follows that $\mathbb{E}(f|\mathfrak{G})$ is integrable over A and that

$$\begin{aligned}\int_A \mathbb{E}(f|\mathfrak{G})\,d\mathbb{P} &= \int_A \mathbb{E}(f^+|\mathfrak{G})\,d\mathbb{P} - \int_A \mathbb{E}(f^-|\mathfrak{G})\,d\mathbb{P}\\ &= \int_A f^+ d\mathbb{P} - \int_A f^- d\mathbb{P}\\ &= \int_A f d\mathbb{P}.\end{aligned}$$

□

1.6 Kernel Operators

In this section we give an introduction into kernel operators and a summary of their properties. We will mention only those concepts that will be useful later. We refer the reader to [33, sections 93, 94 and 95] as well as [25].

Let (X, Σ, μ) and (Y, Λ, ν) be σ-finite measure spaces and let $t(x, y)$ be a real valued $\mu \otimes \nu$-measurable function on $X \times Y$, where $\mu \otimes \nu$ denotes the product measure of μ and ν. For any $f \in L^0(Y, \Lambda, \nu)$ the function $t(x, y) f(y)$ is $\mu \otimes \nu$-measurable, which implies that for almost every $x \in X$ the function $t(x, y) f(y)$ is ν-measurable as a function of y. It then follows that

$$h(x) = \int_Y |t(x, y) f(y)|\, d\nu(y) \tag{1.6.1}$$

makes sense for all $x \in X$ such that $t(x, y) f(y)$ is ν-measurable as a function of y. By Fubini's theorem $h(x)$ is a μ-measurable function on X. The set of all $f \in L^0(Y, \Lambda, \nu)$ for which the function $h(x)$ is finite valued μ-a.e. on X

will be called the Y-*domain* of $t(x,y)$ and will be denoted by $\mathrm{dom}_Y(t)$. For $f \in \mathrm{dom}_Y(t)$ the function h is finite μ-a.e. on X and so

$$g(x) = \int_Y t(x,y) f(y)\, d\nu(y)$$

is also finite μ-a.e on X. The function g is μ-measurable since

$$g(x) = \int_Y [t(x,y) f(y)]^+ \, d\nu(y) - \int_Y [t(x,y) f(y)]^- \, d\nu(y), \qquad (1.6.2)$$

where, $\int_Y [t(x,y) f(y)]^+ \, d\nu(y)$ and $\int_Y [t(x,y) f(y)]^- \, d\nu(y)$ are, by Fubini's theorem, μ-measurable. Equation (1.6.2) then defines a linear operator $T : f \to g$ from $\mathrm{dom}_Y(t)$ into $L^0(X,\Sigma,\mu)$. The operator T is called a *kernel operator* or *integral operator*. The function $t(x,y)$ is called the *kernel* of T. The set of kernel operators with kernel k will be denoted by $\mathcal{L}_k(L,M)$.

If T is a kernel operator with kernel $t(x,y)$ and if L and M are ideals in $L^0(Y,\Lambda,\nu)$ and $L^0(X,\Sigma,\mu)$, respectively, then T is said to be a kernel operator from L into M if $L \subset \mathrm{dom}_Y(t)$ and $\int_Y t(x,y) f(y)\, d\nu(y) \in M$, for all $f \in L$. In this case $\int_Y |t(x,y)f(y)| d\nu(y) \in M$ and

$$\begin{aligned} (T_1 f)(x) &= \int_Y t^+(x,y) f(y)\, d\nu(y) \\ (T_2 f)(x) &= \int_Y t^-(x,y) f(y)\, d\nu(y) \end{aligned}$$

are also kernel operators from L into M. Hence we have that $T = T_1 - T_2$ and T_1 and T_2 are positive operators. Thus the set of all kernel operators from L into M is a linear subspace of $\mathcal{L}_b(L,M)$.

We extend this notion to the concept of τ-kernel operators. For this we refer the reader to [8, Section 5]

Let $(X, \Sigma, \mu), (Y, \Lambda, \nu)$ and (Z, Γ, γ) be σ-finite measure spaces and suppose that $L \subseteq L^0(Y, \Lambda, \nu)$ and $M \subseteq L^0(X, \Sigma, \mu)$ are ideals with carriers Y and X respectively. Let $\tau : X \times Z \to Y$ be a $(\Sigma \otimes \Gamma, \Lambda)$-measurable null-preserving mapping with respect to $\mu \otimes \gamma$ and ν.

Definition 1.6.1 A function $k \in L^0(X \times Z, \Sigma \otimes \Gamma, \mu \otimes \gamma)$ is called an *absolute τ-kernel for L and M* if

$$\int_Z |k(\cdot, z) f(\tau(\cdot, z))| d\gamma(z) \in M$$

for all $f \in L$.

The collection of all such τ-kernels will be denoted by $\mathfrak{K}(L, M)$. For $k \in \mathfrak{K}(L, M)$ and $f \in L$ the operator

$$Kf(x) = \int_Z k(x, z) f(\tau(x, z)) d\gamma(z)$$

is μ-a.e well defined on X and $Kf \in M$. This defines a linear, order bounded, order continuous operator K from L into M, i.e., $K \in \mathcal{L}_n(L, M)$. The operator K thus defined is called an *absolute τ-kernel operator* with kernel $k(x, z)$. We will denote the collection of absolute τ-kernel operators by $\mathcal{L}_{\tau,k}(L, M)$. We have that $\mathcal{L}_{\tau,k}(L, M) \subseteq \mathcal{L}_n(L, M)$ and that if $k \geq 0$, $\mu \otimes \gamma$-a.e. on $X \times Z$ then $K \geq 0$.

We note that if $(Z, \Gamma, \gamma) = (Y, \Lambda, \nu)$ and we choose $\tau(x, z) = z$ then the absolute τ-kernel operator K is an absolute kernel operator.

As an example of an absolute τ-kernel operator we consider the partial integral operator as introduced by Kalitvin and Zabrejko in [12] and the previous mentioned authors together with Appell in [4]. The reader is also referred to [3], [2], and [11]:

Consider σ-finite measure spaces $(\overline{X}, \Sigma, \mu)$ and $(\overline{Y}, \Lambda, \nu)$. Let L and M be ideals in $L^0(\overline{X} \times \overline{Y}, \Sigma \otimes \Lambda, \mu \otimes \nu)$ with carrier $\overline{X} \times \overline{Y}$.

Definition 1.6.2 For a function $k \in L^0(\overline{X} \times \overline{Y} \times \overline{X}, \Sigma \otimes \Lambda \otimes \Sigma, \mu \otimes \nu \otimes \mu)$ such that

$$\int_{\overline{X}} |k(\cdot, \cdot, z) f(z, \cdot)| d\mu(z) \in M \text{ for all } f \in L$$

we define the operator $K : L \to M$ by

$$K(f)(x,y) = \int_{\overline{X}} |k(x,y,z) f(z,y)| d\mu(z) \ \mu \otimes \nu \text{ a.e on } \overline{X} \times \overline{Y} \text{ for all } f \in L. \tag{1.6.3}$$

We call such an operator K an *absolute partial integral operator.* We will denote the collection of all partial integral operators of the form (1.6.3) by $\mathfrak{P}_{\overline{X}}(L, M)$.

If in the above definition we put $X = Y = \overline{X} \times \overline{Y}$, $Z = \overline{X}$ and $\tau : (X \times Z \to Y)$ defined by $\tau(x,y,z) = (z,y)$ we get that partial integral operators defined by (1.6.3) become τ-kernel operators.

Similarly, if $t \in L^0(\overline{X} \times \overline{Y} \times \overline{Y}, \Sigma \otimes \Lambda \otimes \Lambda, \mu \otimes \nu \otimes \nu)$ is such that

$$\int_{\overline{Y}} |t(\cdot, \cdot, z) f(\cdot, z)| d\nu(z) \in M \text{ for all } f \in L,$$

The corresponding partial integral operator is defined by

$$T(f)(x,y) = \int_{\overline{Y}} |t(x,y,z) f(x,z)| d\nu(z) \ \mu \otimes \nu \text{ a.e on } \overline{X} \times \overline{Y} \text{ for all } f \in L. \tag{1.6.4}$$

We will denote the collection of all operators of the form (1.6.4) by $\mathfrak{P}_{\overline{Y}}(L, M)$. Again these operators are τ-integral operators if we take $X = Y = \overline{X} \times \overline{Y}, Z = \overline{Y}$ and $\tau : X \times Z \to Y$ defined by $\tau(x, y, z) = (x, z)$.

We list a number of properties of absolute kernel operators proofs of which can be found in [33, Chapter 13] and [25].

Theorem 1.6.3 *Let (Y, Λ, ν) and (X, Σ, μ) be σ-finite spaces, L and M ideals in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$, respectively. Denote the carrier of L by Y_L. If T is a kernel operator from L into M with kernel $t(x, y)$ then the following hold:*

(i) T is positive if and only if $t(x, y) \geq 0$ $\mu \otimes \nu$-a.e. on $X \times Y_L$

(ii) T is the null operator, 0, if and only if $t(x, y) = 0$ $\mu \otimes \nu$-a.e. on $X \times Y_L$

(iii) If $0 \leq S \leq T$ in $\mathcal{L}_b(L, M)$ then S is a kernel operator.

(iv) If S and T are kernel operators from L to M with kernels $s(x, y)$ and $t(x, y)$, respectively, then $\sup(T, S)$ is a kernel operator with its kernel given by the pointwise a.e. supremum of $s(x, y)$ and $t(x, y)$.

(v) The set $\mathcal{L}_k(L, M)$ is a band in $\mathcal{L}_b(L, M)$.

(vi) If Y does not contain any atom then the identity operator I is disjoint to the band $\mathcal{L}_k(L, L)$ in $\mathcal{L}_b(L, L)$. It then follows that I is not a kernel operator.

Again let (Y, Λ, ν) be a σ-finite measure space and L an ideal in $L^0(Y, \Lambda, \nu)$ such that the carrier of L is Y. Let $L_n^\sim$ be the band of order continuous linear functionals on L which we identify, as usual with L' as discussed in page 22. Let (X, Σ, μ) be another σ-finite measure space and M be an ideal in $L^0(X, \Sigma, \mu)$. Consider the functions $g \in L'$ and $h \in M$, then the $\mu \otimes \nu-$measurable function $t(x, y) = h(x)g(y)$ is the kernel of an absolute kernel operator from L into M, for, for every $f \in L$

$$\int_Y |t(x,y)f(y)|d\nu(y) = |h(x)| \int_Y |g(y)f(y)|d\nu(y) \in M.$$

Any finite linear combination of kernel operators of this elementary kind is called a *kernel operator of finite rank.* We will use the notation $L_n^\sim \otimes M$ for the set of kernel operators of finite rank. The set $L_n^\sim \otimes M$ is a linear subspace of $\mathcal{L}_b(L, M)$. The band generated by $L_n^\sim \otimes M$ in $\mathcal{L}_b(L, M)$ equals $(L_n^\sim \otimes M)^{dd}$. The band $(L_n^\sim \otimes M)^{dd}$ is contained in the band of all absolute kernel operators since $L_n^\sim \otimes M$ consists of absolute kernel operators.

The following is from [33]:

Theorem 1.6.4 *If the carrier of L' is the whole of Y then $\mathcal{L}_k(L, M) = (L_n^\sim \otimes M)^{dd}$, i.e., the set of absolute kernel operators is the band generated by the kernel operators of finite rank.*

We now take a look at continuity of kernel operators. We assume that Y is the carrier of both L and L'. Given a sequence f_n of measurable functions in L and a measurable subset E of Y. Let $\epsilon > 0$ be given and put $\overline{E} := \{y \in E : |f_n(y)| \geq \epsilon\}$. We say that (f_n) *converges in measure* on E to 0 if

$$\lim_{n\to\infty} \nu(\overline{E}) = 0$$

for every $\epsilon > 0$. It is said that (f_n) *star-converges* to 0 if every subsequence of (f_n) contains a subsequence that converges pointwise to 0 a.e. on Y. We will denote this by $f_n \xrightarrow{*} 0$. We state the following theorem, taken from [33, 25], that shows the connection between convergence in measure and star-convergence.

Theorem 1.6.5 *Let $0 \leq u_n \leq u$ in L. Then the following are equivalent.*

(a) $u_n \xrightarrow{*} 0$ as $n \to \infty$.

(b) (u_n) converges to 0 in measure on every subset of Y of finite measure.

(c) For every $E \subset Y$ such that $\mathbf{1}_E \in L'$ we have

$$\int_E u_n d\nu \to 0 \text{ as } n \to \infty.$$

We note that the conditions (a), (b) and (c) in the previous theorem are weaker than the condition that $u_n(y) \to 0$ a.e. on Y. We state the following result that relates kernel operators to star-convergence, again see [33, 25]

Theorem 1.6.6 (Bukhvalov) *For a positive linear operator $T : L \to M$ and a sequence $0 \leq u_n \leq u$ in L the following are equivalent:*

(a) T is a kernel operator

(b) $u_n \xrightarrow{*} 0$ implies that $Tu_n(x) \to 0$ a.e. on X.

The condition (b) in the above theorem can be weakened further. We state the following as in [25].

Theorem 1.6.7 *For a positive linear operator $T : L \to M$, a sequence $0 \leq u_n \leq u$ in L and a sequence of sets (A_n) in Y we have that T is a kernel operator if and only if*

(i) $u_n(y) \to 0$ *for almost every* y *in* Y *implies that* $Tu_n(x) \to 0$ *for almost every* x *in* X.

(ii) $0 \leq \mathbf{1}_{A_n} \leq u$ *in* L *and* $\nu(A_n) \to 0$ *implies that* $T\mathbf{1}_{A_n}(x) \to 0$ *for every* x *in* X.

Chapter 2

MCE operators

In this chapter we explore the notion of Multiplication Conditional Expectation operators (MCE operators) and their characteristics. We will look at those operators that can be represented as Multiplication Conditional Expectation operators or (MCE-representable operators). We will show that Riesz homomorphisms and Conditional Expectation operators are the most fundamental operators in the theory of Riesz spaces mainly because a large class of operators can be expressed as products of Riesz homomorphisms and Conditional Expectation operators.

Grobler and de Pagter in [8] defined MCE-representable operators using Riesz homomorphisms that operate between σ-finite measure spaces. We give an alternate definition of MCE-representable operators, thus in a way, giving necessary and sufficient conditions for Riesz homomorphisms and ideals in those σ-finite measure spaces to be used to define MCE-representable operators.

We first have a look at these operators and their properties on ideals of measurable spaces. The bulk of this is from [8].

2.1 MCE operators on ideals

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, let $\mathfrak{F}_L$ and $\mathfrak{F}_M$ be sub-σ-algebras of $\mathfrak{F}$ and let $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be ideals, both with carriers Ω. Put

$$\mathfrak{M}(L, M) = \{m \in L^0(\Omega, \mathfrak{F}, \mathbb{P}) : \mathbb{E}(|mf||\mathfrak{F}_M) \in M \ \forall f \in L\}.$$

For all $m \in \mathfrak{M}(L, M)$ and $f \in L$ we have that $mf \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{F}_M)$ since $M \subset L^0(X, \Sigma, \mu)$. For any m in $\mathfrak{M}(L, M)$ define an operator $S_m : L \to M$ by

$$S_m f = \mathbb{E}(mf|\mathfrak{F}_M) \ \forall f \in L.$$

We have that $\mathfrak{M}(L, M)$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ since $\mathrm{dom}\mathbb{E}(\cdot|\mathfrak{F}_M)$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$. Indeed if, $0 \le |p| \le |m|$, $m \in \mathfrak{M}(L, M)$ then

$$\mathbb{E}(|pf||\mathfrak{F}_M) \le \mathbb{E}(|mf||\mathfrak{F}_M) \in M,$$

thus $\mathbb{E}(|pf||\mathfrak{F}_M) \in M$, so $p \in \mathfrak{M}(L, M)$. Thus $\mathfrak{M}(L, M)$ is solid and hence an ideal.

We have that S_m is a well defined operator. If $0 \le m \in \mathfrak{M}(M, L)$ then $S_m \ge 0$ and if $m \in \mathfrak{M}(M, L)$ then

$$\begin{aligned} |S_m f| &= |\mathbb{E}(mf|\mathfrak{G})| \\ &\le \mathbb{E}(|m||f||\mathfrak{G}) \\ &= S_{|m|}|f| \end{aligned}$$

and so

$$|S_m| \leq S_{|m|}.$$

Hence S_m is order bounded.

Define $\mathcal{L}_m(L, M) = \{S_m \mid m \in \mathfrak{M}(L, M)\}$. We have thus constructed a space of functions $\mathfrak{M}(L, M) \subset L^0(\Omega, \mathfrak{F}, \mathbb{P})$ such that for an $m \in \mathfrak{M}(L, M)$ there exists an operator S_m which is order bounded and order continuous, hence $S_m \in \mathcal{L}_m(L, M) \subset \mathcal{L}_n^b(L, M)$. The mapping $m \mapsto S_m$ is positive, i.e., $m \geq 0 \Rightarrow S_m \geq 0$.

The following property is from Grobler and de Pagter, [8].

Proposition 2.1.1 *Let $\mathfrak{F}_0 = \sigma(\mathfrak{F}_M, \mathfrak{F}_L)$ be σ-algebra generated by $\mathfrak{F}_M$ and $\mathfrak{F}_L$. Let $\mathfrak{M}_0(L, M) = L^0(\Omega, \mathfrak{F}_o, \mathbb{P}) \cap \mathfrak{M}(L, M)$. Then we have*

$$\mathfrak{L}_{\mathfrak{M}}(L, M) = \mathfrak{L}_{\mathfrak{M}0}(L, M).$$

Proof It will suffice to show that given $0 \leq m \in \mathfrak{F}(M, L)$ there exists $0 \leq m_0 \in \mathfrak{M}_0(L, M)$ such that

$$S_m f = S_{m_0} f \text{ for all } 0 \leq f \in L.$$

Let $0 \leq m \in \mathfrak{F}(M, L)$ be given and set

$$m_0 = \mathbb{E}(m|\mathfrak{F}_0) \in M^+(\Omega, \mathfrak{F}_0, \mathbb{P}).$$

For $0 \leq f \in L$ we have

$$\begin{aligned} S_m f &= \mathbb{E}(mf|\mathfrak{F}_M) \\ &= \mathbb{E}(\mathbb{E}(mf|\mathfrak{F}_0)|\mathfrak{F}_M). \end{aligned}$$

Note that f is $\mathfrak{F}_M$ measurable and $\mathfrak{F}_M \subset \mathfrak{F}_0$. Thus f is $\mathfrak{F}_0$ measurable and so $\mathbb{E}(mf|\mathfrak{F}_0) = \mathbb{E}(m|\mathfrak{F}_0)f$. This gives us that

$$\begin{aligned} S_m f &= \mathbb{E}(\mathbb{E}(m|\mathfrak{F}_0)f|\mathfrak{F}_M) \\ &= \mathbb{E}(m_0 f|\mathfrak{F}_M) \\ &= S_{m_0} f \text{ for all } 0 \leq f \in L. \end{aligned}$$

In particular, $\mathbb{E}(m_0 f|\mathfrak{F}_M) \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ and thus $m_0 f \in L^0(\Omega, \mathfrak{F}_0, \mathbb{P})$ for all $0 \leq f \in L$. Since the carrier of L is Ω, this implies that $0 \leq m_0 \in L^0(\Omega, \mathfrak{F}_0, \mathbb{P})$. We then have that $0 \leq m_0 \in \mathfrak{M}_0(L, M)$ and $S_m f = S_{m_0} f$ for all $0 \leq f \in L$. □

This result shows that, for the study of the space $\mathfrak{L}_{\mathfrak{M}}$, we may assume, without loss of generality, that $\mathfrak{F} = \sigma(\mathfrak{F}_M, \mathfrak{F}_L)$.

