

MANAGEMENT AND AUDITING OF BANK ASSETS AND CAPITAL

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Thesis submitted in fulfilment of the requirements for the
degree Philosophiae Doctor in
Applied Mathematics
at the
North-West University (Potchefstroom Campus)

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12 May 2008
Potchefstroom

Preface

One of the contributions made by North-West University (Potchefstroom Campus) to the activities of the applied mathematics community in South Africa has been the establishment of an active research group that has an interest in financial mathematics. Under the guidance of my supervisor, Prof. Mark A. Petersen, this group has recently made valuable contributions to the existing knowledge about the stochastic control of financial systems in pensions, insurance and banking.

The work in this thesis originated from our interest in the connections between concepts that arise in systems and (stochastic) control theory and financial models. In this regard, the interests of the group lie with the stochastic controllability of interest rate models, stochastic control of continuous- and discrete-time pension funds, the solvency of dividend equalization funds and the solvency, profitability and operational control of commercial banks.

The most important outcomes of this project were collected in 3 peer-reviewed international journal articles (1 appeared, 2 submitted) and 2 accepted peer-reviewed conference proceedings papers. The published paper [26] led to an invitation from Nova Publishers (New York) to contribute to the book [24].

Declaration

This thesis is being submitted in fulfilment of the requirements for the degree Philosophiae Doctor in Applied Mathematics at the North-West University (Potchefstroom Campus). It has not been submitted before for any degree or examination to any other University.

I declare that, apart from the assistance acknowledged, the outcomes of this thesis have resulted from my own unaided research. Nobody, including Prof. MA. Petersen (Supervisor), but myself is responsible for the final version of this thesis.

Signature.....

Date.....

Summary

As a result of the new regulatory prescripts for banks, known as the Basel II Capital Accord, there has been a heightened interest in the auditing process. We consider this issue with a particular emphasis on the auditing of reserves, assets and capital in both a random and non-random framework. The analysis relies on the stochastic dynamic modeling of banking items such as loans, shares, bonds, cash, reserves, Treasuries, outstanding debts, bank capital and government subsidies. In this regard, one of the main novelties of our contribution is the establishment of optimal bank reserves and a rate of depository consumption that is of importance during a (random) audit of the reserve requirements. Here the specific choice of a power utility function is made in order to obtain an analytic solution in a Lévy process setting.

Furthermore, we provide explicit formulas for the shareholder default and regulator closure rules, for the case of a Poisson-distributed random audit. A property of these rules is that they define the standard for minimum capital adequacy in an implicit way. In addition, we solve an optimal auditing time problem for the Basel II capital adequacy requirement by making use of Lévy process-based models. This result provides information about the optimal timing of an internal audit when the ambient value of the capital adequacy ratio (CAR) is taken into account and the bank is able to choose the time at which the audit takes place.

Finally, we discuss some of the economic issues arising from the analysis of the stochastic dynamic models of banking items and the optimization procedure related to the auditing process.

Uittreksel

As gevolg van die nuwe wetlike voorskrifte vir banke, bekend as die "Basel II Capital Accord", is daar 'n hernieude belangstelling in die ouditeringsproses. Ons beskou hierdie aangeleentheid met spesiale klem op die ouditering van reserwes, bates en kapitaal in 'n onreëlmatige sowel as reëlmatige raamwerk. Die ontleding berus op die stogastiese dinamiese modellering van bankwese-items soos lenings, aandele, verbande, kontant, reserwes, Tesourieë, uitstaande skulde, bankkapitaal en regeringsubsidies. In hierdie verband, is een van die vernaamste nuwe bydraes wat gemaak is, die daarstelling van gunstige bankreserwes en die tempo waarteen inlêgeld gebruik word wat belangrik geag word gedurende 'n (toevalige) ouditering van die reserwevereistes. Hier word 'n bepaalde keuse van 'n "power utility function" gebruik om 'n analitiese oplossing in 'n Lévy metode omgewing te verkry.

Vervolgens, voorsien ons uitdruklike formules vir die wanprestasie van aandeelhouers en uitsluitelvoorskrifte vir reguleerders in die geval van 'n "Poisson-verspreide" onbepaalde audit. 'n Eienskap van hierdie reëls is dat hulle die standaard vir minimum voldoende kapitaal implisiet omskryf. Daarbenewens bied ons 'n oplossing vir 'n optimale ouditeringstydprobleem vir die Basel II kapitaaltoereikendheidsvereistes deur van die Lévy-proses-gebaseerde metode gebruik te maak. Hierdie uitslag voorsien inligting oor die optimale tydsberekening vir 'n interne audit wanneer die omringende waarde van die kapitaalvereisteverhouding (CAR) in berekening geneem word en die bank in staat gestel word om 'n tyd te kies wanneer 'n audit plaasvind.

Ten slote, bespreek ons van die ekonomiese aangeleenthede wat spruit uit die ontleding van die stogastiese dinamiese modellering van bankwese-items en die optimaliseringsprosedure met betrekking tot die ouditeringsproses.

Acknowledgements

All praise and thanks is due to the Almighty, Most Gracious, Most Merciful for His grace in enabling me to complete this thesis.

I would like to acknowledge the emotional support provided by my immediate family; father Rashaad, mother Fatima, wife Razaan, brother Qasim, sister Nadiya and uncle Ebrahim. The interest shown by my brother-in-law Saleem and sister-in-law Sihaam is greatly appreciated. Also, my nephews and nieces Kauthar, Maryam, Muhammed Ilyaas and Ruqaiyah showed their support for this venture.

I am indebted to my supervisor, Prof. Mark A. Petersen of the School of Computer, Mathematical and Statistical Sciences at the North-West University (Potchefstroom Campus), for the guidance provided during the completion of this thesis. Also, I would like to thank my colleagues from Santam Mr. Russell Drummond, Mr. Attie Blaauw and Dr. Shahiem Ganief for providing moral support during my studies.

Furthermore, I would like to express my gratitude towards Mr. Barry Danvers from the South African Reserve Bank (SARB) for facilitating the binding of this thesis at the SARB printing department.

Finally, I am grateful to the National Research Foundation (NRF) for providing me with funding during the duration of my PhD studies. In 2002, during my masters studies, I also received NRF funding under projects with GUN No.'s 2053343 and 2074218.

Index of Abbreviations

OECD - Organization for Economic Co-operation and Development
 LLP - Loan Loss Provision
 GDP - Gross Domestic Product
 PD - Probability Default
 LGD - Loss Given Default
 NPL - Non-Performing Loans
 TA - Total Assets
 VaR - Value-at-Risk
 GKW - Galtchouck-Kunita-Watanable
 TCR - Total Capital Ratio
 CRC - Credit Risk Charge
 ADR - Assets-to-Debt Ratio

Index of Symbols

$SIGN_{it}$ - One-Year-Ahead Changes of Earnings Before Taxes and Loan Loss Provisions
 ER_{it} - Positive Correlation Between Earnings Before Taxes and Loan Loss Provisions
 y'_{it} - Annual Growth Rate of GDP
 P_{it} - Ratio of Loan Loss Provisions to Total Assets at the End of Year t for Bank i
 Λ - Loans
 T - Treasuries
 R - Reserves
 K - Capital
 L - Lévy Process
 D - Deposits
 ϕ - Characteristic Function of a Distribution
 ψ - Lévy or Characteristic Exponent of L
 x - Variable
 γ - Drift of a Process
 X - Value Process
 Z - Standard Brownian Motion
 $Q(dt, dx)$ - Poisson measure
 dt - Lebesgue measure

M - Martingale
 ξ - Doléans-Dade Exponential
 P - Total Provision
 A - Assets
 r^A - Loan Rate
 c^d - Risk Premium
 c^a - Administrative Cost
 S^e - Expected Loan Losses
 S^u - Unexpected Loan Losses
 ν - Lévy Measure
 B - Borel Sets
 S - Aggregate Loan Losses
 T - Terminal Time
 P' - Net Loan Loss Provisioning
 ϱ - Net Instantaneous Return of a Value Process
 σ - Volatility of a Value Process
 μ - Mean of a Value Process
 π - Provisioning Strategy
 k_d - Depository Value
 D - Depository Contracts
 L_{it} - Provisions for Loan Losses-to-Total Assets Ratio
 n^T - Number of Treasuries
 n^R - Number of Reserves
 $\widehat{V}(\pi)$ - Provisioning Portfolio Value Process
 c^o - Cost Process
 Λ^c - Probability of Insolvency to Occur
 C_T - Cost of Insolvency
 N^L - Number of Loan Losses
 l - Unexpected Loan Losses sizes
 r^R - Loan Loss Reserve Rate
 P^π - Total Loan Loss Provisioning Under Strategy π .
 R^l - Loan Loss Reserve
 P'^π - Net Loan Loss Provisioning Under π
 r^R - Deterministic Rate of (Positive) Return on Reserves
 f^R - Fraction of the Reserves Consumed by Deposit Withdrawals
 σ^R - Volatility in the Level of Reserves
 G - Girsanov Parameter
 $M^Q(dt, dx)$ - Compensated Jump Measure of L^R Under \mathbf{Q}_g

W - Sum of Treasuries and Reserves
 D^c - Sum of Cohort Deposits
 w_{x+t} - Withdrawal Rate Function
 N^I - Number of Withdrawals
 M^I - Compensated Counting Process
 w^{un} - Unanticipated Deposit Withdrawals
 $f(w^{un})$ - Probability Density Function
 c^l - Cost of Liquidation
 r_t^P - Penalty Rate on Deposit Withdrawals
 $c^{w^{un}}$ - Cost of Deposit Withdrawals
 ρ^r - Relative Risk Ratio

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Chapter 1

INTRODUCTION

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Most commercial and investment banks are subjected to regular internal and/or external (for example, by regulators) auditing of operational items such as assets, liabilities and capital. An audit involves the evaluation of the adequacy of the bank's systems of internal control, the extent of compliance with established procedures and regulations and the effectiveness of banking operations. Despite the importance of this process, it has not been discussed very extensively in the banking literature when compared to other financial decision making problems such as asset quality and credit risk assessment. We separately discuss scenarios in which audits may take place on a random (particularly in the case of external audits) and non-random (especially for internal audits) basis (see, for instance, [19], [42], [43], [65], [69] and [79]). Importantly, we note that the latter option invariably allows the bank owner to exert some influence over the timing of the audit. Of course, the vast majority of audits undertaken in the banking industry are random and involve external auditing. Usually the conclusions drawn from internal audits are made available to external auditors and as a result has some effect on the outcome of the latter type of audit. Three operational requirements that play an important role in auditing are related to the reserves and capital held by the bank and the setting of rules for its default and closure. With regard to the former, it is important for auditors to be able to measure the volume of (Treasuries and) reserves that the bank holds in order to cater for anticipated and unanticipated deposit withdrawals. This auditing criterion is known as the *reserve requirement* and considers the nature of the reserves that are available to meet the bank's obligations to depositors. Both auditors and banks are interested in establishing the optimal level of reserves on demand deposits that the bank must hold. By setting a bank's individual optimal level of reserves, auditors assist in mitigating the costs of financial distress. For instance, if the minimum level of required reserves exceeds a bank's optimally determined level of reserves, this may lead to deadweight losses. Secondly, we explore the connection between auditing and *asset requirements* that are formulated by the bank's shareholders and regulators. In this regard, in the case of a Poisson-distributed random audit, the unlevered asset value, A , is allowed to evolve without any restriction on time until it becomes less than a critical asset value, A^s , that is chosen by the shareholders and initiates the default process. In addition, we discuss a related prescribed asset value, A^r , set by the regulator, that is instrumental in determining a threshold for bank closure and reorganization. The problem of determining and characterizing A^s and A^r and their interrelationship is sometimes called the *asset threshold problem*. Finally, auditors are interested in the *capital requirement* that aims to determine the sufficiency of the capital held by the bank for reducing the default risks on deposits, as well as in incentivising sensible risk-taking. For these issues, empirical and theoretical linkages between credit and

market risk and approaches to the measurement and management of credit, market, liquidity and operational (see, for instance, [13]) risk are of considerable interest. Also, the scope for integrated banking regulation between different types of risk and theoretical and empirical management of economic versus regulatory capital impacts management techniques. Banks are among the most heavily regulated of all financial institutions. In particular, reserve and capital requirements have become important components of regulation and supervision in the banking industry. As from June 1999, the Basel Committee on Banking Supervision (BCBS) released several proposals (see, for instance, [7], [8] and [9]) to reform the original 1988 Basel Capital Accord (see [6]). These efforts culminated in the Basel II Capital Accord (see, for instance, [11], [12], [15] and [16]) which is based on three pillars (see [86] for a discussion on the interaction between these pillars). Pillar 1 intends to provide a stronger link between the management of capital requirements and actual risk. Pillar 2 focusses on strengthening the supervisory process, particularly in assessing the quality of risk management in banking institutions and in evaluating whether these institutions have adequate procedures to determine how much capital they need. Pillar 3 involves the improvement of market discipline through increased disclosure of details about the bank's credit exposures, its amount of reserves and capital, the bank owners and the effectiveness of its internal ratings system. Since bank management has become increasingly complicated and supervisors (acting as representatives of the depositors' interests) battle to monitor banking activities, the recourse to market discipline appears to be justified. In this regard, monitoring of banks by professional investors and financial analysts as a complement to banking supervision is being encouraged. However, the manner in which market discipline and the other two pillars are to be managed in concert with each other is a subject of much debate. In Basel II, the ratio of capital to assets, also called the *capital adequacy ratio (CAR)* plays a major role as an index of the sufficiency of capital held by banks. This ratio is the centerpiece of the minimum capital requirement (Pillar 1) and has the form

$$\text{Capital Adequacy Ratio} = \frac{\text{Indicator of Absolute Amount of Capital}}{\text{Indicator of Absolute Level of Risk}}.$$