Consider a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and two sub-algebras of $\mathfrak{F}, \mathfrak{F}_1$ and $\mathfrak{F}_2$, such that $\mathfrak{F} = \sigma(\mathfrak{F}_1, \mathfrak{F}_2)$. Let

$$\Gamma = \{A \cap B \mid A \in \mathfrak{F}_1, B \in \mathfrak{F}_2\}.$$

Since $\mathfrak{F}_1, \mathfrak{F}_2 \subset \Gamma$, we have that $\sigma(\Gamma) = \mathfrak{F}$ and Γ is a semi-ring. Therefore $\mathfrak{F}$ is a monotone class generated by the finite union of sets in Γ. Thus if $f \in L^1(\Omega, \mathfrak{F}, \mathbb{P})$ such that $\int_G f d\mathbb{P} \geq 0$ for all $G \in \Gamma$ it follows that $\int_E f d\mathbb{P} \geq 0$ for all $E \in \mathfrak{F}$ since $\mathfrak{M} = \{C \in \mathfrak{F} \mid \int_C f d\mathbb{P} \geq 0\}$ is a monotone class which contains all finite disjoint unions of sets in Γ. Therefore, if $\int_{A\cap B} f d\mathbb{P} \geq 0$, for all $A \in \mathfrak{F}_1$ and $B \in \mathfrak{F}_2$, then $f \geq 0$ $\mathbb{P}-$a.e. on Γ.

The following result is also from [8].

Lemma 2.1.2 *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, let $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be order dense ideals of measurable functions and suppose that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$. For $m \in \mathfrak{M}(L, M)$ we have*

(i) if $S_m \geq 0$ then $m \geq 0$ $\mathbb{P}$-a.e. on Ω,

(ii) if $S_m = 0$ then $m = 0$ $\mathbb{P}$-a.e. on Ω.

Proof (i) Put $A_0 \in \mathfrak{F}_L$ such that $\mathbf{1}_{A_0} \in L$. Let $B_0 \in \mathfrak{F}_M$ be such that $\int_{B_0} |m\mathbf{1}_{A_0}| d\mathbb{P} < \infty$, this is possible since the carrier of L is the whole of Ω.

We show that $m \geq 0$ $\mathbb{P}$-a.e. on $A_0 \cap B_0$. For $A \in \mathfrak{F}_L$ with $A \subset A_0$ and $B \in \mathfrak{F}_M$ with $B \subset B_0$ we have that

$$\int_B |m\mathbf{1}_A| d\mathbb{P} < \infty.$$

Thus

$$\begin{aligned} \int_B m\mathbf{1}_A d\mathbb{P} &= \int_B \mathbb{E}(m\mathbf{1}_A | \mathfrak{F}_M) d\mathbb{P} \\ &= \int_B S_m(\mathbf{1}) d\mathbb{P} \\ &\geq 0. \end{aligned}$$

This shows that $\int_{A \cap B} m d\mathbb{P} \geq 0$ for all such A and B.

Now define the following σ-algebras of subsets of $A_0 \cap B_0$

$$\begin{aligned} \mathfrak{F}_M^0 &= \{A_0 \cap B \mid B \in \mathfrak{F}_M, B \subseteq B_0\}, \\ \mathfrak{F}_L^0 &= \{A \cap B_0 \mid A \in \mathfrak{F}_L, A \subseteq A_0\}, \\ \mathfrak{F}^0 &= \{C \in \mathfrak{F} \mid C \subseteq A_0 \cap B_0\}. \end{aligned}$$

We have that $\sigma(\mathfrak{F}^0_M, \mathfrak{F}^0_L) = \mathfrak{F}^0$ and $\int_{A\cap B} m d\mathbb{P} \geq 0$ for all $A \in \mathfrak{F}^0_L$ and $B \in \mathfrak{F}^0_M$.

From the remarks preceeding this lemma, it follows that $\int_C m d\mathbb{P} \geq 0$ for all $C \in \mathfrak{F}^0$, i.e., $m \geq 0$ $\mathbb{P}$-a.e. on $A_0 \cap B_0$. Now fix $A_0 \in \mathfrak{F}_L$ such that $\mathbf{1}_{A_0} \in L$. Since $m\mathbf{1}_{A_0} \in \mathrm{dom}\mathbb{E}(\cdot|\mathfrak{F}_M)$, by Proposition 1.5.6 on page 30, there is a sequence $B_n \in \mathfrak{F}_M$ with $B_n \uparrow \Omega$ such that $\int_{B_n} |m\mathbf{1}_{A_0}| d\mathbb{P} < \infty$. We have that $m \geq 0$ $\mathbb{P}$-a.e. on $B_n \cap A_0$ for each $n = 1, 2, \cdots$. Thus $m \geq 0$ $\mathbb{P}$-a.e. on $\Omega \cap A_0$ since $\cup B_n = \Omega$. Hence $m \geq 0$ $\mathbb{P}$-a.e. on A_0. Since the carrier of L is Ω there exists a sequence $A_n \in \mathfrak{F}_L$ such that $A_n \uparrow \Omega$ and $\mathbf{1}_{A_0} \in L$. Therefore $m \geq 0$ $\mathbb{P}$-a.e. on A_n, and hence, $m \geq 0$ $\mathbb{P}$-a.e. on Ω.

(ii) From $m \mapsto S_m$ we get that $-m \mapsto S_{-m}$. Thus $\mathbb{E}(-mf|\mathfrak{F}_M) = -\mathbb{E}(mf|\mathfrak{F}_M)$. We already have that if $S_m \geq 0$ then $m \geq 0$. Now let $S_m = 0$, then $-S_m \geq 0$. This implies that $S_{-m} \geq 0$ and so $-m \geq 0$, which gives that $m \leq 0$. Thus $m = 0$. □

We also look at this result which is also from [8]:

Lemma 2.1.3 *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be ideals of measurable functions. If $S_\alpha \in \mathcal{L}_{\mathfrak{M}}(L, M)$ is an upwards directed net and if $0 \leq S_\alpha \uparrow S$ in $\mathcal{L}_b(L, M)$ then $S \in \mathcal{L}_{\mathfrak{M}}(L, M)$.*

Proof Without loss of generality, we may assume that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$. Write $S_\alpha = S_{m_\alpha}$ with $m_\alpha \in \mathfrak{M}(L, M)$. By the preceeding proposition we have that $0 \leq m_\alpha \uparrow$ in $M^+(\Omega, \mathfrak{F}, \mathbb{P})$. Then $m := \sup_\alpha m_\alpha$ exists in

$M^+(\Omega, \mathfrak{F}, \mathbb{P})$ and there exists a sequence (m_n) composed of terms of (m_α) such that $m_n \uparrow m$ (this is a slight variation of Lemma 94.4 in [33]). For each $0 \leq f \in L$ we have

$$Sf \geq S_{m_n} f = \mathbb{E}(m_n f | \mathfrak{F}_M) \uparrow \mathbb{E}(mf | \mathfrak{F}_M) \in M^+(\Omega, \mathfrak{F}_M, \mathbb{P}).$$

Now, $Sf \in M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ implies $\mathbb{E}(mf|\mathfrak{F}_M) \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ and consequently, $mf \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$. Since L has carrier Ω, this implies that $m \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ and hence $m \in \mathfrak{M}(L, M)$. By Proposition 1.5.3 (iv), $m_n \uparrow m$ in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ implies that $S_{m_n} \uparrow S_m$ in $\mathcal{L}_b(L, M)$ and hence $S_m \leq S$. On the other hand, $m \geq m_\alpha$ and so $S_m \geq S_{m_\alpha}$ for all α and thus $S_m \geq S$. Hence, $S = S_m \in \mathcal{L}_{\mathfrak{M}}(L, M)$. □

The following result is also found in [8]

Lemma 2.1.4 *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space and let $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be ideals of measurable functions. If $S \in \mathcal{L}_n(L, M)$ such that $0 \leq S \leq S_m$ for some $m \in \mathfrak{M}(L, M)$, then $S \in \mathcal{L}_{\mathfrak{M}}(L, M)$.*

Proof We first prove the proposition under the additional assumption that $\mathfrak{F}_L = \mathfrak{F}$. Then L is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ and the operator S_m is interval preserving (i.e., it has the Maharam property). Indeed, if $f \in L$ and $0 \leq g \leq S_m f$ in M, define the function σ by

$$\sigma = \begin{cases} g/S_m f & \text{if } S_m f \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $0 \leq \sigma \leq 1$ and $\sigma \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ we have $0 \leq \sigma f \leq f$ in L and

$$\begin{aligned} S_m(\sigma f) &= \mathbb{E}(m\sigma f|\mathfrak{F}_M) \\ &= \sigma\mathbb{E}(mf|\mathfrak{F}_M) \\ &= \sigma S_m f \\ &= g. \end{aligned}$$

It follows from the Luxemburg-Schep Radon-Nikodym theorem (see [18]) that $S = S_m\pi$ for some $0 \leq \pi \leq I$ in the centre $Z(L)$ of L. Now π is multiplication by some function $0 \leq p \leq 1$ on Ω (see [33] Example 141.3) and, since $\mathfrak{M}(L, M)$ is an ideal in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$, it follows that $S = S_{pm} \in \mathcal{L}_{\mathfrak{M}}(L, M)$.

For general L, let $I(L)$ be the ideal generated in $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ by L. We have that $\mathfrak{M}(I(L), M) = \mathfrak{M}(L, M)$. Indeed, if $m \in \mathfrak{M}(I(L), M)$ then $\mathbb{E}(|mg||\mathfrak{F}_M) \in M$ for all $g \in I(L)$. But, since $I(L) \supset L$, $\mathbb{E}(|mg||\mathfrak{F}_M) \in M$ holds for all $g \in L$; hence $m \in \mathfrak{M}(L, M)$. On the other hand, if $m \in \mathfrak{M}(L, M)$ then $\mathbb{E}(|mf||\mathfrak{F}_M) \in \mathfrak{F}_M$ for all $f \in L$. Let $g \in I(L)$, then there is some $f \in L$ such that $|g| \leq |f|$. This then gives us that $\mathbb{E}(|mg||\mathfrak{F}_M) \leq \mathbb{E}(|mf||\mathfrak{F}_M) \in M$. Thus $m \in \mathfrak{M}(I(L), M)$. Define $\overline{S}_m : I(L) \to M$ by $\overline{S}_m f = \mathbb{E}(mf|\mathfrak{F}_M)$ for all $f \in I(L)$. Then $\overline{S}_m$ is an extension of S_m and $\overline{S}_m \in \mathcal{L}_{\mathfrak{M}}(I(L), M)$. It follows from the Kantorovich extension theorem (see [1] Theorem I.2.2), that there exists an extension $0 \leq \overline{S} \in \mathcal{L}_b(I(L), M)$ of S such that $0 \leq \overline{S} \leq \overline{S}_m$. By the first part of the proof $\overline{S} \in \mathcal{L}_{\mathfrak{M}}(I(L), M)$. Since $\mathfrak{M}(I(L), M) =$

$\mathfrak{M}(L,M)$, the restriction of $\overline{S}$ to L belongs to $\mathcal{L}_{\mathfrak{M}}(L,M)$, i.e., $S \in \mathcal{L}_{\mathfrak{M}}(L,M)$. This completes the proof. □

We combine the three previous results to prove the following, as was done by Grobler and de Pagter in [8].

Proposition 2.1.5 *Consider a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and ideals of measurable functions $L \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$. Then $\mathcal{L}_{\mathfrak{M}}(L,M)$ is a band in $\mathcal{L}_n(L,M)$. Moreover, assuming that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$, the mapping $m \to S_m$ is a Riesz isomorphism from the ideal $\mathfrak{M}(L,M)$ onto $\mathcal{L}_{\mathfrak{M}}(L,M)$.*

Proof First we show that $\mathcal{L}_{\mathfrak{M}}(L,M)$ is a Riesz subspace of $\mathcal{L}_n(L,M)$. Take $S_m \in \mathcal{L}_{\mathfrak{M}}(L,M)$. As observed earlier, $|S_m| \leq S_{|m|}$. By Lemma 2.1.4 this implies that $|S_m| \in \mathcal{L}_{\mathfrak{M}}(L,M)$. Hence, $\mathcal{L}_{\mathfrak{M}}(L,M)$ is a Riesz subspace. Using the above Lemma 2.1.4 once more, we see that $\mathcal{L}_{\mathfrak{M}}(L,M)$ is actually an ideal in $\mathcal{L}_n(L,M)$. Moreover, Lemma 2.1.3 yields that $\mathcal{L}_{\mathfrak{M}}(L,M)$ is a band in $\mathcal{L}_n(L,M)$. Assuming that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$, it follows from Lemma 2.1.2 that the mapping $m \mapsto S_m$ is a bi-positive bijection from $\mathfrak{M}(L,M)$ onto $\mathcal{L}_{\mathfrak{M}}(L,M)$, consequently this mapping is a Riesz isomorphism. □

2.2 MCE-representable operators

We now investigate a class of operators which factorizes through MCE operators. Our aim is to extend the MCE operators to the case of σ-finite measure spaces. This class of operators includes operators such as kernel operators, order continuous Riesz homomorphisms as well as τ-kernel operators.

We consider σ-finite measure spaces (X, Σ, μ) and (Y, Λ, ν). Let $L \subseteq L^0(Y, \Lambda, \nu)$ be an ideal with carrier Y and $M \subseteq L^0(X, \Sigma, \mu)$ be an ideal with carrier X.

Definition 2.2.1 A linear operator $T : L \to M$ is called *Multiplication Conditional Expectation representable* or (MCE-representable) if there exist

(i) a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and sub-σ-algebras $\mathfrak{F}_L$ and $\mathfrak{F}_M$ such that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$,

(ii) order dense ideals $L_\Omega \subseteq L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M_\Omega \subseteq L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ and order continuous Riesz homomorphisms $\phi_L : L \to L_\Omega$ and $\psi_M : M_\Omega \to M$ with ϕ_L surjective and ψ_M a Riesz isomorphism onto an ideal in M and

(iii) $m \in \mathfrak{M}(L_\Omega, M_\Omega)$

such that

$$T = \psi_M S_m \phi_L,$$

i.e., such that the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{T} & M \\ {\scriptstyle\phi_L}\downarrow & & \uparrow{\scriptstyle\psi_M} \\ L_\Omega & \xrightarrow[S_m]{} & M_\Omega \end{array}$$

$\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \phi_L, \psi_M)$ is called a *representation triple for the operator* T and m a Φ-*kernel of* T. The set of all linear operators $T : L \to M$ which are MCE-representable via a fixed triple Φ will be denoted by $\mathcal{L}_\Phi(L, M)$ and we will also say that Φ is a representation triple for the class $\mathcal{L}_\Phi(L, M)$. An operator $T \in \mathcal{L}_\Phi(L, M)$ will be called *Φ-representable.*

Consider a σ-finite measure space (X, Σ, μ) and let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. If ϕ is a σ-order continuous Riesz isomorphism from $L^0(X, \Sigma, \mu)$ into $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ such that $\phi(\mathbf{1}) = \mathbf{1}$ then the restriction of ϕ to the characteristic functions of elements of Σ gives a Boolean σ-homomorphism $\widehat{\phi} : \Sigma \to \mathfrak{F}$ such that $\mathbb{P}(\widehat{\phi}(A)) = 0$ if and only if $\mu(A) = 0$ for $A \in \Sigma$. Such a mapping is called a bi-null preserving mapping or non-singular. If $\mathfrak{F}_L = \widehat{\phi}(\Sigma)$, then $\mathfrak{F}_L$ is a sub-σ-algebra of $\mathfrak{F}$ and $\operatorname{ran}(\phi) = L^0(\Omega, \mathfrak{F}_L, \mathbb{P}) \subseteq L^0(\Omega, \mathfrak{F}, \mathbb{P})$.

Lemma 2.2.2 *Let L be an ideal in $L^0(X, \Sigma, \mu)$ with carrier of L being X and ϕ a σ-order continuous Riesz isomorphism from $L^0(X, \Sigma, \mu)$ into $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ such that $\phi(\mathbf{1}) = \mathbf{1}$. Put*

$$L_\Omega = \phi(L) \subseteq L^0(\Omega, \mathfrak{F}_L, \mathbb{P}) \subseteq L^0(\Omega, \mathfrak{F}, \mathbb{P}).$$

Then L_Ω is order dense.

Proof Since ϕ is a Riesz isomorphism from $L^0(X, \Sigma, \mu)$ onto $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$, we have, by Proposition 1.4.3 (iii), that L_Ω is an ideal in $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$. We have that the carrier of L is X and therefore there exists a sequence of sets $X_n \uparrow X$ such that $\mathbf{1}_{X_n} \in L$ and so $\phi(\mathbf{1}_{X_n}) \in L_\Omega$. We have that $\mathbf{1}_{X_n} \uparrow \mathbf{1}_X$ and since ϕ is σ-order continuous we get that $\phi(\mathbf{1}_{X_n}) \uparrow \phi(\mathbf{1}_X)$. If we write $\phi(\mathbf{1}_{X_n}) = \phi(\mathbf{1}_X)\mathbf{1}_{\Omega_n} \in L_n$ we have that

$$\Omega_n = \widehat{\phi}(X_n) \uparrow \widehat{\phi}(X) = \Omega.$$

This implies that the carrier of L_Ω is Ω and so L_Ω is order dense.

Grobler and de Pagter in [8] gave a somewhat different definition which is as follows

Definition 2.2.3 Let (X,Σ,μ) and (Y,Λ,ν) be σ-finite measure spaces and let $L \subseteq L^0(Y,\Lambda,\nu)$ and $M \subseteq L^0(X,\Sigma,\mu)$ be ideals with carriers Y and X respectively. A linear operator $T : L \to M$ is called *Multiplication Conditional Expectation representable* or MCE-representable if there exist:

(1) a probability space $(\Omega,\mathfrak{F},\mathbb{P})$;

(2) a Riesz homomorphism $\phi_L : L^0(Y,\Lambda,\nu) \to L^0(\Omega,\mathfrak{F},\mathbb{P})$ with $\phi_L(\mathbf{1})$ $\mathbb{P}$-a.e. strictly positive on Ω;

(3) a sub-σ-algebra $\mathfrak{F}_M \subset \mathfrak{F}$ and an interval preserving Riesz isomorphism $\psi_M : L^0(\Omega,\mathfrak{F}_M,\mathbb{P}) \to L^0(X,\Sigma,\mu)$;

(4) a function $m \in \mathfrak{M}(L_\Omega, M_\Omega)$,

such that:

$$T = \psi_M S_m \phi_L,$$

where $L_\Omega = \phi_L(L)$, $M_\Omega = \psi_M^{-1}(M)$ and $S_m : L_\Omega \to M_\Omega$ is given by

$$S_m f = \mathbb{E}(mf|\mathfrak{F}_M).$$

According to this definition the following diagram commutes

$$\begin{array}{ccc}
L^0(Y,\Lambda,\nu) \supset L & \xrightarrow{T} & M \subset L^0(X,\Sigma,\mu) \\
\phi_L \downarrow & & \uparrow \psi_M \\
L^0(\Omega,\mathfrak{F}_L,\mathbb{P}) \supset L_\Omega & \xrightarrow[S_m]{} & M_\Omega \subset L^0(\Omega,\mathfrak{F}_M,\mathbb{P})
\end{array}$$

In the next two results we show that Definition 2.2.1 and Definition 2.2.3 are equivalent.