Our study expresses the CAR as

$$\text{Basel CAR } (\rho) = \frac{\text{Value of Bank Capital } (K)}{\text{Value of Risk-Weighted Assets } (a)}. \quad (1.1)$$

In practice, the CAR value, ρ , is allowed to evolve without any constraints on time

until it becomes less than a certain CAR value, ρ^s , that initiates the default process. Usually ρ^s is chosen by the bank's shareholders who generally regard capital adequacy requirements as being indicative of the financial health of the bank. An additional scenario is that an audit may occur and the regulator may decide to close the bank because ρ is below a prescribed value, ρ^r , which is set by the regulator. We note that both the capital thresholds, ρ^s and ρ^r , are exogenously chosen and that invariably $\rho_s \leq \rho_r$. Despite the fact that Basel II gives a precise description of the refinements to the risk-weighted assets (RWAs) to be used in the computation of ρ , it neglects to describe ρ^r . In this regard, Figure 1.1 below (see [41] for more details) provides information about how the Basel CAR from (1.1) and the leverage CAR given by

$$\text{Leverage CAR } (\rho^L) = \frac{\text{Value of Bank Capital } (K)}{\text{Value of Total Assets } (A)}$$

may be categorized in terms of supervisory risk groupings.

| CATEGORIES | ρ | | T1CAR | | ρ^L | | TE |
|---------------------------------------|-------------|-----|-------------|-----|-------------|-----|-------------|
| Well-Capitalized | ≥ 0.1 | and | ≥ 0.06 | and | ≥ 0.06 | | - |
| Adequately Capitalized | ≥ 0.08 | and | ≥ 0.04 | and | ≥ 0.04 | | - |
| Undercapitalized | ≥ 0.06 | and | ≥ 0.03 | and | ≥ 0.03 | | - |
| Significantly Undercapitalized | < 0.06 | or | ≥ 0.03 | or | ≥ 0.03 | and | > 0.02 |
| Critically Undercapitalized | | | | | | | ≤ 0.02 |

Figure 1.1: Categories of Banking Threshold Regulatory Ratios

In Figure 1.1, we have that T1CAR and TE are the abbreviations for the Tier 1 CAR and tangible equity, respectively.

1.1 RELATION TO PREVIOUS LITERATURE

In this section, we briefly comment on selected literature related to bank assets, bank liabilities, bank capital, bank auditing, Lévy processes and Jump diffusions, and optimization.

1.1.1 Bank Assets

The assets we consider include, cash, bonds, shares, loans, Treasuries, reserves and intangible assets. In practice, valuing the off-balance sheet item, intangible assets, constitutes one of the principal difficulties with the process of bank valuation by a stock analyst. The reason for this is that intangibles may be considered to be risky assets for which the future service potential is hard to measure. Despite this, our model assumes that the measurement of these intangibles is possible (see, for instance, [52] and [94]).

1.1.2 Bank Liabilities

According to the paper [37], traditional asset-liability management techniques limit banks' abilities to structure their balance sheets - but more recently, financial innovations have allowed banks the chance to manage interest rate risk without constraining their asset-liability choices. Using canonical correlation analysis, we examine how the relationships between asset and liability accounts at U.S. commercial banks changed between 1990 and 2005. Importantly, we show that asset-liability linkages are weaker for banks that are intensive users of risk-mitigation strategies such as interest rate swaps and adjustable loans. Perhaps surprisingly, we find that asset-liability linkages are stronger at large banks than at small banks, although these size-based differences have diminished over time, both because of increased asset-liability linkages at small banks and decreased linkages at large banks.

1.1.3 Bank Capital

The most important role of capital is to mitigate the moral hazard problem that results from asymmetric information between banks, depositors and borrowers. In the presence of asymmetrical information about the loan loss provisions (LLP), bank managers may be aware of asset quality problems unknown to outside analysts. Provisioning the assets may convey a clearer picture regarding the worth of these assets and precipitate a (negative) market adjustment. In the absence of information asymmetry, there may be no new asset quality information released as a result of the LLP announcement. The Modigliani-Miller theorem forms the basis for modern thinking on capital structure (see [70]). In an efficient market, their basic result states that, in the absence of taxes, insolvency costs and asymmetric information, the bank value is unaffected by how it is financed. In this framework, it does not matter if bank capital is raised by issuing equity or selling debt or what the dividend policy is. By

contrast, in our contribution, in the presence of loan market frictions, the bank value is dependent on its financial structure (see, for instance, [17], [38], [65] and [85] for banking). In this case, it is well-known that the bank's decisions about lending and other issues may be driven by the capital adequacy ratio (CAR) (see, for instance, [34], [35], [71], [84] and [86]). Further evidence of the impact of capital requirements on bank lending activities are provided by [51] and [90]. Also, our model allows for the fact that the banking industry may not be perfectly competitive either because of collusion, interest ceilings on deposits or government subsidies that provide a tax-shield on interest.

A new line of research into credit models for monetary policy has considered the association between bank capital and loan demand and supply (see, for instance, [3], [23], [28], [30], [91], [92] and [93]). This credit channel is commonly known as the *bank capital channel* and propagates that a change in interest rates can affect lending via bank capital. We also discuss the effect of macro-economic activity on a bank's capital structure and lending activities (see, for instance, [50]). With regard to the latter, for instance, there is considerable evidence to suggest that macro-economic conditions impact the probability of default and loss given default on loans (see, for instance, [4], [50] and [59]). Of all the papers mentioned in this paragraph our contribution has the closest connection with [30]. Chapter 2 extends the said paper in six definite directions. Firstly, taking our lead from the requirements of Basel II, by contrast to [30], the risk weight for the assets appearing on and off the balance sheet may vary with time. Furthermore, we include both Treasuries and reserves as part of the provisions for deposit withdrawals whereas the aforementioned paper only discusses the role of Treasuries. Thirdly, we provide substantive evidence of the relationship between the business cycle and provisioning and profitability for OECD countries. Also, we include loan losses and its provisioning as an integral part of our analysis. In the fifth place, we recognize the important role that intangible assets play in determining bank profit. In essence this means that, unlike the aforementioned contributions, we consider both on- and off-balance sheet items in the computation of profit. Finally, we determine the value of a bank subject to capital requirements based on reported Value-at-Risk (VaR) measures, as in the Basel Committee's Internal Models Approach (see, for instance, [2] and [32]).

1.1.4 Bank Auditing

As far as the literature on auditing in the banking industry is concerned, we highlight the contributions made in [19], [42], [43], [65], [69] and [79]. In the former paper, a model of optimal bank default and closure rules with Poisson-distributed audits of

the bank's asset value by the regulator, with the goal of eliminating the incentives of the levered bank shareholders/managers to take excessive risks in their choice of the underlying assets. The roles of (tax or other) subsidies on deposit interest payments by the bank and of the auditing frequency are examined. The article [42] examines alternative ways to prevent losses from bank insolvencies. In particular, this contribution develops a model that compares two alternative institutions for bank auditing. The first is a system of central bank auditing of national banks. The second is carried out by an international agency that collects and disseminates risk information on banks in all countries. The contribution [43] suggests that auditing systems can be effective devices to counteract tendencies for risk-taking associated with bank safety nets. Results are obtained from an international sample of publicly traded banks after controlling for other regulatory control devices for bank risk such as restrictions on banking activities, minimum regulatory capital requirements and official discipline. The efficacy of auditing systems in controlling bank risk diminishes with bank charter value and increases with moral hazard stemming from a country's deposit insurance. The results also indicate that auditing systems are complements for minimum capital requirements, but substitutes for restrictions on bank activities and official discipline. Furthermore, we consider our discussion on audit times and thresholds to be related to [69]. In the contribution [79], the objective is to explore the potentials of developing multicriteria decision aid models for reproducing, as accurately as possible, the auditors opinion on the financial statements of the firms.

1.1.5 Lévy Processes and Jump Diffusions

Our discussion in Chapter 3 extends aspects of the recent article [45] (see, also, [71] and [72]) by generalizing the description of bank behaviour in a continuous-time Brownian motion framework to one in which the dynamics of bank items may have jumps and be driven by Lévy processes. As far as information on these processes is concerned, Protter in [83, Chapter I, Section 4] and Jacod and Shiryaev in [57, Chapter II] are standard texts (see, also, [18] and [87]). Also, the connections between Lévy processes and finance are embellished upon in [88] (see, also, [58] and [60]).

As far as banking regulation is concerned, we note that formal models that enable aspects of Basel II to be analyzed is suggested in [35] and [86]. In fact, when discussing the main issues that arise from the models in this thesis, we make liberal use of the investigations undertaken in these contributions (see, also, [47], [73] and [74]).

1.1.6 Optimization

Several discussions related to discrete-time optimization problems for banks have recently surfaced in the literature (see, for instance, [50], [65], [72] and [84]). Also, some recent papers using dynamic optimization methods in analyzing bank regulatory capital policies include [80] for Basel II and [5], [32] and [64] for Basel market risk capital requirements. In [84], a discrete-time dynamic banking model of imperfect competition is presented, where the bank can invest in a prudent or a gambling asset. For both these options, a maximization problem that involves the bank value for shareholders is formulated. On the other hand, [72] examines a problem related to the optimal risk management of banks in a continuous-time stochastic dynamic setting. In particular, the authors minimize market and capital adequacy risk that involves the safety of the assets held and the stability of sources of capital, respectively (see, also, [75]). Further optimization problems involving banking activities were solved in a broader framework in [46], [47] and [73].

1.2 PRELIMINARIES

In this section, we provide some preliminaries on the basic model of a bank as well as Lévy processes. In the sequel, we use the notational convention "subscript t or s " to represent (possibly) random processes, while "bracket t or s " are used to denote deterministic processes.

1.2.1 Preliminaries about Bank Balance Sheets

The preliminaries in this subsection mainly apply to the discussion in Chapter 2. Throughout, we suppose that $(\Omega, \mathbf{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ is a filtered probability space. As is well-known, the bank balance sheet consists of assets (uses of funds) and liabilities (sources of funds) that are balanced by bank capital (see, for instance [38]) according to the well-known relation

$$\text{Total Assets } (A) = \text{Total Liabilities } (\Gamma) + \text{Total Bank Capital } (K). \quad (1.2)$$

The main on-balance sheet items in (1.2) can specifically be identified as

$$A_t = \Lambda_t^m + W_t + S_t + B_t + C_t, \quad W_t = T_t + R_t; \quad \Gamma_t = \Delta_t; \quad K_t = n_t E_t + O_t + R_t^l,$$

where Λ^m , S , B , C , τ , R , Δ , n , E , O and R^l are the market value of loans, shares, bonds, cash, Treasuries, reserves, outstanding debt, number of shares, market price of the bank's common equity, subordinate debt and loan loss reserves, respectively.

1.2.2 Preliminaries about Lévy Processes

In this subsection, for sake of completion, we firstly provide a general description of a Lévy process and its measure and then describe the Lévy decomposition that we consider.

In this regard, we assume that $\phi(\xi)$ is the characteristic function of a distribution. If for every positive integer n , $\phi(\xi)$ is also the n -th power of a characteristic function, we say that the distribution is *infinitely divisible*. For each infinitely divisible distribution, a stochastic process $L = (L_t)_{0 \leq t}$ called a *Lévy process* exists. This process initiates at zero, has independent and stationary increments and has $(\phi(u))^t$ as a characteristic function for the distribution of an increment over $[s, s+t]$, $0 \leq s, t$, such that $L_{t+s} - L_s$. Next, we provide important definitions and a useful result.

Definition 1.2.1 (Cádlág Stochastic Process): *A stochastic process X is said to be cádlág if it almost surely (a.s.) has sample paths which are right continuous, with left limits.*

Proposition 1.2.2 (Stopping Time): *Let X be an adapted cádlág stochastic process, and let Λ be a closed set. Then the random variable*

$$T(\omega) = \inf\{t > 0 : X_t(\omega) \in \Lambda \text{ or } X_{t-}(\omega) \in \Lambda\}$$

is a stopping time.

Proof. The proof is contained in [83] and will not be shown here. □

Definition 1.2.3 (Random Partition): *Let ς denote a finite sequence of finite stopping times*

$$0 = T_0 \leq T_1 \leq \dots \leq T_k < \infty.$$

The sequence ς is called a random partition.

Every Lévy process is a semi-martingale and has a càdlàg version (right continuous with left hand limits) which is itself a Lévy process. We will assume that the type of such processes that we work with are always càdlàg. As a result, sample paths of L are continuous a.e. from the right and have limits from the left. The *jump of L_t* at $t \geq 0$ is defined by $\Delta L_t = L_t - L_{t-}$. Since L has stationary independent increments its *characteristic function* must have the form

$$\mathbf{E}[\exp\{-i\xi L_t\}] = \exp\{-t\Psi(\xi)\}$$

for some function Ψ called the *Lévy or characteristic exponent* of L . The *Lévy-Khintchine formula* is given by

$$\begin{aligned} \Psi(\xi) = i\gamma\xi + \frac{c^2}{2}\xi^2 + \int_{|x|<1} \left[1 - \exp\{-i\xi x\} - i\xi x \right] \nu(dx) \\ + \int_{|x|\geq 1} \left[1 - \exp\{-i\xi x\} \right] \nu(dx), \quad \gamma, c \in \mathbb{R} \end{aligned} \quad (1.3)$$

and for some σ -finite measure ν on $\mathbb{R} \setminus \{0\}$ with

$$\int \inf\{1, x^2\} \nu(dx) = \int \inf(1 \wedge x^2) \nu(dx) < \infty.$$

An infinitely divisible distribution has a *Lévy triplet* of the form

$$[\gamma, c^2, \nu(dx)]$$

where the measure ν is called the *Lévy measure*.

The Lévy-Khintchine formula given by (1.3), is closely related to the Lévy process, L . This is particularly true for the *Lévy decomposition* of L which we specify in the rest of this paragraph. From (1.3), it is clear that L must be a linear combination of a Brownian motion and a quadratic jump process X which is independent of the Brownian motion. We recall that a process is classified as *quadratic pure jump* if the continuous part of its quadratic variation $\langle X \rangle^c \equiv 0$, so that its quadratic variation becomes

$$\langle X \rangle_t = \sum_{0 < s \leq t} (\Delta X_s)^2,$$

where $\Delta X_s = X_s - X_{s-}$ is the jump size at time s . If we separate the Brownian component, Z , from the quadratic pure jump component X we obtain

$$L_t = X_t + cZ_t,$$

where X is quadratic pure jump and Z is standard Brownian motion on \mathbb{R} . Next, we describe the Lévy decomposition of Z . Let $Q(dt, dx)$ be the Poisson measure on $\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$ with expectation (or intensity) measure $dt \times \nu$. Here dt is the Lebesgue measure and ν is the Lévy measure as before. The measure $dt \times \nu$ (or sometimes just ν) is called the compensator of Q . The Lévy decomposition of X specifies that

$$\begin{aligned} X_t &= \int_{|x| < 1} x \left[Q((0, t], dx) - t\nu(dx) \right] + \int_{|x| \geq 1} x Q((0, t], dx) + t\mathbf{E} \left[X_1 - \int_{|x| \geq 1} x\nu(dx) \right] \\ &= \int_{|x| < 1} x \left[Q((0, t], dx) - t\nu(dx) \right] + \int_{|x| \geq 1} x Q((0, t], dx) + \gamma t, \end{aligned} \quad (1.4)$$

where

$$\gamma = \mathbf{E} \left[X_1 - \int_{|x| \geq 1} x\nu(dx) \right].$$

The parameter γ is known as the *drift of X* . In addition, in order to describe the Lévy decomposition of L , we specify more conditions that L must satisfy. The most important supposition that we make about L is that

$$\mathbf{E}[\exp\{-hL_1\}] < \infty, \quad \text{for all } h \in (-h_1, h_2), \quad (1.5)$$

where $0 < h_1, h_2 \leq \infty$. This implies that L_t has finite moments of all orders and in particular, $\mathbf{E}[X_1] < \infty$. In terms of the Lévy measure ν of X , we have, for all $h \in (-h_1, h_2)$, that

$$\int_{|x| \geq 1} \exp\{-hx\} \nu(dx) < \infty;$$

$$\int_{|x| \geq 1} x^\alpha \exp\{-hx\} \nu(dx) < \infty, \quad \forall \alpha > 0;$$

$$\int_{|x| \geq 1} x \nu(dx) < \infty.$$

The above assumptions lead to the fact that (1.4) can be rewritten as

$$X_t = \int_{\mathbf{R}} x \left[Q((0, t], dx) - t\nu(dx) \right] + t\mathbf{E}[X_1] = M_t + at,$$

where we have that

$$M_t = \int_{\mathbf{R}} x \left[Q((0, t], dx) - t\nu(dx) \right]$$

is a martingale and $a = \mathbf{E}[X_1]$.

In the specification of our model, we assume that the Lévy measure $\nu(dx)$ of L satisfies

$$\int_{|x| > 1} |x|^3 \nu(dx) < \infty. \quad (1.6)$$

As in the above, this allows a decomposition of L of the form

$$L_t = cZ_t + M_t + at, \quad 0 \leq t \leq T, \quad (1.7)$$

where $(cZ_t)_{0 \leq t \leq \tau}$ is a Brownian motion with standard deviation $c > 0$, $a = \mathbf{E}(L_1)$ and the martingale

$$M_t = \int_0^t \int_{\mathbf{R}} x M(ds, dx), \quad 0 \leq t \leq T,$$

is a square-integrable. Here, we denote the *compensated Poisson random measure* on $[0, \infty) \times \mathbb{R} \setminus \{0\}$ related to L by $M(dt, dx)$. Subsequently, if $\nu = 0$ then we will have that $L_t = Z_t$, where Z_t is appropriately defined Brownian motion.

1.2.3 Preliminaries about Jump Diffusions

Our contribution concentrates on the dynamics of banking items with jump diffusions. Such processes have an advantage over the more traditional modelling tools such as Brownian motion in that they allow for the non-continuous evolution of the value of such items. In order to formalize this notion, we suppose for $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that $(\Omega, \mathbf{F}, \mathbb{F}, \mathbf{P})$ is a filtered probability space. Also, we recall that an \mathcal{F}_t -adapted process $\{L_t\}_{t \geq 0} \subset \mathbb{R}$ with $L_0 = 0$ a.s. is known as a *Lévy process* if L_t is continuous in probability and has stationary, independent increments. Furthermore, we let \mathbf{B}_0 be the family of Borel sets $\mathcal{U} \subset \mathbb{R}$ with $0 \notin \mathcal{U}$. In addition, for $\mathcal{U} \subset \mathbf{B}_0$, we define the *Poisson random or jump measure* of L as

$$N(t, \mathcal{U}) = N(t, \mathcal{U}, \omega) = \sum_{0 < s \leq t} \mathcal{X}_{\mathcal{U}}(\Delta L_t).$$

This means that $N(t, \mathcal{U})$ is the number of jumps of size $\Delta L_t \in \mathcal{U}$ which occur before or at time t . The differential form of $N(t, \mathcal{U})$ is $N(dt, dx)$. The set function

$$\nu(\mathcal{U}) = \mathbf{E}[N(1, \mathcal{U})],$$

where $\mathbf{E} = \mathbf{E}^{\mathbf{P}}$ denotes expectation with respect to \mathbf{P} , defines a σ -finite measure on \mathbf{B}_0 called the *Lévy measure* of $\{L_t\}_{t \geq 0}$. Finally, the *compensated Poisson random measure* of $\{L_t\}_{t \geq 0}$ is given by

$$\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt. \tag{1.8}$$

Subsequently, if $\nu = 0$ then we will have that $L_t = Z_t$, where Z_t is appropriately defined Brownian motion.

By considering the discussion above, we may assume that the bank's unlevered asset value, A , has Lévy-process driven dynamics of the form

$$dA_t = A_t \left\{ \left[\mu(A_t, t) - f_t \right] dt + \sigma_t dZ_t \right\} + A_{t-} \int_{-1}^{\infty} x \tilde{N}(dt, dx), \quad (1.9)$$

where $\mu(A_t, t)$ is the total expected rate of return on A , f is the fraction of A paid to shareholders, i.e., net cash outflow, σ is the proportional volatility of A per unit time and dZ is the increment of a standard Brownian motion.

In the case of the random audit process, A , the regulator's audits of the bank's asset value, A , may be assumed to be stochastic and follow a Poisson process where the mean number of audits per time unit is denoted by m . This may be expressed in a formal way via the stochastic process

$$dA = dv, \quad A = \begin{cases} 1, & \text{Audit Occurs;} \\ 0, & \text{Otherwise.} \end{cases}$$

Note that under the above assumptions the probability that a random audit occurs in the time interval dt is mdt . On the other hand, the probability of no audit is $1 - mdt$ and the probability of more than one random audit is of the order $o(dt)$. Furthermore, we suppose that the two stochastic processes dZ and dv are independent. This description of the random audit has particular relevance for the analysis of the bank default and closure thresholds, A^s and A^r , respectively. This procedure is via unlevered assets that are exogenously set by the shareholders and regulators. In our contribution, it is of lesser importance for the random auditing of reserve requirements.

In the discussion on capital requirements, we cater for the possibility that the bank is able to choose the time at which a non-random audit takes place. Although this seems to violate the principles on which the (external) auditing process is based, we make this assumption to cater for the situation that may arise for internal audits. It may be that a worst stopping time analysis is more appropriate in this case. However, that approach is beyond the scope of this thesis but would make for an interesting topic for future research.

1.3 OUTLINE OF THE THESIS

The current chapter is introductory in nature.

1.3.1 Outline of Chapter 2

Chapter 2 describes a stochastic model for banks. In Section 2.1, we discuss related items such as cash, bonds, shares, loans, Treasuries, reserves, intangible assets, total unweighted assets and risk-weighted assets. Next, in Section 2.2 we discuss total bank capital, binding capital constraints and capital adequacy ratios. Section 2.3, the only liability that we consider is deposits as it pertains to outstanding debt. In the final section of Chapter 2 we describe the bank assets-to-outstanding debt model that are important for the ensuing analysis.

1.3.2 Outline of Chapter 3

Chapter 3 describes the optimal auditing in the banking industry. In Section 3.1, we discuss the random auditing reserve requirements. The main optimization result for bank reserve allocation and depository consumption is stated in Theorem 3.1.2. In Section 3.2 we discuss the non-random auditing capital requirements. In Theorem 3.2.2 we derive a solution for the optimal auditing time problem.

1.3.3 Outline of Chapter 4

In Chapter 4 we discuss stochastic control of banking systems. Section 4.1 describes the optimal stochastic control for the simplified auditing model.

1.3.4 Outline of Chapter 5

Chapter 5 we consider numerical and illustrative examples for the asset-to-outstanding debt ratio.