Proposition 2.2.4 *Consider a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, sub-$\sigma$-algebras $\mathfrak{F}_L$ and $\mathfrak{F}_M$ such that $\mathfrak{F} = \sigma(\mathfrak{F}_L, \mathfrak{F}_M)$. Let L_Ω and M_Ω be order dense ideals in $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$, respectively, $\phi_L : L \subseteq L^0(Y, \Lambda, \nu) \to L_\Omega$ and $\psi_M : M_\Omega \to M \subseteq L^0(X, \Sigma, \mu)$ be order continuous Riesz homomorphisms with ϕ_L surjective and ψ_M a Riesz isomorphism onto an ideal in M. Then*

(i) ϕ_L and ψ_M can be extended to order continuous Riesz homomorphisms that respectively map $L^0(Y, \Lambda, \nu)$ into $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ into $L^0(X, \Omega, \mu)$.

(ii) For the extended ϕ_L we have that $\phi_L(\mathbf{1})$ is $\mathbb{P}-$a.e. strictly positive on Ω.

(iii) The extended ψ_M is an interval preserving Riesz isomorphism.

Proof (i) Since the carrier of the ideal $L \subseteq L^0(Y, \Lambda, \nu)$ is Y we have that L is order dense in $L^0(Y, \Lambda, \nu)$. Similarly M is order dense in $L^0(X, \Sigma, \mu)$. Also, ϕ_L and ψ_M are order continuous Riesz homomorphisms. So by Proposition 1.4.3 (iv) ϕ_L and ψ_M can be extended, uniquely, to order continuous Riesz homomorphisms that respectively map $L^0(Y, \Lambda, \nu)$ into $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ into $L^0(X, \Omega, \mu)$.

(ii) Suppose that $\phi_L(\mathbf{1})$ is not strictly positive. Then there exists some $B \subset \Omega$ such that $\mathbf{1}_B \neq 0$ in $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $\mathbf{1}_B \wedge \phi(\mathbf{1}) = 0$. L_Ω

is order dense in $L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and so there exists $0 < h \in L_\Omega$ such that $0 < h \leq \mathbf{1}_B$. Now ϕ_L is a surjection from L to L_Ω and so there is some $0 \neq f \in L$ such that $\phi_L(f) = h$. Since ϕ_L is a Riesz homomorphism we have

$$\begin{aligned} \phi_L(f^+) &= \phi_L(f \vee 0) \\ &= \phi_L(f) \vee \phi_L(0) \\ &= h. \end{aligned}$$

Thus we have that there is $0 < f \in L$ such that $\phi_L(f) = h$. From $f \wedge n\mathbf{1} \uparrow f$ we get, by order continuity of ϕ_L, that

$$\phi_L(f \wedge n\mathbf{1}) \uparrow \phi_L(f) = h > 0.$$

On the other hand, for all n

$$\phi_L(f) \wedge \phi_L(n\mathbf{1}) = h \wedge n\phi_L(\mathbf{1}) = 0.$$

Therefore $\phi_L(\mathbf{1})$ is $\mathbb{P}$ strictly positive on Ω.

(iii) We first show that the extended ψ_M is a $1-1$ and thus a Riesz isomorphism. Let $f \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ be such that $\psi_M(f) = 0$. This gives us that $0 = |\psi_M(f)| = \psi_M(|f|)$. We may assume that $f \geq 0$. Since M_Ω is order dense there exists a sequence (f_n) in M_Ω such that $f_n \uparrow f$. Since ψ_M is order continuous we have that $0 \leq \psi_M(f_n) \uparrow \psi_M(f) = 0$. Thus $\psi_M(f_n) = 0$ for all n. Since ψ_M is injective we get that $f_n = 0$ for all n. Thus $f = 0$. Hence ψ_M is a Riesz isomorphism of $L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ into $L^0(X, \Sigma, \mu)$.

Next we show that ψ_M is interval preserving. We want to show that if $[u,v]$ is an interval in $L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ and $f\in[\psi_M(u),\psi_M(v)]$ in $L^0(X,\Sigma,\mu)$ then there exists some $t\in[u,v]$ such that $\psi_M(t)=f$. To that end, we first consider the interval $[0,v]$ in $L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ and let $f\in[0,\psi_M(v)]$.

Since M_Ω is order dense there exists a sequence (v_n) in M_Ω such that $0\le v_n\uparrow v$. We have that $\psi_M(M_\Omega)$ is an ideal and so there exists a sequence (g_n) in M_Ω such that $\psi_M(g_n)=f\wedge\psi_M(v_n)$ for each n. ψ_M is a homomorphism and so $\psi_M(g_n^+\vee v_n)=\psi_M(g_n)$. We may therefore assume that $0\le g_n\le v_n$. Also, for $m\ge n$, $\psi_M(g_n\vee g_m)=\psi_M(g_m)$ and so we may assume that $g_n\uparrow$. Let $g=\sup g_n$. Then $0\le g\le v$, which implies that $g\in L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$. By the order continuity of ψ_M we get that $\psi_M(g_n)\uparrow\psi_M(g)$; but

$$\psi_M(g_n)=f\wedge\psi_M(v_n)\uparrow f\wedge\psi_M(v)=f.$$

Hence $\psi_M(g)=f$. Thus for the interval $[0,v]\subset L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ and $f\in[0,\psi_M(v)]$ there exists a $g\in[0,v]$ such that $\psi_M(g)=f$.

Now let us consider the interval $[u,v]$ in $L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ with $u<v$. Put $f\in[\psi_M(u),\psi_M(v)]$, thus $f-\psi_M(u)\in[0,\psi_M(v-u)]$. By the preceding argument there exists a $g\in[0,v-u]$ such that $\psi_M(g)=f-\psi_M(u)$. But $g\in[0,v-u]$ implies that $g+u\in[u,v]$. The proof is completed by putting $t=g+u$. □

Proposition 2.2.5 *Consider a probability space $(\Omega,\mathfrak{F},\mathbb{P})$, sub-$\sigma$-algebras $\mathfrak{F}_L$ and $\mathfrak{F}_M$ such that $\mathfrak{F}=\sigma(\mathfrak{F}_L,\mathfrak{F}_M)$ and ideals $L\subseteq L^0(Y,\Lambda,\nu)$ and $M\subseteq$*

$L^0(X,\Sigma,\nu)$ with carriers Y and X respectively. Let $\phi_L : L^0(Y,\Lambda,\nu) \to L^0(\Omega,\mathfrak{F},\mathbb{P})$ be a Riesz homomorphism such that $\phi_L(\mathbf{1})$ is $\mathbb{P}$-a.e. strictly positive on Ω and $\psi_M : L^0(\Omega,\mathfrak{F}_M,\mathbb{P}) \to L^0(X,\Sigma,\mu)$ be an interval preserving Riesz isomorphism. Let $L_\Omega = \phi_L(L)$ and $M_\Omega = \psi_M^{-1}(M)$. Then

(i) ϕ_L and ψ_M can be restricted to order continuous Riesz homomorphisms mapping L to $L_\Omega \subset L^0(\Omega,\mathfrak{F}_L,\mathbb{P})$ and $M_\Omega \subset L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$ to M, respectively, with the restricted ϕ_L surjective and the restricted ψ_M a Riesz isomorphism onto an ideal in M.

(ii) The restricted ψ_M is injective.

(iii) L_Ω and M_Ω are order dense.

Proof (i) We have that ϕ_L is a Riesz homomorphism from $L^0(Y,\Lambda,\nu)$ into $L^0(\Omega,\mathfrak{F}_L,\mathbb{P})$ (which are σ-finite measure spaces) and so it is order continuous by Proposition 1.4.3 (i). Its restriction to a mapping from L to L_Ω will be an order continuous Riesz homomorphism as well, which is, moreover, surjective.

We show that $\psi_M(M_\Omega)$ is an ideal. Let $0 \le f \in \psi_M(M_\Omega)$ and $0 \le |g| \le f$. Now $f = \psi_M(v)$ for some $v \in M_\Omega^+$. Since ψ_M is interval preserving, there exists some $t \in M_\Omega$ with $0 \le t \le v$ such that $|g| = \psi_M(t)$. Thus $|g| \in \psi_M(M_\Omega)$. Similar argument shows, since, $g^+ \le |g|$ and $g^- \le |g|$, that $g^+ \in \psi_M(M_\Omega)$ and $g^- \in \psi_M(M_\Omega)$. We then have that $g = g^+ - g^- \in \psi_M(M_\Omega)$.

(ii) Since ψ_M is a Riesz isomorphism, it is injective and so its restriction will be injective as well.

(iii) L is an ideal in $L^0(Y,\Lambda,\mu)$ and ϕ_L is a Riesz homomorphism from $L^0(X,\Sigma,\mu)$ to $L^0(\Omega,\mathfrak{F}_L,\mathbb{P})$ with carrier of L being Y. By Lemma 2.2.2 we then have that $L_\Omega = \phi_L(L)$ is order dense.

Let $0 < f \in L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$. Since ψ_M is $1-1$, we have that $0 < \psi_M(f) \in L^0(X,\Sigma,\mu)$. But M is dense in $L^0(X,\Sigma,\mu)$ and so there exists an element $g \in M$ such that $0 < g \leq \psi_M(f)$. Also, since ψ_M is interval preserving, there is an element $0 < f_1$ such that $\psi_M(f_1) = g$ and $0 < f_1 \leq f$. Hence, since $\psi_M(f_1) \in M$, we have that $f_1 \in \psi_M^{-1}(M) = M_\Omega$. Hence, M_Ω is order dense in $L^0(\Omega,\mathfrak{F}_M,\mathbb{P})$. □

We note that Proposition 2.2.4 and Proposition 2.2.5 give necessary and sufficient conditions that Riesz homomorphisms ϕ_L and ψ_M and ideals L, L_Ω, M and M_Ω should satisfy for them to be used in defining MCE-representable operators.

We conclude by showing that order continuous Riesz homomorphisms, kernel operators and τ-kernel operators are amongst the class of operators that are MCE-representable.

Example 2.2.6 (1) Let $L \subset L^0(Y,\Lambda,\nu)$ be an ideal of measurable functions on the σ-finite measure space (Y,Λ,ν) and $M \subset L^0(X,\Sigma,\mu)$ be an ideal of measurable functions on the σ-finite measure space (X,Σ,μ)

such that L and M have carriers Y and X respectively. Let $T : L \to M$ be an order continuous Riesz homomorphism. We want to show that T is MCE-representable. We first use Proposition 1.4.3 (iv) to extend T into an order continuous Riesz homomorphism $\phi_L : L^0(Y, \Lambda, \nu) \to L^0(X, \Sigma, \mu)$. We have that $\{\phi_L(\mathbf{1}_Y)\}^{dd}$ is a band and so there exists some $X_T \in \Sigma$ such that $\{\phi_L(\mathbf{1}_Y)\}^{dd} = \{(\mathbf{1}_{X_T})\}^{dd}$. Let w_M be a strictly positive function on X_T such that $\int_{X_T} w_M d\mu = 1$. Such a function exists, indeed, let $X_n \uparrow X_T$ with $X_n \in \Sigma$ and $\mu(X_n) < \infty$ since Σ is σ-finite. Let w_M be given by $\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(X_n)} \mathbf{1}_{X_n}$. For $x \in X_T$ we have

$$w_M(x) \geq \frac{1}{2^n} \frac{1}{\mu(X_n)} > 0.$$

and

$$\begin{aligned}
\int_{X_T} w_M d\mu &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(X_n)} \int_{X_T} \mathbf{1}_{X_n} d\mu \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(X_n)} \mu(X_n) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= 1.
\end{aligned}$$

Put $\Omega = X_T$, $\mathbb{P} = w_M \mu$, $\mathfrak{F} = \{A \cap X_T : A \in \Sigma\}$ and $\mathfrak{F}_M = \mathfrak{F}$. Let ψ_M be the canonical embedding of $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ into $L^0(X, \Sigma, \mu)$. We have that ϕ_L is a Riesz homomorphism from $L^0(Y, \Lambda, \nu)$ into $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ and is such that $\phi_L(\mathbf{1})$ is strictly positive on Ω. Let $m = \mathbf{1}$. Then for all

$f \in L$ we have

$$\begin{aligned}
\psi_M S_m \phi_L(f) &= \psi_M \mathbb{E}\left(\phi_L(f)\mathbf{1}|\mathfrak{F}_M\right) \\
&= \psi_M \mathbb{E}\left(\phi_L(f)|\mathfrak{F}_M\right) \\
&= \psi_M \mathbb{E}\left(\phi_L(f)|\mathfrak{F}\right) \\
&= \psi_M \phi_L(f) \\
&= \phi_L(f) \\
&= Tf
\end{aligned}$$

(2) We show that kernel operators are MCE-representable. Consider σ-finite measure spaces (X, Σ, μ) and (Y, Λ, ν) and let $L \subset L^0(Y, \Lambda, \nu)$ and $M \subset L^0(X, \Sigma, \mu)$ be ideals with carriers Y and X respectively. Let $K : L \to M$ be the absolute kernel operator from L into M defined by

$$Kf(\omega) = \int_Y k(\omega, x) f(x) d\nu(x) \text{ for all } f \in L,$$

where $k \in L^0(X \times Y, \Sigma \otimes \Lambda, \mu \otimes \nu)$ is such that

$$\int_Y |k(\cdot, x) f(x)| d\nu(x) \in M \text{ for all } f \in L.$$

We show that K is MCE-representable:

Let $w_1 \in L^1(X, \Sigma, \mu)$ and $w_2 \in L^1(Y, \Lambda, \nu)$ be strictly positive and satisfy $\int_X w_1\, d\mu = \int_Y w_2\, d\nu = 1$.

Taking $\Omega := X \times Y$, $\mathfrak{F} := \Sigma \otimes \Lambda$, and putting $\mathbb{P} := w_1\mu \otimes w_2\nu$, we have that $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space, for,

$$\begin{aligned}
\int_\Omega d\mathbb{P} &= \int_{X\times Y} w_1\mu \otimes w_2\nu d\mu(x)d\nu(y) \\
&= \int_X w_1(x)\left[\int_Y w_2(y)d\nu(y)\right]d\mu(x) \\
&= \int_X w_1(x)d\mu(x)\int_Y w_2 d\nu(y) \\
&= 1.
\end{aligned}$$

Define $(\phi_L f)(x,y) := f(y)\mathbf{1}_X(x)$ for all $f \in L$ and $\mathfrak{F}_M = \{A \times Y : A \in \Sigma\}$. Then $L^0(\Omega, \mathfrak{F}_M, \mathbb{P}) = \{f \in L^0(\Omega, \mathfrak{F}, \mathbb{P}) : f(x,y) = g(x)\mathbf{1}_Y(y)$ for some $g \in L^0(X, \Sigma, \mu)\}$. If $f = g\mathbf{1}_Y \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$, define $\psi_M(f) = g$. We have $\phi_L(\mathbf{1}) = \mathbf{1}$ and ψ_M is a bijection from $L^0(\Omega, \mathfrak{F}, \mathbb{P})$ onto $L^0(X, \Sigma, \mu)$ and therefore interval preserving. Taking $m := w_2^{-1}k$ we get that, for all $f \in L$,

$$\begin{aligned}
\mathbb{E}(mf|\mathfrak{F}_M)(x,y) &= \mathbb{E}(w_2^{-1}kf|\mathfrak{F}_M)(x,y) \\
&= \int_Y w_2^{-1}(y)k(x,y)f(y)\mathbf{1}_X(x)w_2(y)d\nu(y) \\
&= \int_Y k(x,y)f(y)d\nu(y)\mathbf{1}_Y(y) \\
&= \int_Y k(x,y)f(y)d\nu(y)
\end{aligned}$$

Hence, the triple $\Phi := ((\Omega, \mathfrak{F}, \mathbb{P}), \phi_L, \psi_M)$ represents K with Φ-kernel $m := w_2^{-1}k$.

(3) Let $(X, \Sigma, \mu), (Y, \Lambda, \nu)$ and (Z, Γ, λ) be σ-finite measure spaces and suppose that $L \subseteq L^0(Y, \Lambda, \nu)$ and $M \subseteq L^0(X, \Sigma, \mu)$ are ideals with carriers

Y and X respectively. Suppose that $\tau : X \times Z \to Y$ is a $(\Sigma \otimes \Gamma, \Lambda)$-measurable null-preserving mapping. We want to show that every absolute τ-kernel operator $T \in \mathcal{L}_{\tau k}(L, M)$ is MCE-representable via a fixed representation triple Φ. We first introduce the following sub-σ-algebras of $\Sigma \otimes \Gamma$:

(a) $\Lambda_0 := \{\tau^{-1}(A) : A \in \Lambda\} = \{\tau_*(A) : A \in \Lambda\}$;

(b) $\Sigma_0 := \{A \times Z : A \in \Sigma\}$.

Let $0 \le w_1 \in L^1(X, \Sigma, \mu)$ be such that $w_1(x) > 0$ μ-a.e. and $\int_X w_1 \, d\mu = 1$ and let $0 \le w_2 \in L^1(Z, \Gamma, \lambda)$ be such that $w_2(z) > 0$ λ-a.e. and $\int_Z w_2 \, d\lambda = 1$. Define $\Omega := X \times Z$, $\mathfrak{F} = \Sigma \otimes \Gamma$ and $\mathbb{P} = (w_1\mu) \otimes (w_2\lambda)$. Then $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space and $\mathbb{P}$ is equivalent to the product measure $\mu \otimes \lambda$. Define

$$\phi_L := \phi_\tau : L^0(Y, \Lambda, \nu) \to L^0(\Omega, \mathfrak{F}, \mathbb{P}).$$

Then ϕ_L is a Riesz homomorphism satisfying $\phi_L(\mathbf{1}) = \mathbf{1}$ and $\mathfrak{F}_L = \Lambda_0$ in the notation defined above. Let $\mathfrak{F}_M = \Sigma_0$ and note that $f \in L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$ if and only $f = g\mathbf{1}_Z$ with $g \in L^0(X, \Sigma, \mu)$. Define $\psi_M : L^0(\Omega, \mathfrak{F}_M, \mathbb{P}) \to L^0(X, \Sigma, \mu)$ by $\psi_M(f) = \psi_M(g\mathbf{1}_Z) = g$. Then ψ_M is a Riesz isomorphism which is interval preserving (in fact it is a surjection). We will use $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \phi_L, \psi_M)$ as a representation triple of a τ-kernel operator.