1.3.5 Outline of Chapter 6

Section 6.2 contains the analysis of the auditing process with emphasis on reserve, asset and capital requirements. In this regard, one of the main novelties of the thesis is encapsulated by Theorem 3.1.2 in Section 3.1, where the optimal proportion of bank reserves and rate of depository consumption is determined via Lévy processes. In this case, the *rate of depository consumption* is defined as the rate at which Treasuries and reserves are consumed by the taking, holding and anticipated withdrawal of deposits. Typically the bank owner has to make decisions about deposit taking via the fixing of costs related to cheque clearing and bookkeeping, the holding of deposits

by means of the choice of the deposit and coupon rate and anticipated withdrawals via the provisioning provided by Treasuries and reserves. Examples of anticipated withdrawals are stop orders, anticipated living expenses and certain payments. Also, the specific choice of a power utility function for the bank owner is made in order to obtain an analytic solution. Theorem 3.2.2 in Subsection 3.2.2 solves an optimal (non-random) auditing problem in terms of the CAR in a Lévy process setting. This result provides information about the optimal timing of an audit by the regulator when the ambient value of the CAR is taken into account and the bank is able to choose the time at which the audit takes place. In Section 6.1, we analyze the main economic issues arising from the stochastic banking model and the auditing process.

1.3.6 Outline of Chapter 7

Chapter 7 contains the conclusion that we can draw from the study. We also point out what further research problems may be addressed by future students.

1.3.7 Outline of Chapter 8

The bibliography in Chapter 8 contains all the articles, books and other sources used throughout the thesis.

1.3.8 Outline of Chapter 9

In the appendix in Chapter 9 results that are central to the analysis in this thesis is discussed.

Chapter 2

STOCHASTIC MODELS FOR BANKS

2.1 ASSETS

2.1.1 Shares, Bonds and Cash

2.1.1.1 Shares

2.1.1.2 Bonds

2.1.1.3 Cash

2.1.2 Loans

2.1.3 Treasuries and Reserves

2.1.4 Intangible Assets

2.1.5 Total Unweighted Bank Assets

2.1.6 Risk-Weighted Bank Assets

2.2 BANK CAPITAL

2.2.1 Total Bank Capital

2.2.2 Binding Capital Constraints

2.2.3 Capital Adequacy Ratios

2.3 LIABILITIES

2.3.1 Deposits

2.4 BANK ASSETS-TO-OUTSTANDING DEBT MODEL

2.4.1 Description of the Bank Assets-to-Outstanding Debt Model

2.4.2 Description of the Simplified Bank Assets-to-Outstanding Debt Model

The Basel II capital accord allows us to construct a continuous-time stochastic dynamic model that consists of assets, A , (uses of funds), liabilities, Γ , (sources of funds) and bank capital, K_t , (see, for instance [38]).

The main problem investigated in this chapter can be stated as follows.

Problem 2.0.1 (Stochastic Modeling of Banking Activities): *Can we describe and deduce models for bank shares, bonds, cash, loans, Treasuries, reserves, intangible assets, total unweighted assets and risk-weighted assets ?*

2.1 ASSETS

In this section, the bank assets that we discuss are shares, bonds, cash, loans, Treasuries, reserves, total unweighted assets and risk-weighted assets. In the sequel, we suppose that $(\Omega, \mathbf{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ is a filtered probability space.

2.1.1 Shares, Bonds and Cash

The first three classes of banking assets are shares, bonds and cash. Because they have different characteristics and tend to perform differently as economic conditions vary, each one can be an important part of a balanced portfolio for banks. Diversification among these asset classes, or asset allocation, is one of the best methods available for making the bank's investment strategy work. The bank's circumstances usually change over time, so that a periodic review of their financial strategy becomes necessary.

2.1.1.1 Shares

Of the three asset classes, shares, S , have historically been the most powerful performers over the long term. Since the returns from shares usually exceed the returns from both bonds and cash and have significantly outpaced inflation, they are important to a portfolio for growth of capital over time. Over the short term, however, shares can be volatile and as a result there is regulation related to banks holding shares.

2.1.1.2 Bonds

Whereas shares represent equity, or part ownership of the companies that issue them, bonds, B , represent debt. Municipalities and governments all use bonds as a way to

raise cash. When banks buy bonds, they are lending money to the issuer in exchange for fixed interest payments over a set number of years and a promise to pay the original amount back in the future. Bonds are valuable to banks more for the income they provide than for growth potential. Since the income they pay is fixed it is generally reliable and steady. The primary risk in bond market investing comes from interest rate changes. When interest rates rise, a bond's market value decreases. Another potential risk of owning bonds is default, which can occur when the bond issuer is no longer able either to pay the interest or repay the principal. The latter is negated by the fact that banks mainly buy government and municipal bonds with a very small likelihood of default.

2.1.1.3 Cash

Cash, C , is a term assigned to very short-term savings instruments such as money market securities. These investments can be used to meet near-term financial needs or to protect a portion of an investment portfolio from price fluctuation. The downside of cash securities is that they offer no real opportunities for long-term growth. Though economic conditions and factors such as changing interest rates can impact both stocks and bonds, these markets perform independently of each other and can therefore serve as a balance within the portfolio of a bank.

2.1.2 Loans

We suppose that, after providing liquidity, the bank grants loans at the *interest rate on loans* or *loan rate*, r_t^Λ . Profit maximizing banks set their loan rates, r_t^Λ , as a sum of the risk-free rate, r_t , the *expected loan loss ratio*, $\mathbf{E}(d)$, and the *risk premium*, u . We express r_t as a function of the risk-free interest rates, i.e.,

$$r_t = f(r_t^C, r_t^B, r_t^T, r_t^R),$$

where r_t^T and r_t^R are the rates of return of the Treasuries and reserves, respectively. Furthermore, expressing the expected losses, $\mathbf{E}(d)$, as a rate of return per unit time, we obtain the expression

$$r_t^\Lambda = r_t + u + \mathbf{E}(d).$$

The sum $r_t + u$ provides the remuneration for the cost of monitoring and screening of loans and of capital, c^Λ . The $\mathbf{E}(d)$ component is the amount of provisioning that is needed to match the average expected losses faced by the loans. The representation of the bank's interest setting shows that banks will experience positive returns in good times when the *actual rate of default*, r^d , is lower than the provisioning for expected losses, $\mathbf{E}(d)$, and may not be able to cover their expected losses when $r^d > \mathbf{E}(d)$. In the latter case, bank capital may be needed to cover these excess (and unexpected) losses. If this capital is not enough then the bank will face insolvency.

In this paragraph, we provide a brief discussion of loan demand and supply. Firstly, we introduce the generic variable, M_t , that represents the level of macroeconomic activity in the bank's loan market. We suppose that the macroeconomic process, $M = \{M_t\}_{t \geq 0}$, follows the Lévy process

$$dM_t = M_t \left\{ \mu_t^M dt + \sigma_t^M dZ_t^M \right\} + M_{t-} \int_{-1}^{\infty} x^M \tilde{N}^M(dt, dx^M),$$

where σ_t^M and Z_t^M denote volatility in macroeconomic activity and the Brownian motion driving the macroeconomic activity, respectively. In fact, in the sequel, the actual default rate, r^d , can be considered to be dependent on macroeconomic conditions, M , and be denoted by $r^d(M_t)$.

Taking our lead from the equilibrium arguments in [91], we denote both the supply and demand credit price processes by $\Lambda = \{\Lambda_t\}_{t \geq 0}$. In this regard, the bank faces a *Hicksian demand for loans* given by

$$\Lambda_t = l_0 - l_1 \int_0^t r_s^\Lambda ds + \int_0^t \sigma_s^\Lambda dZ_s^\Lambda + l_2 M_t, \quad (2.1)$$

where σ_t^Λ and Z_t^Λ denote volatility in the loan demand and the Brownian motion driving the demand for loans (which may be correlated with the macroeconomic activity), respectively. We note that the loan demand in (2.1) is an increasing function of M and a decreasing (increasing) function of $\int_0^t r_s^\Lambda ds > 0$ (< 0). Also, we assume that the *price process of the loan supply*, $\Lambda = (\Lambda)_{t \geq 0}$, follows the geometric Lévy process

$$d\Lambda_t = \Lambda_t \left\{ \mu_t^\Lambda dt + \sigma_t^\Lambda dZ_t^\Lambda \right\} + \Lambda_{t-} \int_{-1}^{\infty} x^\Lambda \tilde{N}^\Lambda(dt, dx^\Lambda) - f_t A_t dt, \quad (2.2)$$

where f_t is the fraction of A paid to shareholders, $\mu_t^\Lambda = r_t^\Lambda - c^\Lambda - r^d(M_t)$, $\sigma_t > 0$ denotes the volatility in the loan supply and Z_t is a standard Brownian motion with respect to a filtration, $(\mathcal{F}_t)_{t \geq 0}$, of the probability space $(\Omega, \mathbf{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. The *value of the bank's investment in loans*, λ , at t is expressed as

$$\lambda_t = n_t^\Lambda \Lambda_t, \quad (2.3)$$

where n_t^Λ is the number of loans at t . For sake of argument, in the sequel, we assume that $n_t^\Lambda = n^\Lambda = 1$ in (2.3) so that $\lambda_t = \Lambda_t$, for all t .

2.1.3 Treasuries and Reserves

Treasury securities, T , are bonds issued by national Treasuries. They are the debt financing instruments of the federal government, and are often referred to as "Treasuries." There are four types of Treasuries, viz., Treasury bills, Treasury notes, Treasury bonds and savings bonds. All of the Treasuries besides savings bonds are very liquid and are heavily traded on the secondary market. We denote the *interest rate on Treasuries* or *Treasury rate* by r_t^T and assume that for all t we have

$$\mu_t^\Lambda = r_t^\Lambda - c^\Lambda - r^d(M_t) > r_t^T.$$

Bank reserves are the deposits held in accounts with a national agency (for instance, the Federal Reserve for banks) plus money that is physically held by banks (vault cash). Such reserves are constituted by money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits will be needed as reserves. As a result of this description, we may introduce a reserve-deposit ratio, γ , for which

$$R_t = \gamma \Delta_t.$$

The bank uses the remaining deposits to earn profit, either by issuing loans or by investing in assets such as Treasuries and stocks. The individual rationality constraint implies that reserves may implicitly earn at least their opportunity cost through certain bank operations and Federal government subsidies. For instance, members

of the Federal Reserve in the United States may earn a return on required reserves through government debt trading, foreign exchange trading, other Federal Reserve payment systems and affinity relationships (outsourcing) between large and small banks. We note that vault cash in the Automated Teller Machines (ATMs) network also qualifies as required reserves. Also, the investment of bank reserves in the market via bonds and stocks is still possible in many countries. The conclusion is that reserves actually have a stochastic nature and banks may earn a positive return on them.

In the sequel, we take the above discussion into account when assuming that the dynamics of the Treasuries and reserves are described by

$$\begin{aligned} dT_t &= r^T T_t dt, \quad T_0 = b > 0; \\ dR_t &= R_t \left\{ \mu_t^R dt + \sigma_t^R dZ_t \right\} dt + R_t \left\{ \int_{-1}^{\infty} x^R \tilde{N}^R(dt, dx^R) \right\}, \quad R_0 = r > 0, \end{aligned} \quad (2.4)$$

respectively. Here μ^R is the rate of (positive) return earned by the bank and σ^R is the volatility in the level of reserves. In order to have $R_t > 0$, we assume that $\sigma^R \Delta R_t > -1$ a.s. for all t . Next, we suppose that (2.4) holds and π is the (possibly time-dependent) proportion of the sum of Treasuries and reserves that is held in reserves. Taking our lead from (2.4), we assume that the stochastic dynamics of the sum of Treasuries and reserves, W , is given by

$$\begin{aligned} dW_t &= W_t \left[\left(r_t^T + \pi_t (\mu_t^R - r_t^T) \right) dt + \pi_t \sigma_t^R dZ_t^R \right] - k(t) dt + \pi_t W_{t-} \int_{-1}^{\infty} x^R \tilde{N}^R(dt, dx^R); \\ W_{0-} &= b + r = w \geq 0, \quad W_t = W_t^u = T_t^u + R_t^u \geq 0, \quad \text{for all } t \geq 0, \end{aligned} \quad (2.5)$$

where the *rate of depository consumption*, $k(t)$, is the rate at which Treasuries and reserves are consumed by anticipated deposit withdrawals. Moreover, the solution of (2.5) may be given by

$$\begin{aligned} W_t &= W_0 \exp \left\{ \left[\mu_t^R - \frac{1}{2} (\sigma_t^R)^2 \right] t + \sigma_t^R Z_t + \int_0^t \int_{|x^R| < \infty} \left\{ \ln(1 + x^R) - x^R \right\} \nu(dx^R) ds \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} \ln(1 + x^R) \tilde{N}^R(ds, dx^R) \right\}. \end{aligned}$$

The generator G^w of the controlled process

$$Q_t = \begin{bmatrix} s+t \\ W_t \end{bmatrix}; \quad t \geq 0, \quad Q_{0^-} = q = \begin{bmatrix} s \\ w \end{bmatrix}$$

is given by

$$\begin{aligned} G^w \varphi(q) &= \frac{\partial \varphi}{\partial s} + \left([r_t^T(1-\pi) + \mu_t^R \pi]w - k \right) \frac{\partial \varphi}{\partial w} + \frac{1}{2} (\sigma^R)^2 \pi^2 w^2 \frac{\partial^2 \varphi}{\partial w^2} \\ &+ \int_{-1}^{\infty} \left\{ \varphi(s, w + \pi w x^R) - \varphi(s, w) - \pi w x^R \frac{\partial \varphi}{\partial w}(s, w) \right\} \nu(dx^R). \end{aligned} \quad (2.6)$$

2.1.4 Intangible Assets

In the contemporary banking industry, shareholder value is often created by *intangible assets* which consist of patents, trademarks, brand names, franchises and economic goodwill (more specifically, core deposit customer relationships, customer loan relationships as differentiated from the loans themselves, etc.). Economic goodwill consists of the intangible advantages a bank has over its competitors such as an excellent reputation, strategic location, business connections, etc. In addition, such assets can comprise a large part of the bank's total assets and provide a sustainable source of wealth creation. Intangible assets are used to compute Tier 1 bank capital and have a risk weight of 100 % according to Basel II regulation (see Table 2.1 below). As we mentioned in Chapter 1, we denote the value of intangible assets in the planning period by I_t and the return on these assets by $r_t^I I_t$.

2.1.5 Total Unweighted Assets

Suppose that the total unweighted assets, A , whose dynamics is given by (1.9), is a function, g , of Λ , S , B , C , I and W characterized by (2.2) and (2.5), respectively, i.e.,

$$A_t = g(\Lambda_t, S, B, C, I, W_t).$$

We note that in Chapter 3 and related discussions (in the jump diffusion paradigm), we do not consider shares, bonds, cash and intangible assets to be constituent parts of the total unweighted assets, i.e.,

$$A_t = g(\Lambda_t, W_t).$$

In the latter case, for $\pi'_t = \frac{W_t}{A_t}$, the dynamics of A may be represented by

$$dA_t = A_t \left\{ \left[\mu(A_t, t) - f_t \right] dt + \sigma_t dZ_t \right\} + A_{t-} \int_{-1}^{\infty} x \tilde{N}(dt, dx), \quad (2.7)$$

where

$$\begin{aligned} \mu(A_t, t) &= (1 - \pi'_t) \mu_t^\Lambda + \pi'_t \left(r_t^T + \pi_t (\mu_t^R - r_t^T) - k(t) \right); \\ \sigma_t dZ_t &= (1 - \pi'_t) \sigma_t^\Lambda dZ_t^\Lambda + \pi'_t \pi_t \sigma_t^R dZ_t^R; \\ \int_{-1}^{\infty} x \tilde{N}(dt, dx) &= \pi_t \pi'_{t-} \int_{-1}^{\infty} x^R \tilde{N}^R(dt, dx^R) + (1 - \pi'_{t-}) \int_{-1}^{\infty} x^\Lambda \tilde{N}^\Lambda(dt, dx^\Lambda). \end{aligned}$$

We note that the standard Brownian motion, Z , and the compensated Poisson random measure, \tilde{N} , allows for a possible correlation between bank loan value, Λ_t , and the sum of Treasuries and reserves, W .

2.1.6 Risk-Weighted Assets

We consider risk-weighted assets (RWAs) that are defined by placing each on- and off-balance sheet item into a risk category. The more risky assets are assigned a larger weight. Table 2.1 below provides a few illustrative risk categories, their risk-weights and representative items.

As a result of the above, RWAs are a weighted sum of the various assets of the banks. In the sequel, we denote the risk-weights on loans, Treasuries and reserves by ω^Λ , ω^T and ω^R , respectively. With regard to loans, we can identify a special risk-weight denoted by $\omega(M_t) = \omega^\lambda$ that is a decreasing function of current macroeconomic conditions, i.e.,

$$\frac{\partial \omega(M_t)}{\partial M_t} < 0.$$

| Risk Category | Risk-Weight | Banking Items |
|---------------|-------------|-------------------------|
| 1 | 0 % | Cash, Reserves, Bonds |
| 2 | 20 % | Shares |
| 3 | 50 % | Home Loans |
| 4 | 100 % | Intangible Assets |
| 5 | 100 % | Loans to Private Agents |

Table 2.1: Risk Categories, Risk-Weights and Representative Items

This is in line with the procyclical notion that during booms, when macroeconomic activity increases, the risk-weights will decrease. On the other hand, during recessions, risk-weights may increase because of an elevated probability of default (PD) and/or loss given default (LGD) on loans (see, for instance, [22] and [29]).

2.2 BANK CAPITAL

In this subsection, we discuss total bank capital, binding capital constraints and capital adequacy ratios.

2.2.1 Total Bank Capital

In this subsection, we discuss total bank capital, K . It has the form

$$K_t = K_t^{T1} + K_t^{T2} + K_t^{T3}, \quad (2.8)$$

where K_t^{T1} , K_t^{T2} and K_t^{T3} are Tier 1, Tier 2 and Tier 3 capital, respectively. Tier 1 (T1) capital mainly consists of the book value of bank equity. In our contribution, Tier 1 capital is represented at t^- 's market value of the bank equity, $n_t E_t^-$, where n_t is the number of shares and E_t is the market price of the bank's common equity at t . Tier 2 (T2) and Tier 3 (T3) capital consists of preferred stock and short- and long-term subordinate debt (collectively known as supplementary capital). We recall that short- and long-term subordinate debt are constituents of Tier 2 and Tier 3 capital, respectively. Subordinate debt is the subordinate to deposits and hence faces greater default risk. Tier 2 capital, O_t , issued at t^- is represented by bonds that pay an interest rate, r^o , (see [3]).

2.2.2 Binding Capital Constraints

To reflect the book value property of regulatory capital and its market valuation sensitivity, we assume that at t^- , the market value of equity and Treasuries determines the capital constraint to which the bank is subjected at t . While there are several such constraints associated with Basel II, it is easy to show that the binding one is the total capital constraint. This constraint requires

$$\rho_t = \frac{K_t}{a_t} \geq 0.08.$$

For the regulatory ratio of total capital to the weighted sum of loans, Treasuries and reserves denoted as, $\bar{\rho}^r$, a capital constraint may be represented by

$$\bar{\rho}^r \left[\omega^\lambda \lambda_t + \omega^T T_t + \omega^R R_t \right] \leq n_t E_{t^-} + O_t. \quad (2.9)$$

As a result of (2.9), it is not necessary to differentiate between the relative cost of raising debt versus equity. Moreover, when maximizing profits, we consider the regulatory ratio of total capital to risk-weighted loans, ρ^r , as an appropriate capital constraint. This means that we may set $\omega^\lambda = \omega(M_t)$ and $\omega^T = \omega^R = 0$ in (2.9) and express the binding capital constraint as

$$\rho^r \omega(M_t) \lambda_t \leq n_t E_{t^-} + O_t. \quad (2.10)$$

The exact value of the regulatory ratio, ρ^r , may vary quite considerably from institution to institution (see, for instance, [85] and [86]). In fact, subject to an appropriate choice for ρ^r , some banks may consider that equality in (2.10) implies an optimal choice of the investment in loans, λ , so that

$$\lambda_t^* = \frac{n_t E_{t^-} + O_t}{\rho^r \omega(M_t)}.$$

2.2.3 Capital Adequacy Ratios

In this subsection, we discuss the dynamics of the capital adequacy ratio and regulatory thresholds for CARs. In the absence of a reliable procedure for determining $d\rho_t$ from ρ_t and $d\Lambda_t$ given by (1.1) and (2.2), respectively, we suppose that the value

process ρ_t at time t of the CAR is a geometric Lévy process with dynamics

$$\begin{aligned} d\rho_t = & \rho_t \left\{ \mu^\rho dt + \sigma^\rho dZ_t \right\} \\ & + \zeta \rho_t \int_{\mathbb{R}} x^\rho \tilde{N}^\rho(dt, dx^\rho), \quad \rho_0 = z > 0, \end{aligned} \quad (2.11)$$

where ζ is a constant such that $\zeta x^\rho > -1$ a.s. ν . An explicit solution of the SDE (2.11) has the form

$$\begin{aligned} \rho_t = & \rho_0 \exp \left\{ \left[\mu^\rho - \frac{1}{2}(\sigma^\rho)^2 - \zeta \int_{\mathbb{R}} x^\rho \nu(dx^\rho) \right] t \right. \\ & \left. + \int_0^t \int_{\mathbb{R}} \ln(1 + \zeta x^\rho) N^\rho(dt, dx^\rho) + \sigma^\rho dZ_t \right\}. \end{aligned}$$

As a rule, banks strive to maintain economic capital in excess of a regulatory minimum, ρ^r , with supervisors intervening otherwise. Basel II gives a precise description of the TRWAs to be used in the computation of the CAR without providing a description of the associated thresholds. Subsequently, we discuss a CAR threshold, ρ^r , in the context of supervisory auditing or review. If the bank undergoes such an audit at time $s + \tau$ the expected discounted difference between the actual CAR, ρ , and its regulatory threshold, ρ^r , is given by the performance criterion

$$J_\tau(s, z) := \mathbf{E}^{s,z} \left[\exp\{-\delta(s + \tau)\} \left(\rho_\tau - \rho^r \right) \cdot \mathcal{X}_{\tau < \infty} \right], \quad (2.12)$$

where $\delta > 0$ is the discounting exponent. In situations where the value of ρ is smaller than a regulatory threshold, ρ^r , regulators may pressure banks to increase the value of their CARs. This process may involve the withdrawal of insurance coverage, cease-and-desist orders, limits on asset growth and brokered deposits, prohibition of dividend payments and even bank closure. In the USA, the prompt corrective action feature of the Federal Deposit Insurance Corporation Improvement Act (FDICIA) was implemented to improve capital-based incentives by making some of the aforementioned regulatory actions mandatory when CARs fall into certain capitalization categories (refer to the risk-based capital categories and supervisory risk subgroups in Table 2.1). Furthermore, it is possible to find a fixed open set $\mathcal{C} \subset \mathbb{R} \times (0, \infty)$ that represents a region in which the bank is well or adequately capitalized and is known

as the *adequately capitalized category*. In the spirit of the discussion above, we can consider the set

$$\tau_{\mathcal{C}} = \tau_{\mathcal{C}}(z, \omega) = \inf \left\{ t > 0 : (t, \rho_t) \notin \mathcal{C} \right\}$$

to be the time instant at which $\rho_t = \rho^r$. In this case, for

$$X_t = \begin{bmatrix} s + t \\ \rho_t \end{bmatrix}$$

we have that

$$dX_t = \begin{bmatrix} 1 \\ \mu^{\rho} \rho_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma^{\rho} \rho_t \end{bmatrix} dZ_t + \begin{bmatrix} 0 \\ \zeta \rho_{t-} \int_{\mathbf{R}} x^{\rho} \tilde{N}^{\rho}(dt, dx^{\rho}) \end{bmatrix}, \quad X_0 = \begin{bmatrix} s \\ z \end{bmatrix}.$$

Also, the generator G^z of X_t is

$$\begin{aligned} G^z \varphi(s, z) &= \frac{\partial \varphi}{\partial s} + \varrho z \frac{\partial \varphi}{\partial z} + \frac{1}{2} \zeta^2 z^2 \frac{\partial^2 \varphi}{\partial z^2} \\ &\quad + \int_{\mathbf{R}} \left\{ \varphi(s, z + \zeta z x^{\rho}) - \varphi(s, z) - \zeta z x^{\rho} \frac{\partial \varphi}{\partial z} \right\} \nu(dx^{\rho}). \end{aligned}$$

2.3 LIABILITIES

In our study, the only liability that we consider is deposits as it pertains to outstanding debt. In this case, the discussion on bank debt follows [19] (see, also, [65] and [69]) rather closely. For A given by (2.7), we assume that $Z_t = L_t$ in a time-invariant framework, where $A_t = A$, $\alpha_t = \alpha$, $f_t = f$, $\sigma_t = \sigma$ and $Z_t = Z$ (see, for instance, [19], [65] and [69]). In essence this means that (2.7) becomes

$$dA = A \left\{ \left[\mu(A, t) - f \right] dt + \sigma dZ \right\}, \quad (2.13)$$

where

$$\begin{aligned}\mu(A, t) &= (1 - \pi')\mu^\Lambda + \pi' \left(r^T + \pi(\mu^R - r^T) - k(t) \right); \\ \sigma dZ &= (1 - \pi')\sigma^\Lambda dZ^\Lambda + \pi' \pi \sigma^R dZ^R.\end{aligned}$$

2.3.1 Deposits

Outstanding debt, Δ , can be taken to be the sum of deposits held (primary debt) by the bank as well as short- and long-term subordinate debt.