Since $L_\Omega = \{f \circ \tau : f \in L\}$, it follows from the result in Example 1.5.5

on Page 30 that $m \in \mathfrak{M}(L_\Omega, M_\Omega)$ if and only if

$$\mathbb{E}(|mf| \mid \Sigma_0)(x,z) = \int_Z |m(x,z)f(\tau(x,z))w_2(z)| d\lambda(z) < \infty$$

μ-a.e. on X, and

$$\mathbb{E}(|mf| \mid \Sigma_0)(\cdot,z) = \int_Z |m(\cdot,z)f(\tau(\cdot,z))w_2(z)| d\lambda(z) \in M \ \forall\ f \in L.$$

Consequently, $m \in \mathfrak{M}(L_\Omega, M_\Omega)$ if and only if $m \cdot w_2 \in \mathfrak{K}_\tau(L, M)$. Moreover, if $m \in \mathfrak{M}(L_\Omega, M_\Omega)$ then

$$\psi_M S_m \, \phi_L f(x) = \int_Z m(x,z)f(\tau(x,z))w_2(z)\, d\lambda(z),$$

which shows that $\psi_M S_m \, \phi_L \in \mathcal{L}_{\tau k}(L, M)$ with kernel $k(x,z)$ given by $m(x,z)w_2(z)$.

Chapter 3

Operators defined by random measures

In this chapter we look at operators generated by a class of measures known as random measure. We start by introducing random measures as studied by Sourour [28, 27] and Weis [29, 30]. We then look at operators that these random measures generate which are sometimes called pseudo-integral operators. In addition to Sourour and Weis who studied these operators in the setting of ideals of measurable functions over standard spaces, they were also studied in L_2 by Arveson [5] and in L_p $(0 < p < 1)$ by Kalton in [14, 15, 13].

Sourour in [27] showed that the lattice properties of operators generated by random measures are closely related to the properties of the generating (signed) measures in standard measure spaces, where a Borel space X is called standard if X is Borel isomorphic to a Borel subset of a separable complete metric space (see [20] or [5]). He assumed that the total variation of a random measure that generates an order bounded operator is again a random measure and showed that it generates the absolute value of that operator. We show that for an order bounded operator that can be expressed as a difference of two positive operators there exists a signed random measure which is a difference of two measures and that the total variation of that random measure generates the absolute value of that operator in more general measure spaces. In order to do this we make use of the Luxemburg-Schep Radon-Nikodym theorem [18] and the Kantorovich extension theorem [1].

We also develop the concept of random measure-representable operators analogous to MCE-representable operators that were introduced by Grobler and de Pagter in [8].

3.1 Random measures

Sourour in [27] studies operators generated by random measures. He called random measures kernels and the operators that they generate pseudo-integral operators. Weis in [29] called them operators represented by random measures. We will call them operators generated by random measures. Both Sourour and Weis use the same implicit assumption that $\nu(x,\cdot)$ and $|\nu|(x,\cdot)$ are Borel functions, see [28, Definition 1.1 (ii)] and [29, Remark 2.2 (i)]. Our results give a more general setting than those of Sourour and Weis as we do not make use of their assumption but actually show that the mapping $x \mapsto |\nu|(x, B)$ is a measurable function for each $B \in \Lambda$.

We consider measure spaces (Y, Λ, λ) and (X, Σ, μ) with (X, Σ, μ) complete σ-finite, i.e., subsets of μ-a.e. zero sets in Σ are also μ-a.e. zero sets.

Definition 3.1.1 A class of measures $\nu(x,\cdot)$ defined on Λ for each $x \in X$

such that $\nu(x, B)$ is Σ-measurable as a function of x for each fixed $B \in \Lambda$, is called a *random measure* on Y.

Definition 3.1.2 A random measure $\nu(x,\cdot)$ is called *uniformly σ-finite on Y* if $Y = \bigcup Y_n$ with $Y_n \in \Lambda$ and $\nu(x, Y_n) \leq k_n < \infty$ for all $x \in X$, where k_n is a sequence of positive real constants.

For a uniformly σ-finite random measure $\nu(x,\cdot)$ on Y, there exists (see [6] Theorem 2.6.2) a unique measure ν on $\Sigma \otimes \Lambda$ such that

$$\nu(A \otimes B) = \int_A \nu(x, B) d\mu(x) \text{ for all } A \in \Sigma \text{ and } B \in \Lambda. \tag{3.1.1}$$

This measure is given by

$$\nu(E) = \int_X \nu(x, E(x)) d\mu(x) \text{ for } E \in \Sigma \otimes \Lambda.$$

This measure is σ-finite on $\Sigma \otimes \Lambda$ and is a probability measure if μ and $\nu(x,\cdot)$ are probability measures.

We will make use of the following weaker condition.

Definition 3.1.3 A random measure $\nu(x,\cdot)$ is said to be *σ-finite* on Y if $Y = \bigcup Y_n$ and $\nu(x, Y_n) < \infty$ $\mu-$a.e. on X.

We can use σ-finite random measure $\nu(x,\cdot)$ to construct a unique σ-finite measure ν as in (3.1.1).

We will use the notation $\mathcal{F} = \Sigma \otimes \Lambda$.

Lemma 3.1.4 *Let $\nu(x, \cdot)$ be a σ-finite random measure. Then the function $x \mapsto \nu(x, C(x))$ belongs to $M^+(X, \Sigma, \mu)$ for every $C \in \mathcal{F}$.*

Proof Let X_0 be such that $\mu(X - X_0) = 0$ and $\nu(x, Y_n) < \infty$ for every $x \in X_0$ and $n \in \mathbb{N}$. Put $C = A \times B$ with $A \in \Sigma$ and $B \in \Lambda$. Then $\nu(x, C(x)) = \mathbf{1}_A(x)\nu(x, B)$, which, by definition, is in $M^+(X, \Sigma, \mu)$. Let $Z_n = X \times Y_n$, $C_n = Z_n \cap C$ for every $n \in \mathbb{N}$ and $C \subset X \times Y$. Put

$$\mathcal{D}_n = \{C \in \mathcal{F} | \nu(x, C_n(x)) \in M^+(X, \Sigma, \mu)\}.$$

Then $\mathcal{D}_n$ contains the measurable rectangles $A \times B \in \mathcal{F}$. Also, if B and C are in $\mathcal{D}_n$ with $B \subset C$, we have, from the fact that $\nu(x, \cdot)$ is finite on subsets of Y_n, for every $x \in X_0$, that

$$\begin{aligned} \nu(x, (Z_n \cap (C - B))(x)) &= \nu(x, (C_n(x) - B_n(x))) \\ &= \nu(x, C_n(x)) - \nu(x, B_n(x)) \text{ a.e.} \end{aligned}$$

Hence $\nu(x, (Z_n \cap (C - B))(x))$ is equal to a measurable function except possibly on a subset of a μ-null set. By the completeness of μ, $\nu(x, (Z_n \cap (C - B))(x))$ is in $M^+(X, \Sigma, \mu)$. Thus $C - B \in \mathcal{D}_n$. If $C_k \uparrow C$ with $C_k \in \mathcal{D}_n$ then $\nu(x, C_{k,n}(x)) \uparrow_k \nu(x, C_n(x))$, and so $C \in \mathcal{D}_n$. Thus, by the Dynkin principle, $\mathcal{D}_n$ contains the σ-algebra $\mathcal{F}$. From the fact that $\nu(x, C_n(x)) \uparrow \nu(x, C(x))$, we then have that $\nu(x, C(x)) \in M^+(X, \Sigma, \mu)$. □

Theorem 3.1.5 *Let $\nu(x, \cdot)$ be a σ-finite random measure on (Y, Λ) and let μ be a σ-finite measure on X. For every $C \in \mathcal{F}$ define*

$$\nu(C) = \int_X \nu(x, C(x)) d\mu.$$

Then ν is a σ-finite measure on $(X \times Y, \mathcal{F})$, moreover, it is a unique σ-finite measure that satisfies

$$\nu(A \times B) = \int_A \nu(x, B) d\mu$$

for all $A \in \Sigma$ and $B \in \Lambda$.

Proof The proof that ν is a measure on $\mathcal{F}$ is exactly as for the case of a uniformly σ-finite measure (see [6]). We have that $\nu(A \times B) = \int_A \nu(x, B) d\mu$ for all $A \in \Sigma$ and $B \in \Lambda$. We now show that it is a σ-finite measure. Let $X_0 = \{x \in X : \nu(x, Y_n) < \infty \text{ for all } n\}$. We therefore have, by assumption, that $\mu(X - X_0) = 0$. Let $X_n \uparrow X$ and $Y_n \uparrow Y$ be such that $\mu(X_n) < \infty$ and $\mu(Y_n) < \infty$ μ-a.e. on X. Let $X_{n,k} = \{x \in X_n \cap X_0 : \nu(x, Y_n) \leq k\}$ for $n, k = 1 \cdots$. Then for each n we have that $X_{n,k} \uparrow_k X_n \cap X_0$, and so,

$$\begin{aligned} \bigcup_{n,k} (X_{n,k} \times Y_n) \cup (X_0^c \times Y) &= (X_0 \times Y) \cup (X_0^c \times Y) \\ &= X \times Y. \end{aligned}$$

Furthermore,

$$\begin{aligned} \nu(X_{n,k} \times Y_n) &= \int_{X_{n,k}} \nu(x, Y_n) d\mu \\ &\leq k\mu(X_n) \\ &< \infty \end{aligned}$$

and $\nu(X_0^c \times Y) = \int_{X_0^c} \nu(x, Y) d\mu$. The uniqueness of ν follows from the fact that it is a σ-finite measure. □

Corollary 3.1.6 *For every $p \in M^+(X \times Y, \mathcal{F})$ the integral*

$$\int_Y p(x,y)\nu(x,dy)$$

exists and is in $M^+(X,\Sigma,\mu)$. Moreover,

$$\int_{X\times Y} p(x,y)d\nu(x,y) = \int_X \left(\int_Y p(x,y)\nu(x,dy)\right) d\mu(x)$$

For a proof the reader may consult the relevant part of the proof of Fubini's theorem in [6].

Definition 3.1.7 The set $B \subset Y$ is a *null set* with respect to the random measure $\nu(x,\cdot)$ if $\nu(x,B) = 0$ μ-a.e. on X. This is equivalent to saying that $\int_A \nu(x,B)d\mu(x) = 0$ for all $A \in \Sigma$.

A subset of a null set is also a null set. This follows from the fact that if $A \subset B$ then $\nu(x,A) \leq \nu(x,B)$. The union of a countable number of null sets is again a null set.

3.2 Operators generated by random measures

We now take a closer look at operators that are generated by random measures.

Definition 3.2.1 Let $L \subset L^0(Y,\Lambda,\lambda)$ and $M \subset L^0(X,\Sigma,\mu)$ be order dense ideals of measurable functions. We say the positive operator $T \in \mathcal{L}_b(L,M)$ is *generated by the random measure* $\nu(x,\cdot)$ if

$$Tf(x) = \int_Y f(y)\nu(x,dy) \quad \text{for almost every} \quad x \in X.$$

The space of the differences $T_1 - T_2$, where T_1 and T_2 are positive operators from L to M with T_1 generated by the random measure $\nu_1(x, \cdot)$ and T_2 generated by the random measure $\nu_2(x, \cdot)$, will be denoted by $\mathcal{L}_{rm}(L, M)$. It is obvious that this is a subspace of order bounded linear operators from L to M, i.e., $\mathcal{L}_{rm}(L, M) \subset \mathcal{L}_b(L, M)$.

We give examples of operators generated by random measures (taken from [31]).

Example 3.2.2 (1) Let (Y, Λ, λ) and (X, Σ, μ) be σ-finite measure spaces and $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ be ideals with carriers Y and X, respectively. Let $T : L \to M$ be a positive, or, more generally, a regular operator. If T is a kernel operator defined by $Tf(x) = \int_Y f(y)t(x, y)d\lambda(y)$ for all $f \in Y$ with kernel $t(x, y)$ then if we put $\nu_T(x, \cdot) = t\lambda(\cdot)$, i.e., $\nu_T(x, B) = \int_B t(x, y)d\lambda(y)$ for all $B \in \Lambda$. Then T is an operator generated by the random measure $\nu_T(x, \cdot)$

(2) Let K_1 and K_2 be locally compact Hausdorff spaces and $T : C(K_2) \to C(K_1)$ be a lattice homomorphism of the form $Tf(x) = g(x)f(\sigma(x))$, with $\sigma : K_1 \to K_2$ a continuous map and $g \in C(K_1)$, see [21][Theorem 3.2.10]. Then T is an operator generated by the random measure $\nu_T(x, \cdot) = g(x)\delta_{\sigma(x)}$, where, for $B \in K_2$, we have

$$\nu_T(x, B) = \begin{cases} g(x) & \text{if } x \in \sigma^{-1}(B) \\ 0 & \text{if } x \notin \sigma^{-1}(B), \end{cases}$$

i.e., $\nu_T(x, B)$ is a point measure on K_2.

(3) Let η be a measure on a locally compact group G and T be the convolution by the measure η given by $Tf(x) = \int_G f(x-y)d\eta(y)$. Then T is an operator generated by the random measure $\nu_T(x, A) = \eta(A-x)$ for A a Borel subset of G.

(4) If T_t is a semi group generated by the transition probabilities P_t of a Markov process, with $T_t f(x) = \int_Y f(y)dP_t(x, dy)$, then T_t is an operator generated by the random measure $\nu_{T_t}(x, \cdot) = P_t(x, \cdot)$.

We give an example of an operator that is not generated by random measures (see [27])

Example 3.2.3 Let Γ be the unit circle with normalized Lebesgue measure, $L^2 = L^2(\Gamma)$ and H^2 the usual Hardy spaces. The projection P of L^2 onto $(H^2)^\perp$ is not a operator generated by random measure. Suppose, to the contrary, that $(Pf)(w) = \int f(z)\mu(w, dz)$ almost everywhere. Let $e_n(z) = z^n$, and remove a set Γ_0 of measure zero such that

$$\int e_n(z)\mu(w, dz) = \begin{cases} e_n(w) & \text{if } n \leq 0 \\ 0 & \text{if } n > 0, \end{cases}$$

for every $w \in \Gamma - \Gamma_0$. Choose one $w \notin \Gamma_0$ and let $\sigma(dz) = \mu(w, dz)$. Therefore the Fourier transform $\hat{\sigma}$ of σ is given by

$$\hat{\sigma} = \begin{cases} w^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

The theorem of F. and M. Riesz in [9, Page 47] implies that $\hat{\sigma}$ is absolutely continuous and the Riemann-Lebesgue lemma (see [13, Page 13]) $\hat{\sigma}(n) \to 0$, i.e. $1 = |w^n| \to 0$, which is a contradiction.

Proposition 3.2.4 *Let $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ be order dense ideals of measurable functions, let λ be a σ-finite measure on Λ and let $T : L \to M$ be a linear operator that is generated by a random measure $\nu(x, \cdot)$. T is well defined if and only if $\lambda(B) = 0$ implies that $\nu(x, B) = 0$ ν-a.e. on X.*

Proof Let T be well defined and let (Y_n) be a sequence in Λ with $Y_n \uparrow Y$ and $\mathbf{1}_{Y_n} \in L$. Let B be a λ-null set. Then $B_n = B \cap Y_n$ is also a $\lambda-$null set. T is well defined and so $0 = T\mathbf{1}_{B_n}(x) = \nu(x, B_n)$ since $\mathbf{1}_{B_n} = 0$ for almost every x. Thus B_n is a $\nu(x, \cdot)$-null set for every n. Now $B_n \uparrow B$ and so B is also a $\nu(x, \cdot)$-null set.

On the other hand, suppose that for $B \in \Lambda$, $\lambda(B) = 0$ implies that $\nu(x, B) = 0$ almost everywhere on X. It then follows that if f is any positive Λ-measurable function then $\int_B f(y)\nu(x, dy) = 0$, μ-a.e. on X. Let $f, g \in L$ be positive functions with $f(y) = g(y)$ λ-a.e. on Y. Then for the set $B = \{y \in Y : f(y) \neq g(y)\}$ we have that

$$Tf(x) = \int_{Y-B} f(y)\nu(x, dy) + \int_B f(y)\nu(x, dy)$$

and

$$Tg(x) = \int_{Y-B} g(y)\nu(x, dy) + \int_B g(y)\nu(x, dy).$$

Therefore $Tf = Tg$, and so T is well defined. □

Lemma 3.2.5 *If two random measures $\nu_1(x, \cdot)$ and $\nu_2(x, \cdot)$ generate the same positive operator T, then $\nu_1(x, \cdot) = \nu_2(x, \cdot)$ for μ almost every $x \in X$.*

Proof Let Y_n be a sequence in Λ with $Y_n \uparrow Y$ and $\mathbf{1}_{Y_n} \in L$. Let $B \in \Lambda$ and put $B_n = B \cap Y_n$. We have that $B_n \uparrow B$ and so $\nu_1(x, B_n) \uparrow \nu_1(x, B)$ and $\nu_2(x, B_n) \uparrow \nu_2(x, B)$ for every $x \in X$. But $\nu_1(x, B_n) = T\mathbf{1}_{B_n}(x) = \nu_2(x, B_n)$ for almost every x. Hence $\nu_1(x, B) = \nu_2(x, B)$ for almost every x. □

Let $T : L \to M$ be a positive operator generated by the random measure $\nu(x, \cdot)$. If $Y_0 \subset Y$ is such that $\mathbf{1}_{Y_0} \in L$ and if Λ_0 is the σ-algebra induced by Λ on Y_0, then the random measure defines a vector-valued measure $\overline{v}$ on (Y_0, Λ_0) by

$$\overline{v}(B) = \nu(x, B) \in M \quad \text{for all } B \in \Lambda_0.$$

In particular, if $L^\infty(Y, \Lambda) \subset L$ then the random measure defines an M-valued measure in (Y, Λ).

Let T_1 be generated by the random measure $\nu_1(x, \cdot)$ and T_2 be generated by the random measure $\nu_2(x, \cdot)$. Then the operator $T_1 + T_2$ is generated by the random measure $\tau := \nu_1(x, \cdot) + \nu_2(x, \cdot)$. We have that $T_1 + T_2$ maps L into M, and since L is an order dense ideal, there is a sequence $Y_n \uparrow Y$ such that $\mathbf{1}_{Y_n} \in L$ and so $\tau(x, Y_n)$ is finite μ-a.e. on X. Let X_0 be a subset of X such that $\tau(X - X_0) = 0$ and $\tau(x, Y_n) < \infty$ for all n and all $x \in X_0$. Let Λ_0 be the ideal in Λ consisting of all $B \in \Lambda$ with $B \subset Y_n$ for some n. For $B \in \Lambda_0$, define

$$\nu(x, B) = \begin{cases} \nu_1(x, B) - \nu_2(x, B) & \forall\, x \in X_0 \\ 0 & \forall\, x \in X - X_0. \end{cases}$$

Then $\nu(x,\cdot)$ is a signed random measure defined on Λ_0 and is σ-additive on Λ_0. Now define

$$\int_Y f(y)\nu(x,dy) = \begin{cases} \int_Y f(y)\nu_1(x,dy) - \int_Y f(y)\nu_2(x,dy) & \forall\ x \in X_0 \\ 0 & \forall\ x \in X - X_0. \end{cases}$$

for all $f \in L$. Then $Tf(x) = \int_Y f(y)\nu(x,dy)$ and we say that T is generated by the signed measure $\nu(x,\cdot)$.