2.4 BANK ASSETS-TO-OUTSTANDING DEBT MODEL

In this section we describe aspects of the Bank that are important for the ensuing analysis. We show that concepts related to the Bank such as rates of deposit withdrawals, investment returns and increase of outstanding debt before deposit withdrawals may be modelled as random variables that are driven by an associated Brownian motion or Wiener process and may be expressed explicitly in terms of their respective rate values and a diffusion term.

2.4.1 Description of the Bank Assets-to-Outstanding Debt Model

Throughout we assume that we are working with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on a time index set $T = [t_0, t_1]$. Furthermore, we are able to produce a system of stochastic differential equations that provide information about the *bank's total assets at time t* with $A : \Omega \times T \rightarrow \mathbb{R}$ denoted by A_t and *bank's outstanding debt at time t* with $\Delta : \Omega \times T \rightarrow \mathbb{R}$ denoted by Δ_t and their relationship. Responsibility for the assets of a bank is usually borne by a group of bank managers and/or trustees who must have the best interests of the bank members at heart. They choose how the bank assets are invested with the particular investment strategy depending on such factors as tax status, maximization of returns, minimization of risk, diversification, security, avoidance of self-investment, and cashflow specifications. The dynamics of the bank's total assets, A_t , is stochastic in nature because it depends in part on the stochastic rates of return of the investments of the bank. Also, the dynamics of the bank's outstanding debt, Δ_t , is stochastic because its value has a reliance on the outstanding debt cash flows and

asset values that both have randomness associated with them. The bank's total assets A_t may be described as the amount that remains when past deposit withdrawals are deducted from risk premiums and investment returns. Furthermore, for $x : \Omega \times T \rightarrow \mathbb{R}^2$ we use the notation x_t to denote

$$x_t = \begin{bmatrix} A_t \\ \Delta_t \end{bmatrix} \quad (2.14)$$

and represent the *total assets-to- outstanding debt ratio* with $s : \Omega \times T \rightarrow \mathbb{R}$ denoted by s_t , of the bank as

$$s_t = A_t / \Delta_t. \quad (2.15)$$

It is important for bank solvency that s_t has to maintain a high value. Obviously, low values of s_t indicate that the bank is struggling to stay solvent.

In the main, the bank is free to choose how the risk premium rate can be varied. The underlying principle governing this decision is that the amount of surplus or deficit has to be taken into account. Roughly speaking, the rate of risk premiums can be reduced during times of surplus and should be increased beyond the normal rate when the bank is in deficit. In the sequel, the stochastic process $u^1 : \Omega \times T \rightarrow \mathbb{R}$ is the *normal rate of risk premiums per unit of the bank's outstanding debt* whose value at time t is denoted by u_t^1 . In this case $u_t^1 dt$ turns out to be the value of risk premiums per unit of bank outstanding debt over the time period $(t, t + dt)$. A notion related to this is the *adjustment to the rate of risk premiums per unit of the bank's outstanding debt for surplus or deficit*, $u^2 : \Omega \times T \rightarrow \mathbb{R}$, whose value at time t is denoted by u_t^2 that will in closed loop be made dependent on the total assets-to-outstanding debt. Here the amount of surplus or deficit is reliant on the excess of assets over outstanding debt. We denote the sum of u^1 and u^2 by the risk premium rate $u^3 : \Omega \times T \rightarrow \mathbb{R}$, i.e.,

$$u_t^3 = u_t^1 + u_t^2, \text{ for all } t. \quad (2.16)$$

The risk premium rate u^3 is assumed to be a predictable process and, as we shall see in the sequel, provides us with a means of controlling the dynamics of the bank. The closed loop system will be defined such that this assumption is met. For the equation (2.16) we could possibly consider choosing the additional risk premium rate

u^2 sufficiently large in order to guarantee the solvency of the bank.

The *rate of deposit withdrawal per unit of the bank's outstanding debt*, $e : \Omega \times T \rightarrow \mathbb{R}$, whose value at time t is denoted by e_t , is given by

$$de_t = r_e(t)dt + \sigma_t^e dL_t^e, \quad e_{t_0} = e_0, \quad (2.17)$$

where $r_e : T \rightarrow \mathbb{R}$, e_t is a random variable, $\sigma^e : T \rightarrow \mathbb{R}$, is the volatility in the deposit withdrawal per unit of the bank's outstanding debt and $L^e : \Omega \times T \rightarrow \mathbb{R}$ is a Lévy process whose value at time t is denoted by L_t^e .

Furthermore, we consider

$$dh_t = r_h(t)dt + \sigma_t^h dL_t^h, \quad h_{t_0} = h_0, \quad (2.18)$$

where the random variable $h : \Omega \times T \rightarrow \mathbb{R}$ in (2.18) whose value at time t is denoted by h_t is the *rate of investment return on bank assets per unit of the bank's assets*, $\sigma^h : T \rightarrow \mathbb{R}$, is the volatility in the rate of investment return on bank assets and $L^h : \Omega \times T \rightarrow \mathbb{R}$ is a Lévy process whose value at time t is denoted by L_t^h .

We suppose from the outset that the bank invests in a financial market with $n + 1$ financial assets. One of these assets is risk free and will be called a money market. Assets 1, 2, ..., n are risky and will be labelled as stocks. These assets evolve continuously in time and are modelled using a n -dimensional Lévy process. In this multidimensional context, the *rate of investment return on bank assets in the k -th asset per unit of the k -th bank's total assets* is denoted by y_t^k , $k \in \mathbb{N}_n = \{0, 1, 2, \dots, n\}$ where $y : \Omega \times T \rightarrow \mathbb{R}^{n+1}$. We can represent y as

$$y = (y_t^0, y_t^1, \dots, y_t^n), \quad (2.19)$$

where y_t^0 is a riskless asset and y_t^1, \dots, y_t^n are risky assets. Furthermore, we can model y as

$$dy_t = r_y(t)dt + \Sigma_t^y dL_t^y, \quad y_{t_0} = y_0, \quad (2.20)$$

with

$$r_y : T \rightarrow \mathbb{R}^{n+1}, \quad L^y : \Omega \times T \rightarrow \mathbb{R}^n, \quad \Sigma_t^y \in \mathbb{R}^{(n+1) \times n},$$

where there are only n scalar Lévy processes due to one of the assets being a riskless asset.

Denote the proportion of the capital invested in the assets by

$$\pi_t = (\pi_t^0, \pi_t^1, \dots, \pi_t^n)^T, \quad \pi : T \rightarrow \mathbb{R}^{n+1}.$$

The rate of investment return on bank assets is then $h : \Omega \times T \rightarrow \mathbb{R}$,

$$dh_t = \pi_t^T dy_t = \pi_t^T (r_y(t) + a \Sigma_t^y) dt + \pi_t^T c \Sigma_t^y dZ_t^y + \pi_t^T \Sigma_t^y dM_t^y.$$

Due to the fact that the proportions of the investment and, the components of the vector π_t , sum to 1 for all $t \in T$, the notation can be simplified. Denote,

$$\begin{aligned} r_0(t) &= r_{y,0}(t), \quad r_0 : T \rightarrow \mathbb{R}, \quad \text{the return of the riskless asset,} \\ r_y(t) &= (r_0(t), \tilde{r}_y(t)^T + r_0(t)1_n)^T, \quad \tilde{r}_y : T \rightarrow \mathbb{R}^n, \\ \pi_t &= (\pi_t^0, \tilde{\pi}_t^T)^T = (\pi_t^0, \pi_t^1, \dots, \pi_t^k)^T, \quad \tilde{\pi} : T \rightarrow \mathbb{R}^k, \\ \Sigma_t^y &= \begin{pmatrix} 0 & \dots & 0 \\ & \tilde{\Sigma}_t^y & \end{pmatrix}, \quad \tilde{\Sigma}_t^y \in \mathbb{R}^{n \times n}, \\ \tilde{C}_t &= \tilde{\Sigma}_t^y (\tilde{\Sigma}_t^y)^T. \quad \text{Then,} \\ \pi_t^T r_y(t) &= \pi_t^0 r_0(t) + (\tilde{\pi}_t^j)^T \tilde{r}_y(t) + (\tilde{\pi}_t^j)^T r_0(t) 1_n = r_0(t) + \tilde{\pi}_t^T \tilde{r}_y(t), \\ \pi_t^T \Sigma_t^y dL_t^y &= \tilde{\pi}_t^T \tilde{\Sigma}_t^y dL_t^y, \\ dh_t &= [r_0(t) + \tilde{\pi}_t^T \tilde{r}_y(t)] dt + \tilde{\pi}_t^T \tilde{\Sigma}_t^y dL_t^y, \quad h_{t_0} = h_0. \end{aligned}$$

Next, we take $i : \Omega \times T \rightarrow \mathbb{R}$ as the rate of increase of outstanding debt before deposit withdrawals per unit of outstanding debt whose value at time t is denoted by i_t , the volatility in the increase of outstanding debt before deposit withdrawals by σ^i , and $L^i : \Omega \times T \rightarrow \mathbb{R}$ as a Lévy process whose value at time t is denoted by L_t^i . Then, we set

$$di_t = r_i(t) dt + \sigma^i dL_t^i, \quad i_{t_0} = i_0. \quad (2.21)$$

The random variable i_t in (2.21) may typically originate from outstanding debt that have recently been accrued or instability in the value of pre-existing outstanding debt that may result from factors such as, for example, inflation.

We can choose from two approaches when modeling our bank in a stochastic setting.

The first is a realistic model that incorporates all the aspects of the bank like salary growth, individual mortality and individual members. Alternatively, we can develop a simple model which acts as a proxy for something more realistic and which emphasizes features that are specific to our particular study. In our situation we choose the latter option, with the model for Bank assets A_t and outstanding debt Δ_t at time t and their relationship being derived as

$$\begin{aligned} dA_t &= A_t dh_t + \Delta_t u_t^3 dt - \Delta_t de_t \\ &= [r_0(t)A_t + A_t \tilde{\pi}_t^T \tilde{r}_y(t) + \Delta_t u_t^1 + \Delta_t u_t^2 - \Delta_t r_e(t)]dt + \\ &\quad + [A_t \tilde{\pi}_t^T \tilde{\Sigma}_t^y dL_t^y - \Delta_t \sigma^e dL_t^e], \end{aligned} \quad (2.22)$$

$$\begin{aligned} d\Delta_t &= \Delta_t di_t - \Delta_t de_t \\ &= \Delta_t [r_i(t)dt + \sigma^i dL_t^i] - \Delta_t [r_e(t)dt + \sigma^e dL_t^e] \\ &= \Delta_t [r_i(t) - r_e(t)]dt + \Delta_t [\sigma^i dL_t^i - \sigma^e dL_t^e]. \end{aligned} \quad (2.23)$$

The stochastic differential equations (2.22) and (2.23) may be rewritten into matrix-vector form in the following way.