Proposition 3.2.6 *If $L \subset L^0(Y,\Lambda,\lambda)$ and $M \subset L^0(X,\Sigma,\mu)$ are ideals of measurable functions and $T \in \mathcal{L}_{rm}(L,M)$. Then*

(i) The operator T is order continuous.

(ii) If $0 \leq S \leq T \in \mathcal{L}_{rm}(L,M)$ then $S \in \mathcal{L}_{rm}(L,M)$.

(iii) $\mathcal{L}_{rm}(L,M)$ is a Riesz subspace of $\mathcal{L}_b(L,M)$.

(iv) $\mathcal{L}_{rm}(L,M)$ is an ideal in $\mathcal{L}_b(L,M)$.

Proof (i) Let $T \in \mathcal{L}_{rm}(L,M)$ with $T = T_1 - T_2$. In order to show that T is order continuous we have to show that T_1 and T_2 are order continuous. We may therefore assume that T is positive and is generated by the random measure $\nu(x,\cdot)$. Since L is super Dedekind complete it is sufficient to show that T is σ-order continuous. Let $\{f_n\}$ be a sequence in L such that $f_n \downarrow 0$ $\nu-$a.e. There exists a set Y_0 such that $\lambda(Y - Y_0) = 0$ and $f_n(y) \downarrow 0$ for all $y \in Y_0$. But then $Y - Y_0$ is, by Proposition 3.2.4, a $\nu(x,\cdot)$-null set, and so, for almost every x, $f_n(y) \downarrow 0$ $\nu(x,\cdot)$-a.e. on Y and $\int_Y f_1(y)\nu(x,dy) < \infty$ μ-a.e. By the Lebesgue theorem $\int_Y f_n(y)\nu(x,dy) \downarrow 0$ for almost every x, i.e., $Tf_n(x) \downarrow 0$ for almost every x. Thus $Tf_n \downarrow 0$ and so T is order continuous.

(ii) Let T be generated by the random measure $\nu(x,\cdot)$ and let ν be the σ-finite measure on $\mathcal{F}$ generated by $\nu(x,\cdot)$ and μ as constructed in Theorem 3.1.5. Consider the map $f(y) \to \overline{f}(x,y) = \mathbf{1}_X(x)f(y)$, $f \in L$. This map is well defined, for, if $f(y) = 0$ λ-a.e., then, if $C = \{(x,y) : \mathbf{1}_X(x)f(y) \neq 0\}$ we have that $C = X \times N$, with $N = \{y : f(y) \neq 0\}$. Hence $C(x) = N$ for all x. Since $\lambda(N) = 0$, we have, by assumption, that $\nu(x,N) = 0$ μ-a.e. on X and so

$$\nu(C) = \int_X \nu(x,N)d\mu = 0.$$

Taking $C = \{(x,y) : \mathbf{1}_X(x)f(y) = \infty\}$, a similar argument shows that this map maps L into $L^0(X \times Y, \Sigma \otimes \Lambda, \nu)$. It is also clear that this map is a Riesz homomorphism. Let $\overline{L}$ be the image of L in $L^0(X \times Y, \Sigma \otimes \Lambda, \nu)$ under this Riesz homomorphism. Put $I(\overline{L})$ to be the ideal generated in $L^0(X \times Y, \Sigma \otimes \Lambda, \nu)$ by $\overline{L}$. We extend the operator T to $\overline{T}$ on $I(\overline{L})$ by

$$\overline{T}h(x) = \int_Y h(x,y)\nu(x,dy) \text{ for } h \in I(\overline{L}).$$

Now $\overline{T}$ is well defined, since if $h(x,y) = 0$ ν-a.e. and if $C = \{(x,y) : h(x,y) \neq 0\}$, it follows from $0 = \nu(C) = \int_Y \nu(x,C(x))d\mu$, that $\nu(x,C(x)) = 0$ μ-a.e. on X. Hence, for μ-almost every x, the set $C(x)$ is a $\nu(x,\cdot)$-null set. It then follows that $h(x,y) = 0$

$\nu(x,\cdot)-$a.e. on Y for μ-almost every x. This then gives that $\int_Y h(x,y)\nu(x,dy)=0$, i.e., $\overline{T}h(x)=0$ for almost every x in X.

We can use a similar argument to show that for $g,h\in I(\overline{L})$ with $g(x,y)\leq h(x,y)$ $\nu-$a.e. we have

$$\int_Y g(x,y)\nu(x,dy)\leq \int_Y h(x,y)\nu(x,dy)$$

holds $\mu-$a.e. on X. In particular, if $h\in I(\overline{L})$, then $|h(x,y)|\leq \mathbf{1}_X f(y)$ for some $f\in L$. Therefore

$$\begin{aligned} 0 &\leq |\overline{T}h(x)| \\ &= \left|\int_Y h(x,y)\nu(x,dy)\right| \\ &\leq \int_Y |h(x,y)|\nu(x,dy) \\ &\leq \int_Y f(y)\nu(x,dy)\in M, \end{aligned}$$

Which shows that $\overline{T}: I(\overline{L})\to M$

Now, if $h_n(x,y)\downarrow 0$ ν-a.e., we have that

$$\int_X\int_Y h_n(x,y)\nu(x,dy)d\mu=\int_{X\times Y} h_n(x,y)d\nu\downarrow 0.$$

This implies that $\int_Y h_n(x,y)\nu(x,dy)d\mu\downarrow 0$ for μ-almost every x, which shows that $\overline{T}$ is order continuous.

Let S be an operator such that $0\leq S\leq T$ and for $\overline{f}(x,y)\in\overline{L}$ define the operator $\overline{S}\,\overline{f}(x)=Sf(x)\in M$. Let $\overline{f}(x,y)=0$ ν-a.e., then $|\overline{S}\,\overline{f}(x)|=|Sf(x)|\leq S|f|(x)\leq T|f|(x)=\overline{T}|\overline{f}|(x)=0$, thus $\overline{S}$ is well defined. We show that $0\leq\overline{S}\leq\overline{T}$. Let $\overline{f}(x,y)\geq 0$ ν-a.e.

and let $C = \{(x,y) : \overline{f}(x,y) < 0\}$. We again have that $C = X \times N$ with $C(x) = \{y : f(y) < 0\} = N$ and $\nu(x,N) = 0$ for almost every $x \in X$. We thus have that $T(\mathbf{1}_N|f|)(x) = \int_N |f|(y)\nu(x,dy) = 0$ for almost every x. From $0 \leq S \leq T$ we get that $S(\mathbf{1}_N|f|)(x) = 0$ for almost every x. Therefore $S(\mathbf{1}_N f)(x) = 0$ for almost every x. We have that $\mathbf{1}_{N^c}f(x)$ is positive for all y and so it is a positive function in L and $0 \leq S(\mathbf{1}_{N^c}f)(x) \leq T(\mathbf{1}_{N^c}f)(x)$ for almost every x. Putting these together we see that $0 \leq \overline{S}\,\overline{f}(x) \leq \overline{T}\,\overline{f}(x)$ μ-a.e. on X.

We use the theorem of Kantorovich to extend $\overline{S}$ as a mapping on $\overline{L}$ to a mapping $\overline{S} : I(\overline{L}) \to M$ such that $0 \leq \overline{S} \leq \overline{T}$. Now $\overline{T}$ is interval preserving, i.e., it has the Maharam property, for, if $0 \leq g \leq \overline{T}h$ put

$$\sigma(x) = \begin{cases} g/\overline{T}h(x) & \text{if } (\overline{T}h) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have $0 \leq \sigma(x) \leq 1$ and it is Σ-measurable, i.e., $\sigma(x)h(x,y) \in I(\overline{L})$, and

$$\begin{aligned} \overline{T}(\sigma h)(x) &= \int_Y \sigma(x)h(x,y)\nu(x,dy) \\ &= \sigma(x)\overline{T}h \\ &= g(x). \end{aligned}$$

By the Luxemburg-Schep Radon-Nikodym theorem we have that $\overline{S} = \overline{T}\pi$ for some orthomorphism $0 \leq \pi \leq I_{I(\overline{L})}$. But it is known that $\pi h = p(x,y)h(x,y)$ for some ν-measurable function $p(x,y)$ satisfying $0 \le p(x,y) \le 1$ ν-a.e. So,

$$\begin{aligned} \overline{S}(h(x)) &= \overline{T}(ph)(x) \\ &= \int_Y h(x,y)p(x,y)\nu(x,dy) \text{ for all } h \in I(\overline{L}). \end{aligned}$$

Therefore,

$$(Sf)(x) = \overline{S}\,\overline{f}(x) = \int_Y f(y)p(x,y)\nu(x,dy)$$

Now define $\nu_S(x,B) = \int_B p(x,y)\nu(x,dy)$. This is a random measure which generates $\overline{S}$ and so it also generates S.

(iii) Let $T_1, T_2 \in \mathcal{L}_{rm}(L,M)$. For $T = T_1 - T_2$ we have that $|T| \le T_1 + T_2$. But $T_1+T_2 \in \mathcal{L}_{rm}(L,M)$. Thus by (ii) above $|T| \in \mathcal{L}_{rm}(L,M)$. Hence $\mathcal{L}_{rm}(L,M)$ is a Riesz subspace of $\mathcal{L}_b(L,M)$.

(iv) Let $0 \le |S| \le T \in \mathcal{L}_{rm}(L,M) \subset \mathcal{L}_b(L,M)$. Then by (ii), $|S| \in \mathcal{L}_{rm}(L,M)$. Again from (ii) and the fact that $S^+ \le |S| \in \mathcal{L}_{rm}(L,M)$ and $S^- \le |S| \in \mathcal{L}_{rm}(L,M)$, we get that $S^+, S^- \in \mathcal{L}_{rm}(L,M)$. Therefore $S = S^+ - S^- \in \mathcal{L}_{rm}(L,M)$. □

From the proof of the proceeding result we obtain the following:

Corollary 3.2.7 *Let $0 \le S \le T$ and let T be generated by the random measure $\nu_T(x,\cdot)$. Then there exists a measurable function $0 \le p(x,y) \le 1$ such that S is generated by the random measure $\nu_S(x,\cdot) = p(x,y)\nu_T(x,\cdot)$, i.e.,*

$$(Sf)(x) = \int_Y f(y)p(x,y)\nu_T(x,dy) \text{ for almost every } x \in X.$$

In particular it follows that there exists a generating measure $\nu_S(x,\cdot)$ for S satisfying $\nu_S(x,B) \le \nu_T(x,B)$ for all $B \in \Lambda$ and all $x \in X$.

The following result characterizes the random measures which generate the operators T^+, T^- and $|T|$ in terms of the random measure which generates T. We first look at the following case: If $L \subset L^0(Y,\Lambda,\lambda)$ and $M \subset L^0(X,\Sigma,\mu)$ are ideals of measurable functions and if T is generated by $\nu(x,\cdot)$, then we show that for every $x \in X$ the total variation $|\nu|(x,\cdot)$ is also a random measure which generates the operator $|T|$. This result was proved by Sourour in the special case where the measure spaces in question are standard measure spaces and in which it is assumed, a priori, that $|\nu|(x,B)$ is a measurable function of x for every $B \in \Lambda$. (see [28, Remark 1.3 (ii)]). Without this assumption one would almost be tempted to use the following reasoning:

Let $B \subset Y$ be such that $\mathbf{1}_B \in L$. For a signed measure ν we have that

$$|\nu|(B) = \sup\left\{\sum_{i=1}^{n} |\nu(B_i)| : B_i \bigcap B_j = \emptyset, \bigcup_{1=1}^{n} B_i = B\right\}.$$

Thus for a fixed x we have

$$\begin{aligned}|\nu|(x,B) &= \sup\left\{\sum_{i=1}^{n} |\nu(x,B_i)| : B_i \bigcap B_j = \emptyset, \bigcup_{1=1}^{n} B_i = B\right\}\\ &= \sup\left\{\sum_{i=1}^{n} |(T\mathbf{1}_{B_i})(x)| : \mathbf{1}_{B_i} \wedge \mathbf{1}_{B_j} = 0, \sum_{i=1}^{n} \mathbf{1}_{B_i} = \mathbf{1}_B\right\}\end{aligned}$$

for μ–a.e. $x \in X$.

$$\begin{aligned}|T|\mathbf{1}_B &= \sup\left\{|Tg_i| : g_i \wedge g_j = 0, \sum_{i=1}^{n} g_i = \mathbf{1}_{B_i}\right\}\\ &= \sup\left\{\sum_{i=1}^{n} |T(\mathbf{1}_{B_i})| : \mathbf{1}_{B_i} \wedge \mathbf{1}_{B_j} = 0, \sum_{i=1}^{n} \mathbf{1}_{B_i} = \mathbf{1}_B\right\}\end{aligned}$$

The problem with this is that the first supremum is a pointwise supremum which holds λ−a.e. and the second supremum is a supremum taken in Riesz space $L^0(X,\Sigma,\mu)$ and, in general, for uncountable families these two suprema need not be equal as the following well known example shows: For $\alpha \in [0,1]$ let

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x = \alpha \\ 0 & \text{if } x \neq \alpha. \end{cases}$$

We have that $f_\alpha = 0$ λ−a.e. on $[0,1]$ for a Lebesgue measure λ and so $\sup_\alpha f_\alpha = 0$ in $L^0([0,1],\lambda)$. On the other hand, the pointwise supremum of $\{f_\alpha\}$ is $\mathbf{1}$.

We present a different approach to show, among others, that if $\nu(x,\cdot)$ generates an operator T then $|\nu|(x,\cdot)$ generates the operator $|T|$.

Theorem 3.2.8 *Let $L \subset L^0(Y,\Lambda,\lambda)$ and $M \subset L^0(X,\Sigma,\mu)$ be ideals of measurable functions and let $T \in \mathcal{L}_{rm}(L,M)$. Then there exists a random signed measure $\nu_T(x,\cdot)$ that generates T, such that $|T|$, T^+ and T^- are generated by the random measures $|\nu_T|(x,\cdot)$, $\nu_T^+(x,\cdot)$ and $\nu_T^-(x,\cdot)$ respectively. Moreover, $S \in \mathcal{L}_{rm}(L,M)$ has a generating random signed measure $\nu_S(x,\cdot)$ such that $(\nu_S \vee \nu_T)(x,\cdot)$ and $(\nu_S \wedge \nu_T)(x,\cdot)$ are random signed measures that generate $S \vee T$ and $S \wedge T$ respectively.*

Proof Assume that $T = T_1 - T_2 \in \mathcal{L}_{rm}(L,M)$ with $\nu_1(x,\cdot)$ and $\nu_2(x,\cdot)$ random measures which generate T_1 and T_2 respectively. Let $\nu(x,\cdot) = \nu_1(x,\cdot) + \nu_2(x,\cdot)$. This is a random measure which generates $T_1 + T_2$. We have that $T^+ \leq T_1 + T_2$ and $T^- \leq T_1 + T_2$, therefore, from

Corollary 3.2.7 there exist functions

$$0 \leq p_{T^+}(x,y) \leq 1 \text{ such that } p_{T^+}(x,y)\nu(x,\cdot) \text{ generates } T^+$$

and

$$0 \leq p_{T^-}(x,y) \leq 1 \text{ such that } p_{T^-}(x,y)\nu(x,\cdot) \text{ generates } T^-.$$

Define $p_T(x,y) = p_{T^+}(x,y) - p_{T^-}(x,y)$. Then $\nu_T(x,\cdot) = p_T(x,y)\nu(x,\cdot)$ generates T, for

$$\begin{aligned}
(Tf)(x) &= (T^+f)(x) - (T^-f)(x) \\
&= \int_Y f(y)p_{T^+}(x,y)\nu(x,dy) - \int_Y f(y)p_{T^-}(x,y)\nu(x,dy) \\
&= \int_Y f(y)p_T(x,y)\nu(x,dy)
\end{aligned}$$

holds μ-a.e. on X. Let $C = \{(x,y) : p_T(x,y) < 0\}$. Then C is $\Sigma \times \Lambda$ measurable and so is $C(x)$ for almost every $x \in X$. For all $B \in \Lambda$ we have that

$$\nu_T(x, B \cap C(x)) = \int_{B \cap C(x)} p_T(x,y)\nu(x,dy) \leq 0$$

and

$$\nu_T(x, B \cap C(x)^c) = \int_{B \cap (Y - C(x))} p_T(x,y)\nu(x,dy) \geq 0.$$

Therefore

$$\nu_T^-(x,B) = \int_{B \cap C(x)} p_T(x,y)\nu(x,dy) = \int_B p_T^-(x,y)\nu(x,dy)$$

and

$$\nu_T^+(x,B) = \int_{B \cap (Y - C(x))} p_T(x,y)\nu(x,dy) = \int_B p_T^+(x,y)\nu(x,dy).$$

This shows that ν_T^- and ν_T^+ are random measures. We now show that they generate T^- and T^+ respectively. Suppose $p_T^+(x,y)\nu(x,\cdot)$ generates S_1 and $p_T^-(x,y)\nu(x,\cdot)$ generates S_2. We then have, since $p_T = p_{T^+}(x,y) - p_{T^-}(x,y)$ and consequently $p_T^+ \leq p_{T^+}$ and that $p_T^- \leq p_{T^-}$, that

$$\begin{aligned} T^+f(x) &= \int_Y f(y)p_{T^+}(x,y)\nu(x,dy) \\ &\geq \int_Y f(y)p_T^+(x,y)\nu(x,dy) \\ &= S_1 f(x). \end{aligned}$$

for all $f \in L^+$. Hence $T^+ \geq S_1$. Similarly, $T^- \geq S_2$. Since $T = S_1 - S_2$ we have that $T^+ \leq S_1$ and $T^- \leq S_2$. Therefore $T^+ = S_1$ and $T^- = S_2$. By putting

$$\begin{aligned} |\nu|(x,\cdot) &= \nu^+(x,\cdot) + \nu^-(x,\cdot) \\ &= p_T^+(x,y)\nu(x,\cdot) + p_T^-(x,y)\nu(x,\cdot) \\ &= |p_T(x,y)|\nu(x,\cdot) \end{aligned}$$

we see that $|\nu|(x,B)$ is measurable for all $B \in \Lambda$, therefore it is a random measure and, moreover, it generates $|T|$. $\square$

From the preceding proof we see that we have a useful "functional calculus" for the generating random measures: Let $\{T_i\}$ be a finite set of elements of $\mathcal{L}_{rm}(L,M)$ and let $\nu(x,\cdot)$ be a generating measure for $\sup_i |T_i|$. Then, for every T in the ideal generated by $\{T_i\}$ in $\mathcal{L}_{rm}(L,M)$ there exists a ν-measurable function $h_T(x,y)$ such that $h_T(x,y)\nu(x,\cdot)$ is a random signed measure which generates T and the map $T \mapsto h_T$ is a Riesz homomorphism.

We can apply this to derive the following result:

Proposition 3.2.9 *Let $0 \leq T \in \mathcal{L}_{rm}(L, M)$ be generated by the random measure $\nu_T(x, \cdot)$ and let $0 \leq S \in \mathcal{L}_{rm}(L, M)$. The following statements are equivalent:*

(i) $S \in \{T\}^{dd}$;

(ii) *there exists a $\Sigma \otimes \Lambda$–measureable function $0 \leq p : X \times Y \to \mathbb{R}$ such that for any generating measure $\nu_S(x, \cdot)$ for S, $\nu_S(x, B) = p\nu_T(x, B)$ for almost every $x \in X$ for all $B \in \Lambda$.*

(iii) *S has a generating measure $\nu_S(x, \cdot)$ satisfying $\nu_S(x, \cdot) \prec\prec \nu_T(x, \cdot)$ for μ-almost every $x \in X$;*

(iv) *S has a generating measure $\nu_S(x, \cdot)$ satisfying $\nu_S \prec\prec \nu_T$ on $\Sigma \otimes \Lambda$;*

Proof (i) $\Rightarrow$ (ii): Let $S \in \{T\}^{dd}$ and define $S_n := S \wedge nT$; then $S_n \uparrow S$ and $S_n \leq nT$, the latter operator generated by the random measure $n\nu_T$. It follows from Corollary 3.2.7 that there exists a $\Sigma \times \Lambda$-measureable function q_n such that $0 \leq q_n \leq 1$ such that S_n is generated by the random measure $q_n(x, y)n\nu_T(x, \cdot) := p_n(x, y)\nu_T(x, \cdot)$, with $0 \leq p_n(x, y) \leq n$ ν_T-a.e on $X \times Y$. Since $S_n \leq S_{n+1}$ we may assume that $p_n(x, y) \leq p_{n+1}(x, y)$ holds ν-almost everywhere on $X \times Y$. Define $p(x, y) := \sup_n p_n(x, y)$ in the space $M^+(X \times Y, \Sigma \otimes \Lambda, \nu_T)$. We then have for every $B \in \Lambda$ that

$$\begin{aligned}\int_A \nu_S(x,B)d\mu(x) &= \int_A S\mathbf{1}_B(x)d\mu(x)\\ &= \sup_n \int_A S_n\mathbf{1}_B d\mu(x)\\ &= \sup_n \int_A \int_B p_n(x,y)\nu_T(x,dy)d\mu(x)\\ &= \sup_n \int_{A\times B} p_n(x,y)d\nu_T\\ &= \int_{A\times B} f(y)p(x,y)d\nu_T\\ &= \int_A \int_B p(x,y)\nu_T(x,dy)d\mu(x).\end{aligned}$$

This holds for every $A \in \Sigma$ and hence, $\nu_S(x,B) = \int_B p(x,y)\nu_T(x,dy)$ for μ-almost every $x \in X$. This proves (ii).

(ii) $\Rightarrow$ (iii): If $\nu_T(x,B) = 0$ for some $B \in \Lambda$, then, $\int_B p(x,y)\nu_T(x,dy) = 0$. It follows that the measure $\nu_S(x,\cdot) := p\nu_T(x,\cdot)$, is a generating measure for S for which we have $\nu_S(x,\cdot) \prec\prec \nu_T(x,\cdot)$ for every $x \in X$ and so (iii) holds.

(iii) $\Rightarrow$ (iv): Let $N \in \Sigma$ be such that $\mu(N) = 0$ and $\nu_S(x,\cdot) \prec\prec \nu_T(x,\cdot)$ for all $x \in X - N$. Take $C \in \Sigma \otimes \Lambda$ such that $\nu_T(C) = 0$. Then there exists a set $M \in \Sigma$ such that $\mu(M) = 0$ and $\nu_T(x, C(x)) = 0$ for all $x \in X - M$. Hence, if $x \in X - (N \cup M)$, then $\nu_S(x, C(x)) = 0$, so $\nu_S(x, C(x)) = 0$ μ-almost everywhere on X. This implies that $\nu_S(C) = \int_X \nu_S(x, C(x))d\mu = 0$.

(iv) $\Rightarrow$ (ii): Applying the Radon-Nikodym theorem there exists some

$0 \leq p(x,y)$ such that $\nu_S = p\nu_T$. In particular, for every $A \in \Sigma$ we have

$$\begin{aligned} \int_A \nu_S(x,B) &= \nu_S(A \times B) \\ &= \int_{X\times Y} \mathbf{1}_{A\times B} p(x,y) d\nu_T \\ &= \int_A \int_B p(x,y)\nu_T(x,dy), \end{aligned}$$

from which it follows that $\nu_S(x,B) = \int_B p(x,y)\nu_T(x,dy)$ for μ-almost every $x \in X$ and we are done.

(ii) $\Rightarrow$ (i):

Let $p_n(x,y) := n \wedge p(x,y)$ and set $S_n f(x) := \int_Y f(y)p_n(x,y)\nu_T(x,dy)$. It then follows that $0 \leq S_n \leq nT$ and so each S_n is in the ideal generated by T in $\mathcal{L}_{rm}(L,M)$. But, since $p_n(x,y) \uparrow p(x,y)$ we have for every $0 \leq f \in L$ that

$$S_n f(x) = \int_Y f(y)p_n(x,y)\nu_T(x,dy) \uparrow \int_Y f(y)p(x,y)\nu_T(x,dy) = Sf(x)$$

holds μ-almost everywhere on X. Thus $S_n \uparrow S$ which proves that $S \in \{T\}^{dd}$. □

Note that the equivalence of (ii), (iii) and (iv) in the above theorem still holds if we replace $\nu_T(x,\cdot)$ by λ, although λ may not generate an operator from L into M. Therefore, taking $\nu_T(x,\cdot) = \lambda$, we have the following result in which condition (ii) translates into the assertion that S is a kernel operator.

Proposition 3.2.10 *Let $0 \leq S : L \to M$ be generated by the random measure $\nu_S(x,\cdot)$. Then the following are equivalent.*

(i) S *is a kernel operator.*

(ii) $\nu_S(x,\cdot) \prec\prec \lambda$ *for almost every* $x \in X$.

(iii) $\nu_S \prec\prec \mu \otimes \lambda$ *on* $\Sigma \otimes \Lambda$.

Proof (i) $\Rightarrow$ (ii): Let S be a kernel operator with kernel $0 \leq p(x,y)$. So $Sf(x) = \int_Y f(y)p(x,y)d\lambda(y)$ for all $f \in L$ showing that S has a generateng measure $\nu_S(x,\cdot) = p\lambda(\cdot)$. If $\lambda(B) = 0$ for some $B \in \Lambda$ then $\int_B p(x,y)d\lambda(y) = 0$ for almost every $x \in X$. It follows that $\nu_S(x,\cdot) \prec\prec \lambda$ for almost every $x \in X$.

(ii) $\Rightarrow$ (iii): Let $N \in \Sigma$ be such that $\mu(N) = 0$ and $\nu_S(x,\cdot) \prec\prec \lambda$ for all $x \in X - N$. Take $C \in \Sigma \otimes \Lambda$ such that $\mu \otimes \lambda(C) = 0$. Then, by Fubini, there exists a set $M \in \Sigma$ such that $\mu(M) = 0$ and $\lambda(C(x)) = 0$ for all $x \in X - M$ with $C(x)$ the section of C at x. Hence if $x \in X - (N \cup M)$, then $\lambda(C(x)) = 0$ and so $\nu_S(x, C(x)) = 0$ μ–a.e. on X. This implies that $\nu_S(C) = \int_X \nu_S(x, C(x))d\mu(x) = 0$ and therefore $\nu_S \prec\prec \mu \otimes \lambda$ in $\Sigma \otimes \Lambda$ for every $x \in X$.

(iii) $\Rightarrow$ (i): By the Radon–Nikodym theorem there exists a function $0 \leq p(x,y)$ such that $\nu_S = p(\mu \otimes \lambda)$. Thus, for $A \in \Sigma$ we have

$$\begin{aligned}
\int_A \nu_S(x,B)d\mu(x) &= \nu_S(A \times B) \\
&= \int_{X \times Y} \mathbf{1}_{A \times B} p(x,y) d(\mu \otimes \lambda) \\
&= \int_A \int_B p(x,y)d\lambda(y)d\mu(x),
\end{aligned}$$

from which it follows that $\nu_S(x,B) = \int_B p(x,y)d\lambda(y)$ for μ-almost every $x \in X$, i.e., $\nu_S(x,\cdot) = p\lambda(\cdot)$. Since S is generated by the random measure $\nu_S(x,\cdot)$ we have

$$\begin{aligned} Sf(x) &= \int_Y f(y)\nu_S(x,dy) \\ &= \int_Y f(y)p(x,y)d\lambda(y) \end{aligned}$$

for almost every $x \in X$ and for all $f \in L$. So S is a kernel operator with kernel $p(x,y)$. □

For the proof of the following result we will make use of the following order separable condition:

Lemma 3.2.11 *Let L and M be two Archimedean Riesz spaces with M order separable. Let (u_n) be a sequence in L^+ such that $\{u_n : n = 1,2,\cdots\}^{dd} = L$. Then the space $\mathcal{L}_n(L,M)$ is order separable.*

Proof Let (T_α) be an upward directed net in $\mathcal{L}_n(L,M)$ such that $0 \le T_\alpha \uparrow T \in \mathcal{L}_n(L,M)$. We have that $0 \le T_\alpha u_n \uparrow Tu_n$. Also, from the fact that M is order separable there exists a sequence $(\alpha_{n,k})$ such that $\sup_k T_{\alpha_{n,k}} u_n = Tu_n$. Put $S = \sup_{n,k} T_{\alpha_{n,k}}$ in $\mathcal{L}_n(L,M)$. We then have that $0 \le T_{\alpha_{n,k}} \le S \le T$ for all n,k. In particular, for any n we have $0 \le T_{\alpha_{n,k}} u_n \le Su_n \le Tu_n$ for all k and so $Su_n = Tu_n$. Now $0 \le T - S$ and so $T - S = 0$ on the ideal generated by $\{u_n : n = 1,2,\cdots\}$. Since $T - S$ is order continuous we get that $T - S = 0$ on L. Thus $T = \sup_{n,k} T_{\alpha_{n,k}}$. □

Theorem 3.2.12 *$\mathcal{L}_{rm}(L, M)$ is a band in $\mathcal{L}_b(L, M)$.*

Proof The space $\mathcal{L}_n(L, M)$ is order separable since the underlying measure spaces are assumed to be σ-finite. It is therefore sufficient to show that if (T_n) is a sequence in $\mathcal{L}_{rm}(L, M)$ such that $0 \leq T_n \uparrow T$ in $\mathcal{L}_n(L, M)$, then it follows that $T \in \mathcal{L}_{rm}(L, M)$. Put $T_0 = 0$ and let $S_n = T_n - T_{n-1}$ for all $n = 1, 2, \cdots$. We have that $0 \leq S_n \leq T_n$ and since $\mathcal{L}_{rm}(L, M)$ is an ideal in $\mathcal{L}_n(L, M)$, it then follows that $0 \leq S_n \in \mathcal{L}_{rm}(L, M)$ and so S_n is generated by some random measure $\tau_n(x, \cdot)$.

Let $\nu_n(x, \cdot) = \sum_{k=1}^{n} \tau_k(x, \cdot)$, then $\nu_n(x, \cdot)$ is a generating measure for T_n for all $n = 1, 2, \cdots$. By definition, $\nu_n(x, B) \uparrow_n$ for all $B \in \Lambda$ and all $x \in X$. Define, for all $B \in \Lambda$ and all $x \in X$

$$\nu(x, B) = \sup_n \nu_n(x, B).$$

The function $x \mapsto \nu(x, B)$ is $\Sigma-$measurable on X for every $B \in \Lambda$ and $\nu(x, \cdot)$ is a random measure on X.

Lastly we show that $\nu(x, \cdot)$ is a generating measure for T.

Let $B \in \lambda$ be such that $f = \mathbf{1}_B \in L$. Then

$$\begin{aligned} Tf(x) &= \sup_n T_n f(x) \\ &= \sup_n \nu_n(x, B) \\ &= \nu(x, B) \\ &= \int_Y f(y) \nu(x, dy) \ \mu - \text{a.e. on } X. \end{aligned}$$

This then implies that $Tf(x) = \int_Y f(y)\nu(x,dy)$ holds $\mu-$a.e. on X for the step function f given by

$$f = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{B_j} \tag{3.2.1}$$

where $0 \leq \alpha_j \in \mathbb{R}$ and $B_j \in \Lambda$ for $j = 1, 2, \cdots, k$. Let $0 \leq f \in L$ be arbitrary, then there exists a sequence (f_n) in L such that each f_n is of the form (3.2.1) and $0 \leq f_n \uparrow f$ holds $\lambda-$a.e. on Y. Since T is order continuous, we also have that $Tf_n \uparrow Tf\mu-$a.e. on X. Using the monotone convergence theorem we then get that

$$\int_Y f_n(y)\nu(x,dy) \uparrow \int_Y f(y)\nu(x,dy)\nu - \text{a.e. on } X.$$

Thus for each n we have that $Tf_n(x) = \int_Y f_n(y)\nu(x,dy)\mu-$a.e. on X. We therefore conclude that $Tf(x) = \int_Y f(y)\nu(x,dy)$ holds $\mu-$a.e. on X and the proof is complete. □

3.3 Random measure-representable operators

Here we develop a theory of random measure-representable operators analogous to the theory of MCE−representable operators. Our main theorem will be that every order continuous operator is random measure-representable. This gives us an extension of the theorem by Sourour [28] to a more general setting.

Consider measure spaces (Y,Λ,λ) and (X,Σ,μ) with (X,Σ,μ) complete σ-finite and let $L \subset L^0(Y,\Lambda,\lambda)$ and $M \subset L^0(X,\Sigma,\mu)$ be order dense ideals of measurable functions.

Definition 3.3.1 We say that a linear operator $T : L \to M$ is *random measure-representable* if there exist σ-finite measure spaces $(\Omega_1, \mathfrak{F}_1, \pi_1)$ and $(\Omega_2, \mathfrak{F}_2, \pi_2)$, order dense ideals $L_{\Omega_2} \subset L^0(\Omega_2, \mathfrak{F}_2, \pi_2)$ and $M_{\Omega_1} \subset L^0(\Omega_1, \mathfrak{F}_1, \pi_1)$ and order continuous Riesz homomorphisms $\phi_L : L \to L_{\Omega_2}$ and $\psi_M : M_{\Omega_1} \to M$ with ϕ_L surjective and ψ_M a Riesz isomorphism onto an ideal in M; such that

$$T = \psi_M \circ T_\nu \circ \phi_L,$$

with $T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})$, i.e., such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "T"] \arrow[d, "\phi_L"'] & M \\ L_{\Omega_2} \arrow[r, "T_\nu"'] & M_{\Omega_1} \arrow[u, "\psi_M"'] \end{tikzcd}$$

As in the case of MCE-representable operators the following also holds.

Proposition 3.3.2 *Let (X,Σ,μ) and (Y,Λ,ν) be σ-finite measure spaces and let $L \subseteq L^0(Y,\Lambda,\nu)$ and $M \subseteq L^0(X,\Sigma,\mu)$ be ideals with carriers Y and X respectively. Then the linear operator $T : L \to M$ is random measure-representable if and only if there exist σ-finite measure spaces $(\Omega_1, \mathfrak{F}_1, \pi_1)$ and $(\Omega_2, \mathfrak{F}_2, \pi_2)$, Riesz homomorphisms $\phi_L : L^0(Y,\Lambda,\nu) \to L^0(\Omega_2, \mathfrak{F}_2, \pi_2)$ and $\psi_M : L^0(\Omega_1, \mathfrak{F}_1, \pi_1) \to L^0(X,\Sigma,\mu)$ with $\phi_L(\mathbf{1})$ strictly positive and ψ_M $1-1$ and interval preserving and $T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})$ with $L_{\Omega_2} = \phi(L)$ and $M_{\Omega_1} =$*

$\psi_M^{-1}(M)$ *order dense ideals such that*

$$T = \psi_M \circ T_\nu \circ \phi_L,$$

i.e., such that the following diagram commutes

$$\begin{array}{ccc} L^0(Y,\Lambda,\nu) \supset L & \xrightarrow{T} & M \subset L^0(X,\Sigma,\mu) \\ \phi_L \Big\downarrow & & \Big\uparrow \psi_M \\ L^0(\Omega_2,\mathfrak{F}_2,\pi_2) \supset L_{\Omega_2} & \xrightarrow[T_\nu]{} & M_{\Omega_1} \subset L^0(\Omega_1,\mathfrak{F}_1,\pi_1) \end{array}.$$

Proof The proof of the corresponding result for MCE-representable operators (Proposition 2.2.4 and Proposition 2.2.5) involve only properties of the ideals and Riesz homomorphisms that are used there, which are similar to those in this theorem. We can, therefore, apply the same argument replacing S_m with T_ν. □

As in the case of MCE-representable operators, putting $\Omega = \Omega_1 \times \Omega_2$, $\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$, and $\pi = \pi_1 \otimes \pi_2$ We call $\Phi = ((\Omega, \mathfrak{F}, \pi), \phi_L, \psi_M)$ a representing triple for the operator T. We will denote the set of all linear operators $T : L \to M$ which are random measure-representable via the triple Φ by $\mathcal{L}_\Phi(L, M)$ and the operator $T \in \mathcal{L}_\Phi(L, M)$ will be called $\Phi-$representable.

The following are examples that show that operators generated by random measures and kernel operators are amongst the class of operators that are random measure-representable.

Example 3.3.3 (1) Let (X, Σ, μ) and (Y, Λ, λ) be σ-finite measure spaces and let $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ be ideals with carriers Y

and X respectively. Let $T \in \mathcal{L}_{rm}(L, M)$. If we take ϕ_L and ψ_M to be identity operators then T is Random measure–representable.

(2) Consider σ-finite measure spaces (X, Σ, μ) and (Y, Λ, λ) and let $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ be ideals with carriers Y and X respectively. Let $K : L \to M$ be the absolute kernel operator from L into M defined by

$$Kf(\omega) = \int_Y k(\omega, x)f(x)d\lambda(x) \text{ for all } f \in L,$$

with kernel k. We have that the $k(\omega, \cdot)\lambda(\cdot)$ is a random measure that generates K (see Example 3.2.2 (1) on Page 68). By (1) above K is random measure-representable.