Definition 2.4.1 (Stochastic System for Bank Asset-to-Outstanding Debt Model): *Define the stochastic system for the banking model as*

$$dx_t = A_t x_t dt + N(x_t) u_t dt + a_t dt + S(x_t, u_t) dL_t, \quad (2.24)$$

with the various terms in this stochastic differential equation being

$$\begin{aligned}
u_t &= \begin{bmatrix} u_t^2 \\ \tilde{\pi}_t \end{bmatrix}, \quad u : \Omega \times T \rightarrow \mathbb{R}^{n+1}, \\
A_t &= \begin{bmatrix} r_0(t) & -r_e(t) \\ 0 & r_i(t) - r_e(t) \end{bmatrix}; \\
N(x_t) &= \begin{bmatrix} \Delta_t & A_t \tilde{r}_y(t)^T \\ 0 & 0 \end{bmatrix}; \quad a_t = \begin{pmatrix} \Delta_t u_t^1 \\ 0 \end{pmatrix}; \\
S(x_t, u_t) &= \begin{bmatrix} A_t \tilde{\pi}_t^T \tilde{\Sigma}_t^y & -\Delta_t \sigma^e & 0 \\ 0 & -\Delta_t \sigma^e & \Delta_t \sigma^i \end{bmatrix}; \\
L_t &= \begin{bmatrix} L_t^y \\ L_t^e \\ L_t^i \end{bmatrix},
\end{aligned}$$

where L_t^y , L_t^e and L_t^i are mutually (stochastically) independent Lévy processes. It is assumed that for all $t \in T$, $\sigma_t^e > 0$, $\sigma_t^i > 0$, and $\tilde{C}_t > 0$. Often the time argument of the functions σ^e and σ^i are omitted.

We can also rewrite (2.24) by making the following computations.

$$\begin{aligned}
N(x_t)u_t &= \begin{bmatrix} \Delta_t \\ 0 \end{bmatrix} u_t^2 + \begin{bmatrix} A_t \tilde{r}_y(t)^T \\ 0 \end{bmatrix} \tilde{\pi}_t \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_t u_t^3 + \sum_{m=1}^n \begin{bmatrix} A_t \tilde{r}_{y,m}(t) \\ 0 \end{bmatrix} \tilde{\pi}_t^m \\
&= B_0 x_t u_t^0 + \sum_{m=1}^n \begin{bmatrix} \tilde{r}_{y,m}(t) & 0 \\ 0 & 0 \end{bmatrix} x_t \tilde{\pi}_t^m \\
&= \sum_{m=0}^n [B_m x_t] u_t^m,
\end{aligned}$$

and

$$\begin{aligned}
S(x_t, u_t)dL_t &= \begin{bmatrix} [\tilde{\pi}_t^T \tilde{C}_t \tilde{\pi}_t]^{1/2} & 0 \\ 0 & 0 \end{bmatrix} x_t dL_t^1 + \\
&+ \begin{bmatrix} 0 & -\sigma^e \\ 0 & -\sigma^e \end{bmatrix} x_t dL_t^2 + \begin{bmatrix} 0 & 0 \\ 0 & \sigma^i \end{bmatrix} x_t dL_t^3 \\
&= \sum_{jj=1}^3 [M_{jj}(u_t)x_t] dL_t^{jj}.
\end{aligned}$$

Then (2.24) becomes

$$dx_t = A_t x_t dt + \sum_{j=0}^n [B_j x_t] u_t^j dt + a_t dt + \sum_{jj=1}^3 [M_{jj}(u_t)x_t] dL_t^{jj}. \quad (2.25)$$

Remark 2.4.2 From (2.24) it is clear that $u = (u^2, \tilde{\pi})$ affects only the stochastic differential equation of A_t but not that of Δ_t . In particular, for (2.24) we have that $\tilde{\pi}$ affects the variance of A_t and the drift of A_t via the term $A_t \tilde{r}_y(t)^T \tilde{\pi}_t$. On the other hand, u^2 affects only the drift of A_t .

2.4.2 Description of the Simplified Bank Assets-to-Outstanding Debt Model

The model can be simplified if attention is restricted to the system with as state the asset-to-outstanding debt ratio, denoted in this subsection by $x_t = A_t/\Delta_t$.

Definition 2.4.3 Define the simplified auditing system by the stochastic differential equation,

$$\begin{aligned}
dx_t &= x_t [r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2 + \tilde{r}_y(t)^T \tilde{\pi}_t] dt \\
&+ [u_t^1 + u_t^2 - r_e(t) - (\sigma^e)^2] dt \\
&+ [(\sigma^e)^2 (1 - x_t)^2 + (\sigma^i)^2 (x_t)^2 + (x_t)^2 \tilde{\pi}_t^T \tilde{C}_t \tilde{\pi}_t]^{1/2} d\bar{L}_t, \quad x_{t_0} = x_0.
\end{aligned} \quad (2.26)$$

The model is derived as follows. The starting point is the two-dimensional stochastic differential equation for $x = [A, \Delta]^T$ as in the equations (2.22), (2.23). Then one calculates

$$\begin{aligned}
d(\Delta_t)^{-1} &= -(\Delta_t)^{-2}d\Delta_t + \frac{1}{2}2(\Delta_t)^{-3}d\langle \Delta, \Delta \rangle (t) \\
&= [-(\Delta_t)^{-1}(r_i(t) - r_e(t)) + (\Delta_t)^{-1}((\sigma^e)^2 + (\sigma^i)^2)]dt \\
&\quad - (\Delta_t)^{-1} \begin{bmatrix} 0 & -\sigma^e & -\sigma^i \end{bmatrix} dL_t, \tag{2.27}
\end{aligned}$$

$$\begin{aligned}
dx_t &= A_t d(\Delta_t)^{-1} + (\Delta_t)^{-1} dA_t + d\langle A, (\Delta)^{-1} \rangle (t) \\
&= [r_0(t)x_t - r_e(t) + u_t^1 + u_t^2 + x_t \tilde{r}_y(t) \tilde{\pi}_t \\
&\quad - x_t[r_i(t) - r_e(t)] + x_t((\sigma^e)^2 + (\sigma^i)^2) - (\sigma^e)^2]dt \\
&\quad + \begin{pmatrix} x_t \tilde{\pi}_t^T \tilde{\Sigma}_t^y & -\sigma^e(1-x_t) & -\sigma^i x_t \end{pmatrix} dL_t \\
&= x_t[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2 + \tilde{r}_y(t)^T \tilde{\pi}_t]dt + \\
&\quad + [u_t^1 + u_t^2 - r_e(t) - (\sigma^e)^2]dt \\
&\quad + \left[(\sigma^e)^2(1-x_t)^2 + (\sigma^i)^2 \Delta_t + \Delta_t \tilde{\pi}_t^T \tilde{C}_t \tilde{\pi}_t \right]^{1/2} d\bar{L}_t, \tag{2.28}
\end{aligned}$$

for a stochastic process $\bar{L} : \Omega \times T \rightarrow \mathbb{R}$ which is a Lévy process.

Note that in the drift of the stochastic differential equation (2.26) the term

$$-r_e(t) + x_t r_e(t) = r_e(t)(x_t - 1),$$

appears because it models the effect of deposit withdrawals both on the assets and on the outstanding debt. Similarly the term $-(\sigma^e)^2 + x_t(\sigma^e)^2 = (\sigma^e)^2(x_t - 1)$ appears.

Chapter 3

OPTIMAL AUDITING IN THE BANKING INDUSTRY

3.1 RANDOM AUDITING: RESERVE REQUIREMENTS

3.2 NON-RANDOM AUDITING: CAPITAL REQUIREMENTS

3.3.1 An Optimal Auditing Time Problem for CARs

3.3.2 Solution to the Optimal Auditing Time Problem

In the ensuing chapter, we discuss random and non-random auditing for reserve, asset and capital requirements. The main results in Sections 3.1 and 3.2 can be verified by using standard techniques in the stochastic analysis of jump diffusions. The main banking indicators that we discuss in the sequel are the sum of the Treasuries and reserves, W , and capital adequacy ratio, ρ .

The main problems investigated in this chapter can be stated as follows.

Problem 3.0.4 (Random Audit: Reserve Requirements): *Can we give a mathematical description of a random audit involving reserve requirements ?*

Problem 3.0.5 (Non-Random Audit: Capital Requirements): *Can we give a mathematical description of a non-random audit involving capital requirements ?*

Problem 3.0.6 (Optimal Bank Reserves and Depository Consumption Rate): *Can we maximize banking reserves and consumption for a commercial bank ?*

3.1 RANDOM AUDITING: RESERVE REQUIREMENTS

Auditors ought to be able to access information about the volume of (Treasuries and) reserves that the bank holds as a provision for deposit withdrawals. This auditing criterion is called the *reserve requirement*. Both auditors and banks are interested in establishing the optimal level of reserves on demand deposits that the bank must hold. By setting a bank's individual optimal level of reserves, auditors assist in mitigating the costs of financial distress.

In the sequel, we discuss how the model (2.5) for Treasuries and reserves built in Subsection 2.1.3 may be used to obtain an important result in the stochastic optimization of the reserve requirement (see, for instance, [1]). From the point of view of the bank owner, we may pick any *utility function*, $U : [0, \infty) \rightarrow \mathbf{R}$, but, in our case, the choice of power utility

$$U(k) = \frac{1}{\alpha} k^\alpha(t), \text{ with } 0 < \alpha < 1, \quad (3.1)$$

leads to an analytic solution. In this case, the *performance criterion* is the objective functional given by

$$J_t(\pi, k) = \mathbf{E}^w \left[\int_0^\infty \exp\{-\delta(s+t)\} U(k) dt \right]. \quad (3.2)$$

In order to determine the optimal reserve allocation, π^* , and depository consumption, k^* , we consider the set of admissible controls

$$\mathcal{A} = \left\{ u(t) = (\pi_t, k(t)) : (2.5) \text{ has unique strong soln and } (3.2) \text{ a finite value} \right\}. \quad (3.3)$$

Also, the *value function* is given by

$$V(t, w) = \sup_{(\pi, k) \in \mathcal{A}} \mathbf{E}^w \left[\int_0^\infty \exp\{-\delta(s+t)\} \frac{1}{\alpha} k^\alpha(t) dt \right]. \quad (3.4)$$

The optimal auditing problem for Treasuries and reserves may be formally stated as follows.

Problem 3.1.1 (Optimal Bank Reserves and Depository Consumption Rate):

Suppose that the sum of the Treasuries and reserves, W , utility, U , and performance criterion, J , and admissible class of control laws, $\mathcal{A} \neq \emptyset$, are described by (2.5), (3.1), (3.2) and (3.3), respectively. In this case, determine $V(t, w)$ in (3.4) and the optimal control law (π^*, k^*) , if it exists.

The main optimization result for bank reserve allocation and depository consumption follows below.

Theorem 3.1.2 (Optimal Bank Reserves and Depository Consumption Rate):

Suppose that the sum of the Treasuries and reserves, W , terminal utility, U , performance criterion, J , set of admissible controls, \mathcal{A} , and value function, V , are characterized by (2.5), (3.1), (3.2), (3.3) and (3.4), respectively. Furthermore, assume that the generator G^w of Q is given by (2.6). Then the optimal bank reserve allocation is π^* which solves the integral equation

$$\Phi(\pi^*) = \mu^R - r^T - \sigma^R \pi(1 - \alpha) - \int_{-1}^{\infty} \left\{ 1 - (1 + \pi x^R)^{\alpha-1} \right\} x^R \nu(dx^R) = 0, \quad \pi^* \in (0, 1]. \quad (3.5)$$

If we assume that

$$K = \frac{1}{\alpha} \left[\frac{1}{1 - \alpha} \left(\delta - \alpha \{ r^T (1 - \pi^*) + \mu^R \pi^* \} + \frac{1}{2} \alpha (1 - \alpha) \sigma^{R2} \pi^{*2} - \int_{\mathbb{R}} \left\{ (1 + \pi^* x)^\alpha - 1 - \alpha \pi^* x \right\} \nu(dx) \right) \right]^{\alpha-1} \quad (3.6)$$

then the optimal depository consumption rate is given by

$$k^* = (\alpha K)^{1/\alpha-1} w. \quad (3.7)$$

Proof. The proof is outlined in Appendix A of Subsection 9.1.2 from Section 9.1.

□

3.2 NON-RANDOM AUDITING: CAPITAL REQUIREMENTS

In this section, our analysis makes use of the SDE for CARs given by (2.11). In the sequel, this ratio plays a central role in an optimal auditing time problem that, from the bank owner's viewpoint, provides an indication of when auditing by the regulator would yield the most favourable outcome.