Proposition 3.3.4 *$\mathcal{L}_\Phi(L, M)$ is a band in $\mathcal{L}_n(L, M)$.*

Proof Let the carrier of ϕ_L be denoted by C_{ϕ_L} and put $L_1 := L \cap C_{\phi_L}$. We have that C_{ϕ_L} is a band in $L^0(Y, \Lambda, \lambda)$ and so L_1 is a band in L. Let ϕ_1 be the restriction of ϕ_L to L_1, then ϕ_1 is a Riesz isomorphism from L_1 to L_{Ω_1}. By Proposition 1.4.3 (ii) we get that $\operatorname{ran}(\psi_M)$ is a band in $L^0(X, \Sigma, \mu)$. If we put $M_1 := M \cap \operatorname{ran}(\psi_M)$ then M_1 is a band in M. let the restriction of ψ_M to M_1 be denoted by ψ_1. Then ψ_1 is a Riesz isomorphism from M_{Ω_1} onto M_1. Using these Riesz isomorphisms the space $\mathcal{L}_n(L, M)$ can be identified with the band in $\mathcal{L}_n(L_1, M_1)$ consisting of all operators $T \in \mathcal{L}_n(L, M)$ for which the carrier of T, denoted by C_T, is in L_1 and $\operatorname{ran}(T)$ is contained in M_1. From Proposition 3.3.2

we get that $\mathcal{L}_\Phi(L,M) \subset \mathcal{L}_n(L_1,M_1)$. It will therefore suffice to show that $\mathcal{L}_\Phi(L,M)$ is a band in $\mathcal{L}_n(L_1,M_1)$. To that end we define

$$\tau : \mathcal{L}_n(L_{\Omega_1}, M_{\Omega_2}) \to \mathcal{L}_n(L_1, M_1)$$

by

$$\tau(S) = \psi_1 \circ S \circ \phi_1$$

for all $S \in \mathcal{L}_n(L_{\Omega_1}, M_{\Omega_2})$. Note that τ is bipositive and therefore a Riesz isomorphism from $\mathcal{L}_n(L_{\Omega_1}, M_{\Omega_2})$ onto $\mathcal{L}_n(L_1,M_1)$. We can now express the relation $T = \psi_M \circ T_\nu \circ \phi_L$ in Proposition 3.3.2 by $T = \tau(S_\nu)$ and so

$$\mathcal{L}_\Phi(L,M) = \tau(\mathcal{L}_{rm}(L_{\Omega_1}, M_{\Omega_2})).$$

By Theorem 3.2.12 $\mathcal{L}_{rm}(L_{\Omega_1}, M_{\Omega_2})$ is a band in $\mathcal{L}_n(L_{\Omega_1}, M_{\Omega_2})$. It therefore follows that $\mathcal{L}_\Phi(L,M)$ is a band in $\mathcal{L}_n(L_1,M_1)$, and hence a band in $\mathcal{L}_n(L,M)$. □

The main result of this section is that every order continuous operator is random measure-representable. As in the last section we consider order dense ideals L and M of almost everywhere finite measurable functions on measure spaces (Y,Λ,λ) and (X,Σ,μ) respectively. We note firstly that for an operator from L into M we may assume without loss of generality that its range is in $L^0(X,\Sigma,\mu)$, for if it is a random measure representable operator from L into the latter space, then it is also representable as an operator from L into M. For the same reason we can consider T defined on its natural domain, which is defined to be $\{f \in L^0(Y,\Lambda,\lambda) : T|f| \in L^0(X,\Sigma,\mu)\}$. From this observation we infer that we may assume without loss of generality that L has a weak order unit. This is a consequence of the next lemma which was communicated to us by A.R. Schep.

Lemma 3.3.5 *Let $0 \le T : L \to M$ and let $0 \le f_n$ be disjoint in L. Then there exists $0 \le g_n \le f_n$ such that $\{g_n\}^{dd} = \{f_n\}^{dd}$ and such that $\sum_{n=1}^{\infty} Tg_n(x) < \infty$ μ-a.e. on X.*

Proof Fix n. Then $(1/k)Tf_n(x) \to 0$ μ-a.e. as $k \to \infty$. It follows that there exists some k_n such that $\mu\{x : (1/k_n)Tf_n(x) > 2^{-n}\} < 2^{-n}$. Let $g_n := f_n/k_n$. Let $\epsilon > 0$ and choose N such that $2^{-N} < \epsilon$. Then $\mu\{x : Tg_n(x) > 2^{-n}\} < 2^{-n}$ for all $n \ge N+1$ implies

$$\begin{aligned}
\mu\{x : \sum_{n=N+1}^{\infty} Tg_n(x) > 1\} &\le \mu\left(\bigcup_{n=N+1}^{\infty} \{x : Tg_n(x) > 2^{-n}\}\right) \\
&\le \sum_{n=N+1}^{\infty} 2^{-n} \\
&= 2^{-N} \\
&< \epsilon.
\end{aligned}$$

Since $Tg_1, \cdots, Tg_N$ are finite μ-a.e., we get that

$$\mu\{x : \sum_{n=1}^{\infty} Tg_n(x) = \infty\} < \epsilon.$$

Since this holds for arbitrary $\epsilon > 0$, we have that $\sum_{n=1}^{\infty} Tg_n(x) < \infty$ holds μ-a.e. on X. □

In our case we apply the lemma to the disjoint functions $\mathbf{1}_{Y_{n+1}\setminus Y_n}$ with $Y_n \uparrow Y$ as sequence such that $\mathbf{1}_{Y_n} \in L$. The function $g := \sum_n g_n$ constructed in the lemma is then strictly positive and belongs to the natural domain of T.

Let $0 \le u \in L$ be a weak order unit for L. The principal ideal generated by u, denoted by L_u, normed by the gauge function of $[-u, u]$ is an AM-space with unit u. By the Kakutani-Krein theorem (see [21, Theorem 2.1.3] or [24, Theorem II.7.4, Corollary 1]), L_u is Riesz and isometric isomorphic to a space $C(\Omega)$ with Ω a compact Hausdorff space. If ϕ denotes this Riesz isomorphism, then $\phi(u) = \mathbf{1}_\Omega$. Since L_u is Dedekind complete, so also is $C(\Omega)$ and this implies that Ω is an extremally disconnected topological space (i.e., the closure of each open set is open).

Since our results are measure theoretic a few facts on the σ-algebra of subsets of Ω inherited from Λ are recalled. Since u is a strictly positive function, its Boolean algebra of components are in one-to-one correspondence with the elements of the measure algebra associated with Λ (we will not distinguish between the elements of Λ and the elements of the measure algebra). On the other hand ϕ establishes a one-to-one correspondence between the components of u and those of $\mathbf{1}_\Omega$. The latter components are of the form $\mathbf{1}_C$ and we have that such a C is both open and closed (clopen). Conversely, for each clopen set $C \subset \Omega$, $\mathbf{1}_C$ is a continuous function and hence a component of $\mathbf{1}_\Omega$ in $C(\Omega)$. It follows that there is a one-to-one correspondence between the measure algebra associated with Λ, and the set of clopen subsets of Ω. From this it follows already that in our case the set $\mathcal{C}$ of clopen subsets of Ω is a σ-algebra (a fact which can also be proven directly using the fact that $C(\Omega)$ is σ-Dedekind complete in this case). For every clopen set C, we have $C = \overline{C}$ and so C is trivially an open F_σ set. Since the σ-algebra $\mathcal{A}(\Omega)$ of Baire subsets of Ω is generated by the open F_σ sets, we have that $\mathcal{C} \subset \mathcal{A}(\Omega)$. Conversely, since every positive $f \in L_u$ can be approximated u-uniformly by linear combinations of components of u (Freudenthal), and since ϕ is order continuous, every positive element of $C(\Omega)$ can be uniformly approximated by $\mathcal{C}$-step functions and so f is $\mathcal{C}$-measurable. However, the smallest σ-algebra of subsets of Ω with respect to which all continuous functions on Ω are measurable is $\mathcal{A}(\Omega)$. Therefore, $\mathcal{A}(\Omega) \subset \mathcal{C}$. Thus, the image of Λ is the σ-algebra $\mathcal{A}(\Omega)$ of Baire subsets of Ω.

It follows from the characterization of $\mathcal{A}(\Omega)$ as $\mathcal{C}$, that every $\mathcal{A}(\Omega)$ step function is continuous and so every positive, bounded Baire-measurable function is continuous. Denoting the image of the measure λ on $\mathcal{A}(\Omega)$ by $\widehat{\lambda}$, it follows from this that $L^\infty(\Omega, \mathcal{A}(\Omega), \widehat{\lambda}) = C(\Omega)$.

Theorem 3.3.6 *Let (Y, Λ, λ) and (X, Σ, μ) be σ–finite measure spaces and let L and M be order dense ideals in $L^0(Y, \Lambda, \lambda)$ and $L^0(X, \Sigma, \mu)$ respectively. If $T : L \to M$ is an order continuous linear operator then T is random measure-representable.*

Proof We assume, without loss of generality that L has a weak order unit, i.e., assume that there exists a strictly positive function u on Y such that $u \in L$. Furthermore it is sufficient to prove the theorem for a positive T. Therefore, let $0 \leq T \in \mathfrak{L}_n(L, M)$, i.e., T is an order continuous operator from L to M, and let $v = Tu \in M$. Denote the principal ideals generated by u and v in L and M respectively by L_u and M_v. Then $T : L_u \to M_v$ is also order continuous.

Let Ω_u and Ω_v be compact Hausdorf spaces such that L_u and M_v are Riesz isometric isomorphic to $C(\Omega_u)$ and $C(\Omega_v)$, respectively, via the Riesz isomorphisms $\phi_u : L_u \to C(\Omega_u)$ and $\phi_v : M_v \to C(\Omega_v)$. The Riesz isomorphisms ϕ_u and ϕ_v are surjections. It follows that they are order continuous Riesz isomorphisms. Indeed, for ϕ_u, if we let $f_n \uparrow f$ in L we have that $\phi_u(f_n) \uparrow$ and $\phi_u(f_n) \leq \phi_u(f)$. Since L_u is Dedekind complete, so is $\phi_u(L_u)$ and hence there exists a $g \in C(\Omega_u)$ such that $\phi_u(f_n) \uparrow g$ in $C(\Omega_u)$, and $g \leq \phi_u(f)$. This implies that $\phi_u^{-1}(\phi_u(f_n)) \uparrow$ and $\phi_u^{-1}(\phi_u(f_n)) \leq \phi_u^{-1}(g)$. Hence $f_n \leq \phi_u^{-1}(g)$ and so $f \leq \phi_u^{-1}(g)$. Thus $\phi_u(f) \leq g$. Therefore $\phi_u(f) = g$, i.e., $\phi_u(f_n) \uparrow \phi_u(f)$ and this proves our assertion. (This actually holds for any Riesz isomorphism which is onto, and so for ϕ_u^{-1} as well). Note that $\phi_u(u) = \mathbf{1}_{\Omega_u}$ and $\phi_v(v) = \mathbf{1}_{\Omega_v}$ and that $\mathcal{A}(\Omega_u)$ and $\mathcal{A}(\Omega_v)$ are images of Λ and Σ under ϕ_u and ϕ_v, respectively. We also have that $L^\infty(\Omega_u, \mathcal{A}(\Omega_u), \widehat{\lambda}) = C(\Omega_u)$

We define the positive linear map $\overline{T} : C(\Omega_u) \to C(\Omega_v)$ by $\overline{T}g = \phi_v \circ T \circ \phi_u^{-1}(g)$ for $g \in C(\Omega_u)$. Since ϕ_v and ϕ_v^{-1} are Riesz isomorphisms onto, it then follows that $\overline{T}$ is order continuous. Furthermore, $\overline{T}\mathbf{1}_{\Omega_u} = \mathbf{1}_{\Omega_v}$, so $\overline{T}$ is a continuous map from the Banach space $C(\Omega_u)$ into $C(\Omega_v)$ and $||\overline{T}|| = 1$.

Let $\overline{T}' : C(\Omega_v)' \to C(\Omega_u)'$ be the adjoint of $\overline{T}$ mapping the dual of $C(\Omega_v)$ into $C(\Omega_u)'$, the dual of $C(\Omega_v)$. For each $x \in \Omega_v$ let $F_x \in C(\Omega_u)'$

be defined by $F_x = \overline{T}'\delta_x$ where $\delta_x(g) = g(x)$ is a continuous linear functional in $C(\Omega_v)$. Then

$$F_x(f) = (\overline{T}'\delta_x)(f) = \delta_x(\overline{T}f) = (\overline{T}f)(x) \text{ and } F_x(\mathbf{1}_{\omega_u}) = 1.$$

By the Riesz Representation theorem (see [6, Theorem 4.3.9]) F_x can be represented by a unique probability measure $\nu(x,\cdot)$ defined on $\mathcal{A}(\Omega_u)$ such that

$$\int_{\Omega_u} f(y)\nu(x,dy) = F_x(f) = (\overline{T}f)(x) \tag{3.3.1}$$

for all $f \in C(\Omega_u) = L^\infty(\Omega_u, \mathcal{A}(\Omega_u), \widehat{\lambda})$. Thus, for each $f \in L_u$ we have that

$$\overline{T}(\phi_u f)(x) = \int_{\Omega_u} \phi_u f(y)\nu(x,dy),$$

and since $Tf = \phi_v^{-1}(\overline{T}\phi_u f)$ we have that

$$[\phi_v(Tf)](x) = \int_{\Omega_u} (\phi_u f)(y)\nu(x,dy).$$

From Equation (3.3.1) it follows that $\nu(x,C)$ is $\mathcal{A}(\Omega_v)$-measurable for every $C \in \mathcal{A}(\Omega_u)$. Therefore $\nu(x,\cdot)$ is a random measure on $\mathcal{A}(\Omega_u)$. Thus $\overline{T} : L^\infty(\Omega_u, \mathcal{A}(\Omega_u), \widehat{\lambda}) \to L^\infty(\Omega_v, \mathcal{A}(\Omega_v)_\Sigma, \widehat{\mu})$ is generated by the random measure $\nu(x,\cdot)$, i.e. the following diagram commutes

$$\begin{array}{ccc} L_u & \xrightarrow{T} & M_v \\ {\scriptstyle\phi_u}\big\downarrow & & \big\downarrow{\scriptstyle\phi_v} \\ C(\Omega_u) & \xrightarrow[\overline{T}]{} & C(\Omega_v) \end{array}$$

Note that the Riesz homomorphism ϕ_u can be extended to an order continuous Riesz homomorphism ϕ_L from $L^0(Y,\Lambda,\lambda)$ into $L^0(\Omega_2, \mathfrak{F}_2, \pi_2)$

and similarly ϕ_v^{-1} can be extended to an order continuous Riesz homomorphism ψ_M from $L^0(\Omega_1, \mathfrak{F}_1, \pi_1)$ into $L^0(X, \Sigma, \mu)$. Since L_u is order dense in L we have, for $0 \leq g \in L$, that there exists a sequence $0 \leq (g_n)$ in L_u such that $g_n \uparrow g$. So $\phi_L(g_n) \uparrow \phi_L(g)$. Let $f_n = \phi_L(g_n)$. We have that $f_n \uparrow \phi_L(g) := f$, so that $\int_{\Omega_u} f_n(y)\nu(x, dy) \uparrow \int_{\Omega_u} f(y)\nu(x, dy)$. But, $\psi_L \int_{\Omega_u} f_n(y)\nu(x, dy) = Tg_n$ and since T is order continuous we have that $Tg_n \uparrow Tg$, so that

$$\begin{aligned} Tg &= \psi_L \int_{\Omega_1} f(y)\nu(x, dy) \\ &= \psi_L \int_{\Omega_1} \phi_L(g(y))\nu(x, dy). \end{aligned}$$

This holds for all $0 \leq g \in L$ and so for all $g \in L$. Thus $T = \psi_L \circ \overline{T} \circ \phi_L$. This completes the proof of the theorem. □

Chapter 4

Random measures and MCE-Operators

There seems to be a strong connection between conditional expectation operators and operators generated by random measures. By investigating the relationship between these operators, we prove the main theorem of this thesis, which is that every order continuous operator is MCE-representable.

We also forge a link amongst order continuous operators, MCE-representable operators and random measure-representable operators.

This solves many open problems involving MCE-representable operators. For instance, it becomes easy, almost trivial, to show that the sum, composition, product, etc of MCE-representable operators are again MCE-representable operators. Direct proofs of these results are not trivial.

Note that if we put $\mathfrak{R}(L, M)$ to be the set of all random measures $\nu(x, \cdot)$ such that $\int_Y f(y)\nu(x, dy) \in M$ for all $f \in L$ then $\mathcal{L}_{rm}(L, M)$ is Riesz isomorphic to $\mathfrak{R}(L, M)$. Thus $\mathfrak{R}(L, M)$ is a Riesz space since $\mathcal{L}_{rm}(L, M)$ is a Riesz space.

4.1 Random measures and MCE-Operators

Before we embark on a proof of the main result we first show that operators generated by random measures are MCE-representable, i.e., we want to show that if $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ are ideals of measurable functions and $T : L \to M$ a positive operator generated by the random measure $\nu(x, \cdot)$, then T is MCE - representable.

The case where $\nu(x, \cdot)$ is a probability measure for each $x \in X$ and μ is also a probability measure on X is trivial.

To show this let $\Omega = X \times Y$. Since $\nu(x, \cdot)$ and μ are probability measures, there exists, by the Product of Measures theorem see [6, Theorem 2.6.2], a unique probability measure $\mathbb{P}$ on $\Sigma \otimes \Lambda$ defined by

$$\mathbb{P}(C) = \int_X \nu(x, C(x)) d\mu(x).$$

The operator T is given by $Tf(x) = \int_Y f(z)\nu(x, dz)$ for almost every $x \in X$. For $f \in L$ define the homomorphism ϕ_L by $\phi_L(f) = f(z)\mathbf{1}_X(x)$ and for $g \in M$ define the homomorphism ψ_M by $\psi_M^{-1}(g) = g(x)\mathbf{1}_Y(y)$. Let $f \in L$, put $\mathfrak{F}_M = \{B \times Y,$ such that $B \in \Sigma\} = \Sigma \otimes Y$. We have that

$$\begin{aligned}\int_{B\times Y} (Tf)(x)\mathbf{1}_Y(y) d\mathbb{P} &= \int_B \int_Y Tf(x)\mathbf{1}_Y(y)\nu(x, dy)d\mu(x) \\ &= \int_B \int_Y \int_Y f(z)\nu(x, dz)\nu(x, dy)d\mu(x),\end{aligned}$$

which by the theorem of Fubini (see [6, Theorem 2.6.6]) we get

$$\int_{B\times Y} (Tf)(x)\mathbf{1}_Y(y) d\mathbb{P} = \int_B \int_Y f(z)\nu(x, dz) \int_Y \nu(x, dy)d\mu(x).$$

Since $\int_Y \nu(x, dy) = 1$ we then have that

$$\begin{aligned}\int_{B\times Y}(Tf)(x)\mathbf{1}_Y(y)d\mathbb{P} &= \int_B\int_Y f(z)\nu(x,dz)d\mu(x)\\ &= \int_{B\times Y} f(z)\mathbf{1}_X(x)d\mathbb{P}.\end{aligned}$$

Since $(Tf)(x)\mathbf{1}_Y(y)$ is $\mathfrak{F}_M$-measurable, this shows that

$$Tf(x)\mathbf{1}_Y(y) = \mathbb{E}(f\mathbf{1}_X(x)|\mathfrak{F}_M).$$

Thus $\psi_M^{-1}(Tf) = \mathbb{E}(\phi_L(f)|\mathfrak{F}_M)$, and so $\psi_M\mathbb{E}(\phi_L(f)|\mathfrak{F}_M) = Tf$.

We present a formal proof for the general case. We will use the following convention:

Convention(A) If $h : \Omega \mapsto \overline{\mathbb{R}}$ is a measurable function we define

$$\frac{1}{h(\omega)} = \begin{cases} \frac{1}{h(\omega)}, & h(\omega) \neq 0 \\ 0, & h(\omega) = 0 \end{cases}$$

The function $\frac{1}{h}$ is thus again a measurable function.

Theorem 4.1.1 *If $L \subset L^0(Y, \Lambda, \lambda)$ and $M \subset L^0(X, \Sigma, \mu)$ are ideals of measurable functions and $T : L \to M$ a positive operator generated by the random measure $\nu(x, \cdot)$, then T is MCE-representable.*

Proof Since L is order dense, there exists a sequence $Y_n \uparrow Y$ such that $\mathbf{1}_{Y_n} \in L$ and $T\mathbf{1}_{Y_n} = \nu(x, Y_n) \in M$ for each n. For $B \in \Lambda$ put $\nu_n(x, B) = \nu(x, B \cap Y_n)$. Following convention (A), we define

$$\overline{\nu}(x,\cdot) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\nu_n(x,Y_n)}\nu_n(x,\cdot).$$

Let $\widetilde{X} = \{x \in X : \nu(x,Y) \neq 0\}$. Since $\nu(x,B) = 0$ for all $x \in \widetilde{X}^c$ we have that $\overline{\nu}(x,\cdot) = 0$ for all $x \in \widetilde{X}^c$. Also, since $\nu(x,Y_n) \uparrow \nu(x,Y) > 0$, we have that $\overline{\nu}(x,\cdot) \gneqq 0$ for all $x \in \widetilde{X}$. Therefore

$$
\begin{aligned}
\overline{\nu}(x,Y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu_n(x,Y_n)} \nu_n(x,Y) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu_n(x,Y_n)} \nu(x,Y \cap Y_n) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\nu_n(x,Y_n)} \nu(x,Y_n) \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= 1.
\end{aligned}
$$

This holds for every $x \in \widetilde{X}$. Put, using convention (A),

$$\overline{\overline{\nu}}(x,\cdot) = \frac{\overline{\nu}(x,\cdot)}{\overline{\nu}(x,Y)}$$

Note that $\overline{\overline{\nu}}(x,\cdot)$ is non-zero for all $x \in \widetilde{X}$. Also, $\overline{\overline{\nu}}(x,Y) = 1$ for all $x \in \widetilde{X}$, i.e., it is a probability measure on Y for all $x \in \widetilde{X}$ and so $\overline{\overline{\nu}}(x,Y) = \mathbf{1}_{\widetilde{X}}(x)$. It follows that $\overline{\overline{\nu}}(x,\cdot)$ is uniformly σ-finite (in fact uniformly finite).

We have that μ restricted to $\widetilde{X}$ is a σ-finite measure and it can be normalized to be a probability measure on $\widetilde{X}$ as we did with the measure ν. We denote this normalized measure by $\overline{\overline{\mu}}$. Hence there exists a unique product measure $\mathbb{P}$ on $\Sigma \times \Lambda$ such that $\mathbb{P}(A \times B) = \int_X \overline{\overline{\nu}}(x,B)\overline{\overline{\mu}}(A)$ for all $A \in \Sigma$ and $B \in \Lambda$. Thus

$$\begin{aligned}
\mathbb{P}(X \times Y) &= \int_X \overline{\mu}(x, Y) d\overline{\overline{\mu}}(x) \\
&= \int_X \mathbf{1}_{\widetilde{X}} d\overline{\overline{\mu}} \\
&= \overline{\overline{\mu}}(\widetilde{X}) \\
&= 1.
\end{aligned}$$

It follows that $\mathbb{P}$ is a probability measure on $\Omega = X \times Y$. Let $\mathfrak{F} = \Sigma \otimes \Lambda$ and g be a step function given by $g = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{E_j}$, where $E_j \in \Lambda$. Then

$$\begin{aligned}
\int_Y g(z) \nu_n(x, dz) &= \sum_{j=1}^{k} \alpha_j \int_{E_j} \nu_n(x, dz) \\
&= \sum_{j=1}^{k} \alpha_j \nu_n(x, E_j) \\
&= \sum_{j=1}^{k} \alpha_j \nu(E_j \cap Y_n) \\
&= \sum_{j=1}^{k} \alpha_j \int_{E_j \cap Y_n} \nu(x, dz) \\
&= \int_Y \mathbf{1}_{Y_n}(z) g(z) \nu(x, dz)
\end{aligned}$$

so that $\int_Y g(z)\nu_n(x,dz) = \int_Y \mathbf{1}_{Y_n}(z)g(z)\nu(x,dz)$. Thus

$$\begin{aligned}
\int_Y g(z)\overline{\overline{\nu}}(x,dz) &= \int_Y g(z)\frac{\overline{\nu}(x,dz)}{\overline{\nu}(x,Y)} \\
&= \int_Y g(z)\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\nu(x,Y_n)}\nu_n(x,dz)\frac{1}{\overline{\nu}(x,Y)} \\
&= \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\nu(x,Y_n)}\int_Y \frac{1}{\overline{\nu}(x,Y)}g(z)\nu_n(x,dz) \\
&= \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\nu(x,Y_n)}\int_Y \frac{1}{\overline{\nu}(x,Y)}\mathbf{1}_{Y_n}(z)g(z)\nu(x,dz) \\
&= \int_Y \frac{1}{\overline{\nu}(x,Y)}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\nu(x,Y_n)}\mathbf{1}_{Y_n}(z)g(z)\nu(x,dz).
\end{aligned}$$

Put, using convention (A),

$$h(x,z) = \frac{1}{\overline{\nu}(x,Y)}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{\nu(x,Y_n)}\mathbf{1}_{Y_n}(z).$$

It follows that

$$\int_Y g(z)\overline{\overline{\nu}}(x,dz) = \int_Y h(x,z)g(z)\nu(x,dz) \tag{4.1.1}$$

for every step function g and so it holds for arbitrary positive functions $g \in L^0(Y,\Lambda,\lambda)$.

For an arbitrary A in $\mathfrak{F}_M = \{A \times Y : A \in \Sigma\}$ and $0 < f \in L$ we have, that

$$\begin{aligned}
\int_{A\times Y}(Tf)(x)\mathbf{1}_Y(y)d\mathbb{P} &= \int_A\int_Y Tf(x)\mathbf{1}_Y(y)\overline{\overline{\nu}}(x,dy)d\overline{\overline{\mu}}(x) \\
&= \int_A\int_Y\int_Y f(z)\nu(x,dz)\overline{\overline{\nu}}(x,dy)d\overline{\overline{\mu}}(x) \\
&= \int_A\int_Y f(z)\nu(x,dz)\int_Y \overline{\overline{\nu}}(x,dy)d\overline{\overline{\mu}}(x) \\
&= \int_A\int_Y f(z)\nu(x,dz)\mathbf{1}_{\tilde{X}}(x)d\overline{\overline{\mu}}(x) \\
&= \int_{A\cap\tilde{X}}\int_Y f(z)\nu(x,dz)d\overline{\overline{\mu}}(x) \\
&\quad + \int_{A\cap\tilde{X}^C}\int_Y f(z)\nu(x,dz)d\overline{\overline{\mu}}(x) \\
&= \int_A\int_Y f(z)\nu(x,dz)d\overline{\overline{\mu}}(x) \\
&= \int_A\int_Y \frac{1}{h(x,z)}h(x,z)f(z)\nu(x,dz)d\overline{\overline{\mu}}(x),
\end{aligned}$$

which, by using Equation (4.1.1) gives

$$\begin{aligned}
\int_{A\times Y}(Tf)(x)\mathbf{1}_Y(y)d\mathbb{P} &= \int_A\int_Y \frac{f(z)}{h(x,z)}\overline{\overline{\nu}}(x,dz)d\overline{\overline{\mu}}(x) \\
&= \int_{A\times Y}\frac{1}{h(x,z)}f(z)d\mathbb{P}.
\end{aligned}$$

We thus have shown that $\mathbb{E}(\frac{1}{h(x,z)}f(z)(x)|\mathfrak{F}_M) = Tf(x)\mathbf{1}_Y(y)$. Putting $m(x,\cdot) = \frac{1}{h(x,\cdot)}$ we get that $\psi_M\mathbb{E}(m\phi_L(f)|\mathfrak{F}_M) = Tf$. This holds for positive $f \in L$ and consequently for all $f \in L$. This completes the proof. □

4.2 Main result

We are now in a position to prove the main result of this thesis.

Theorem 4.2.1 *Let (Y, Λ, ν) and (X, Σ, μ) be σ-finite measure spaces and let L and M be order dense ideals in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$ respectively. If $T : L \to M$ is an order continuous linear operator then T is MCE representable.*

Proof From Theorem 3.3.6 we get that the operator T is random measure-representable. Therefore there exist σ-finite measure spaces $(\Omega_1, \mathfrak{F}_1, \pi_1)$ and $(\Omega_2, \mathfrak{F}_2, \pi_2)$, order dense ideals $L_{\Omega_2} \subset L^0(\Omega_2, \mathfrak{F}_2, \pi_2)$ and $M_{\Omega_1} \subset L^0(\Omega_1, \mathfrak{F}_1, \pi_1)$ and order continuous Riesz homomorphisms $\phi'_L : L \to L_{\Omega_2}$ and $\psi'_M : M_{\Omega_1} \to M$ with ϕ'_L surjective and ψ'_M a Riesz isomorphism onto an ideal in M; such that

$$T = \psi'_M \circ T_\nu \circ \phi'_L,$$

with $T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})$, i.e., such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "T"] \arrow[d, "\phi'_L"'] & M \\ L_{\Omega_2} \arrow[r, "T_\nu"'] & M_{\Omega_1} \arrow[u, "\psi'_M"'] \end{tikzcd}$$

From Theorem 4.1.1 we get that the operator $T_\nu \in \mathcal{L}_{rm}(L_{\Omega_2}, M_{\Omega_1})$ is MCE-representable. Thus there exist a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, order dense ideals $L_\Omega \subset L^0(\Omega, \mathfrak{F}_L, \mathbb{P})$ and $M_\Omega \subset L^0(\Omega, \mathfrak{F}_M, \mathbb{P})$, order continuous Riesz homomorphisms $\phi''_L : L_{\Omega_2} \to L_\Omega$ and $\psi''_M : M_\Omega \to M_{\Omega_1}$, with ϕ''_L surjective and ψ''_M a Riesz isomorphism onto an ideal in M_{Ω_1} and a measurable function $m \in L^0(\Omega, \mathfrak{F}, \mathbb{P})$ such that

$$T_\nu = \psi''_M \circ S_m \circ \phi''_L,$$

i.e., such that the following diagram commutes

$$\begin{array}{ccc} L_{\Omega_2} & \xrightarrow{T_\nu} & M_{\Omega_1} \\ \phi''_L \downarrow & & \uparrow \psi''_M \\ L_\Omega & \xrightarrow[S_m]{} & M_\Omega \end{array}$$

If we take $\phi_L := \phi'_L \circ \phi''_L$ and $\psi_M := \psi''_M \circ \psi'_M$ then ϕ_L and ψ_M are order continuous Riesz homomorphisms with ϕ_L surjective and ψ_M a Riesz isomorphism onto an ideal in M and we have that the following diagram commutes.

$$\begin{array}{ccc} L & \xrightarrow{T} & M \\ \phi_L \downarrow & & \uparrow \psi_M \\ L_\Omega & \xrightarrow[S_m]{} & M_\Omega \end{array}$$

and the proof is complete. □

The following result now follows from the fact that the sum and the composition of order continuous operators are also order continuous.

Corollary 4.2.2 *Let (Y, Λ, ν), (X, Σ, μ) and (Z, Γ, γ) be $\sigma-$finite measure spaces and let L, M and N be order dense ideals in $L^0(Y, \Lambda, \nu)$, $L^0(X, \Sigma, \mu)$ and (Z, Γ, γ), respectively.*

(a) If T_1 and T_2 are order continuous linear operators that map L into M then $T = T_1 + T_2$ is MCE-representable.

(b) If $T_1 : M \to N$ and $T_2 : L \to M$ are order continuous linear operators then $T = T_2 \circ T_1$ is MCE-representable.

Theorem 4.2.3 *Let (Y, Λ, ν) and (X, Σ, μ) be σ-finite measure spaces and let L and M be order dense ideals in $L^0(Y, \Lambda, \nu)$ and $L^0(X, \Sigma, \mu)$ respectively. For a linear operator $T : L \to M$ the following are equivalent*

(i) T is order continuous;

(ii) T is random measure-representable;

(iii) T is MCE-representable.

Proof (i) $\Rightarrow$ (ii)

See Theorem 3.3.6 on Page 94.

(ii) $\Rightarrow$ (iii)

See Theorem 4.1.1 on Page 100.

(iii) $\Rightarrow$ (i)

If T is MCE-representable then by Definition 2.2.1 on Page 49 there is a representation triple $\Phi = ((\Omega, \mathfrak{F}, \mathbb{P}), \phi_L, \psi_M)$ for the operator T and a $\Phi-$kernel m of T such that

$$T = \psi_M \circ S_m \circ \phi_L.$$

Since ϕ_L, S_m and ψ_M are all order continuous we have that T is order continuous. □

We remark that if T is generated by a random measure then T is random measure-representable (and therefore MCE-representable). The converse is true if the measure spaces (Y, Λ, ν) and (X, Σ, μ) are standard Borel measure spaces see [27].

Bibliography

[1] C.D. ALIPRANTIS and O. BURKINSHAW. *Positive Operators.* Academic Press, 1985.

[2] J. APPELL, E.V. FROVOLA, A.S. KALITVIN, and P.P. ZABREJKO. Partial intergal operators on $C([a,b] \times [c,d])$. *Integral Equations and Operator Theory*, **27**:125 – 140, 1997.

[3] J. APPELL, A.S. KALITVIN, and M.Z. NASHED. On some partial integral equations arising in the mechanics of solids. *Journal of Applied Mathematics and Mechanics*, **79**(10):703 – 713, 1999.

[4] J. APPELL, A.S. KALITVIN, and P.P. ZABREJKO. Partial integral operators in Orlicz spaces with mixed norm. *Collo. Math.*, **78**(2):293 – 306, 1998.

[5] W. ARVESON. Operator algebras and invariant subspaces. *Annals of Mathematics*, **100**(2):433 – 532, 1974.

[6] R.B. ASH. *Measure, Integration and Functional Analysis.* Academic Press, London, 1972.

[7] P.G. DODDS, C.B. HUIJSMANS, and B. de PAGTER. Characterizations of conditional expectation - type operators. *Pacific Journal of Mathematics*, **141**(1):55 – 77, 1990.

[8] J.J. GROBLER and B. de PAGTER. Operators representable as multiplication conditional expectation operators. *Journal of Operator Theory*, **48**:15 – 40, 2002.

[9] K. HOFFMAN. *Banach spaces of analytic functions.* Prentice-Hall, Englewood Cliffs, N.J., 1962.

[10] C.B. HUIJSMANS and B. de PAGTER. Averaging operators and positive contractive projection. *Journal of Mathematical Analysis and Applications*, **113**:163 – 184, 1986.

[11] A.S. KALITVIN. Spectral properties of partial integral operators of Volterra and Volterra–Fredholm type. *Journal of Analysis and its Applications [Zeitschrift fr Analysis und ihre Anwendungen]*, **17**(2):297 – 310, 1998.

[12] A.S. KALITVIN and P.P. ZABREJKO. On the theory of partial integral operators. *Journal of Integral Equations and Applications*, **3**(3):351 – 382, 1991.

[13] N. KALTON. The endomorphisms of l_p $(0 < p < 1)$. *Indiana University Mathematics*, **27**:353 – 381, 1978.

[14] N. KALTON. Emmbeding L_1 in Banach space. *Israel Journal of Mathematics*, **32**:209 – 220, 1979.

[15] N. KALTON. Linear operators on L_p for $0 \leq p \leq 1$. *Transactions of the American Mathematical Society*, **259**:319 – 355, 1980.

[16] J.L. KELLEY. Averaging operators on $C_\infty(x)$. *Illinois Journal of Mathematics*, **2**:214 – 223, 1958.

[17] A. LAMBERT. A Hilbert C^*–module view of some spaces related to probabilistic conditional expectation. *Questiones Mathematicae*, **22**:165 – 170, 1999.

[18] W.A.J LUXEMBURG and A.R. SCHEP. A Randon-Nikodym type theorem for positive operators and a dual. *Indag Mathematicae*, **40**(3):357 – 375, 1978.

[19] W.A.J. LUXEMBURG and A.C. ZAANEN. *Riesz Spaces I.* North Holland, Amsterdam, 1983.

[20] G. MACKEY. Borel structures in groups and their duals. *Transactions of the American Mathematical Society*, **85**:134165, 1957.

[21] P. MEYER-NIEBERG. *Banach Lattices.* Springer Verlag, Berlin, 1991.

[22] S–T.C. MOY. Characterization of conditional expectations as a transformation on function spaces. *Pacific Journal of Mathematics*, **4**:47 – 63, 1954.

[23] J. NEVEU. *Discrete - parameter martingales.* North Holland, Amsterdam, 1975.

[24] H.H. SCHAEFER. *Banach Lattices and Positive Operators.* Springer - Verlag, Berlin, 1974.

[25] A.R. SCHEP. *Kernel Operators.* PhD thesis, University of Leiden, 1977.

[26] S. SIDAK. On relations between strict - sence and wide sense conditional expectation. *Theory of Probability and Applications*, **2**(2):267 – 271, 1957.

[27] A.R. SOUROUR. Pseudo-integral operators. *Transactions of the American Mathematical Society*, **253**:339 – 363, 1979.

[28] A.R. SOUROUR. Characterization and order properties of pseudo-integral operators. *Pacific Journal of Mathematics*, **99**(1):145 – 158, 1982.

[29] L. W. WEIS. On the representation of order continuous operators by random measures. *Transactions of the American Mathematical Society*, **285**(2):535 – 563, 1984.

[30] L. W. WEIS. The representation of l_1-operators by stochaic kernels and some operator properties connected with it. *Seminaire Initiation a l'Analyse*, **23**(18):1 – 7, 1984.

[31] L.W. WEIS. Decomposition of positive operators and some of their applications. In K.D. Bierstedt and B. Fuchssteiner, editors, *Functional*

Analysis: Recent Results III. Elsevier Science Publishers, North – Holland, 1984.

[32] A.C. ZAANEN. *Integration.* North - Holland,, Amsterdam, 1967.

[33] A.C. ZAANEN. *Riesz Spaces II.* North Holland, Amsterdam, 1983.

[34] A.C. ZAANEN. *Introduction to Riesz Spaces and Operator Theory.* Springer - Verlag, Berlin, 1997.