3.2.1 An Optimal Auditing Time Problem for CARs

In the analysis that follows, we will concentrate our efforts on finding a precise formula for the value function, $V(s, z)$, for which the performance criterion, J_τ , is defined by (2.12). Here, the set of admissible controls is given by

$$\mathcal{A}^{(1)} = \left\{ (s, z) : (2.11) \text{ has unique strong soln and } (2.12) \text{ a finite value} \right\}. \quad (3.8)$$

and the *value function* has the form

$$V^{(1)}(s, z) = \sup_{(s, z) \in \mathcal{A}^{(1)}} \mathbf{E}^{s, z} \left[\exp\{-\delta(s + \tau)\} \left(\rho_\tau - \rho^r \right) \cdot \mathcal{X}_{\tau < \infty} \right]. \quad (3.9)$$

We are now in a position to state the optimal auditing time problem that we solve in the sequel.

Problem 3.2.1 (Optimal Auditing Time Problem): *Suppose that the SDE for the ρ -dynamics is given by (2.11) and the performance criterion, J , is represented by (2.12). In this case, determine $V^{(1)}(s, z)$ in (3.9) and the optimal auditing time τ^* , if it exists.*

3.2.2 Solution to the Optimal Auditing Time Problem

In this subsection, we derive a solution for Problem 3.2.1 in the case where the dynamics of ρ is characterized by (2.11).

Theorem 3.2.2 (Solution to the Optimal Auditing Time Problem): *Suppose that the ρ -dynamics is described by (2.11), $\rho < \delta$ and*

$$\varrho > \frac{1}{2}\varsigma^2 + \zeta \int_{\mathbb{R}} x\nu(dx) \text{ and } 2\varrho - 2\delta + \varsigma^2 + \zeta^2 \int_{\mathbb{R}} x^2\nu(dx) < 0$$

hold. Then with $\kappa > 1$, z^* and $K^{(1)}$ given by

$$\delta - \kappa\varrho - \frac{1}{2}\varsigma^2\kappa(\kappa - 1) = \int_{-1}^{\infty} \left\{ (1 + \zeta x)^\kappa - 1 - \kappa\zeta x \right\} \nu(dx), \quad z^* = \frac{\kappa}{\kappa - 1}\rho^r \text{ and}$$

$$K^{(1)} = \frac{1}{\kappa}(z^*)^{1-\kappa}, \quad (3.10)$$

respectively, the function

$$\varphi(s, z) = \begin{cases} \exp\{-\delta s\}K^{(1)}z^\kappa; & 0 < z \leq z^* \\ \exp\{-\delta s\}K^{(1)}(z - \rho^r); & z^* \leq z \end{cases} \quad (3.11)$$

corresponds to the value function, $V^{(1)}(s, z)$, given by (3.9). Furthermore, we have that $\tau^* = \tau_{\mathcal{D}}$ is an optimal auditing time where

$$\mathcal{D} = \left\{ (s, z) : 0 < z < z^* \right\}, \quad z^* \geq \frac{\delta}{\delta - \varrho}\rho^r, \quad \tau_{\mathcal{D}} = \inf\{t > 0 : (t, z) \notin \mathcal{D}\}. \quad (3.12)$$

Proof. The outline of the proof is provided in Appendix B in Subsection 9.2.3 of Section 9.2. \square

Chapter 4

OPTIMAL STOCHASTIC CONTROL OF BANKING SYSTEMS

4.1 OPTIMAL STOCHASTIC CONTROL OF A SIMPLIFIED BANKING MODEL

The main problem investigated in this chapter can be stated as follows.

Problem 4.0.3 (Optimal Stochastic Control for the Banking Model): *Can we optimize the premium rate and asset allocation strategy for a commercial bank ?*

In order for a bank manager or trustee to determine an optimal premium rate and asset allocation strategy it is imperative that a well-defined objective function (loss function in our case) with appropriate constraints is considered. The choice has to be carefully made in order to avoid ambiguous solutions to our stochastic control problem. In this particular contribution, we choose to determine a control law $g(t, x_t)$ that minimizes the cost function $J : \mathcal{G}_A \rightarrow \mathbb{R}_+$, where \mathcal{G}_A is the class of admissible control laws

$$\mathcal{G}_A = \{g : T \times \mathcal{X} \rightarrow \mathcal{U} | g \text{ Borel measurable} \quad (4.1)$$

and there exists an unique solution to the closed-loop system},

with the closed-loop system for $g \in \mathcal{G}_A$ being given by

$$\begin{aligned}
 dx_t &= A_t x_t dt + \sum_{j=0}^n B_j x_t g_j(t, x_t) dt + a_t dt \\
 &\quad + \sum_{jj=1}^3 M_{jj}(g(t, x_t)) x_t dL_t^{jj}, \quad x_{t_0} = x_0.
 \end{aligned} \tag{4.2}$$

Furthermore, the cost function, $J : \mathcal{G}_A \rightarrow \mathbb{R}_+$, of the banking problem is given by

$$\begin{aligned}
 J(g) &= \mathbf{E} \left[\int_{t_0}^{t_1} \exp(-r_d(s - t_0)) b(s, x_s, g(s, x_s)) ds \right. \\
 &\quad \left. + \exp(-r_d(t_1 - t_0)) b_1(x_{t_1}) \right],
 \end{aligned} \tag{4.3}$$

where $g \in \mathcal{G}_A$, $T = [t_0, t_1]$ and $b_1 : \mathcal{X} \rightarrow \mathbb{R}_+$ is a Borel measurable function. Furthermore, $b : T \times \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_+$, where for $b_2 : \mathcal{U}_2 \rightarrow \mathbb{R}_+$ and $b_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

$$b(t, x, u) = b_2(u^2) + b_3(x^1/x^2). \tag{4.4}$$

Also, $r_d \in \mathbb{R}_+$ is called the *discount rate*. The functions b_1 , b_2 and b_3 are selected below where various choices are considered. Assume that $\mathcal{G}_A \neq \emptyset$.

We are now in a position to state the stochastic optimal control problem for continuous time banking systems that we solve. The said problem may be formulated as follows.

Problem 4.0.4 Consider the stochastic control system (4.2) for the banking problem with the admissible class of control laws, \mathcal{G}_A , given by (4.1) and the cost function, $J : \mathcal{G}_A \rightarrow \mathbb{R}_+$, given by (4.3). Solve

$$\inf_{g \in \mathcal{G}_A} J(g), \tag{4.5}$$

which amounts to determining the value J^* ,

$$J^* = \inf_{g \in \mathcal{G}_A} J(g), \tag{4.6}$$

and the optimal control law g^* , if it exists,

$$g^* = \arg \min_{g \in \mathcal{G}_A} J(g). \quad (4.7)$$

4.1 OPTIMAL STOCHASTIC CONTROL OF A SIMPLIFIED BANKING MODEL

Consider the simplified stochastic control system (2.26) for the auditing problem with the admissible class of control laws, \mathcal{G}_A , given by (4.1) but with $\mathcal{X} = \mathbb{R}$. In this section, we have to solve

$$J^* = \inf_{g \in \mathcal{G}_A} J(g), \quad (4.8)$$

$$J(g) = \mathbf{E} \left[\int_{t_0}^{t_1} \exp(-r_d(s - t_0)) [b_2(u_t^2) + b_3(x_t)] dt + \exp(-r_d(t_1 - t_0)) b_1(x_{t_1}) \right], \quad (4.9)$$

and $b_1 : \mathbb{R} \rightarrow \mathbb{R}_+$, $b_2 : \mathbb{R} \rightarrow \mathbb{R}_+$, and $b_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are all Borel measurable functions. Next we state and prove the result.

Theorem 4.1.1 (Optimal Control Problem for the Simplified Auditing Model)

Consider the optimal stochastic control problem for the simplified auditing system (2.28). Suppose that the following assumptions hold.

1. The cost function is assumed to satisfy

$$\begin{aligned} b_2(u^2) &\in C^2(\mathbb{R}), \\ \lim_{u^2 \rightarrow -\infty} D_{u^2} b_2(u^2) &= -\infty, \quad \lim_{u^2 \rightarrow +\infty} D_{u^2} b_2(u^2) = +\infty; \\ D_{u^2 u^2} b_2(u^2) &> 0, \quad \forall u^2 \in \mathbb{R}. \end{aligned}$$

2. There exists a function

$$v : T \times \mathbb{R} \rightarrow \mathbb{R}, \quad v \in C^{1,2}(T \times \mathcal{X}),$$

which is a solution of the partial differential equation,

$$\begin{aligned} 0 = & D_t v(t, x) + \frac{1}{2} [(\sigma^e)^2 (1-x)^2 + (\sigma^i)^2 x^2] D_{xx} v(t, x) \\ & + x(r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2) D_x v(t, x) \\ & + [u_t^1 - r_e(t) - (\sigma^e)^2] D_x v(t, x) \\ & + u_t^{2*} D_x v(t, x) + \exp(-r_d(t - t_0)) b_2(u_t^{2*}) \\ & + \exp(-r_d(t - t_0)) b_3(x) - \frac{[D_x v(t, x)]^2}{2D_{xx} v(t, x)} \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t), \end{aligned} \quad (4.10)$$

$$\begin{aligned} v(t_1, x) & = \exp(-r_d(t_1 - t_0)) b_1(x), \end{aligned} \quad (4.11)$$

where u^{2*} is the unique solution of the equation

$$0 = D_x v(t, x) + \exp(-r_d(t - t_0)) D_{u^2} b_2(u_t^2). \quad (4.12)$$

Then the optimal control law is

$$g_2^*(t, x) = u^{2*}, \quad g_2^* : T \times \mathcal{X} \rightarrow \mathbb{R}_+,$$

with $u^{2*} \in \mathcal{U}_2$ the unique solution of the equation (4.12)

$$\tilde{\pi}^* = -\frac{D_x v(t, x)}{x D_{xx} v(t, x)} \tilde{C}_t^{-1} \tilde{r}_y(t), \quad (4.13)$$

$$g_3^*(t, x) = \min\{1, \max\{0, \tilde{\pi}^*\}\}, \quad g_3^* : T \times \mathcal{X} \rightarrow \mathbb{R}^k, \quad (4.14)$$

Furthermore, the value of the problem is

$$J^* = J(g^*) = \mathbf{E}[v(t, x_0)]. \quad (4.15)$$

Proof. It will be proven that the infimization in the dynamic programming equation can be achieved and that there exists a solution of the dynamic programming equation.

(1) Recall from optimal stochastic control theory that the dynamic programming equation (DPE) for the optimal control problem for $w : T \times \mathbb{R} \rightarrow \mathbb{R}$, $w \in C^{1,2}(T \times \mathcal{X})$,

is given by

$$\begin{aligned}
 0 &= D_t w(t, x) + \\
 &+ \inf_{u^2 \in \mathbb{R}, \tilde{\pi} \in [0, 1]} \left[\frac{1}{2} [(\sigma^e)^2 (1-x)^2 + (\sigma^i)^2 x^2 + x^2 \tilde{\pi}^T \tilde{C}_t \tilde{\pi}] D_{xx} w(t, x) \right. \\
 &+ [(r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2 + \tilde{r}_y(t)^T \tilde{\pi}) x] D_x w(t, x) \\
 &\quad + [u_t^1 + u^2 - r_e(t) - (\sigma^e)^2] D_x w(t, x) \\
 &\quad \left. + \exp(-r_d(t - t_0)) [b_2(u^2) + b_3(x)] \right] \quad (4.16)
 \end{aligned}$$

$$w(t_1, x) = \exp(-r_d(t_1 - t_0)) b_1(x). \quad (4.17)$$

Note that this PDE separates additively into terms (1) depending on u^2 , (2) depending on $\tilde{\pi}$, and (3) not depending on either of these variables. The infimization is therefore decomposed into two infimizations.

(2) The infimizations are calculated as follows with the function v

$$\begin{aligned}
 &\inf_{u^2 \in \mathbb{R}} H_2(t, x, u^2), \\
 H_2(t, x, u) &= u^2 D_x v(t, x) + \exp(-r_d(t - t_0)) b_2(u^2). \quad (4.18)
 \end{aligned}$$

$$D_{u^2} H_2(t, x, u) = D_x v(t, x) + \exp(-r_d(t - t_0)) D_{u^2} b_2(u^2) = 0. \quad (4.19)$$

Because of Assumption (1), equation (4.19) for all $(t, x) \in T \times \mathcal{X}$ has an unique solution $u^{2*} \in \mathcal{U} = \mathbb{R}$ and

$$\inf_{u^2 \in \mathbb{R}} H_2(t, x, u^2) = H_2(t, x, u^{2*}) = u^{2*} D_x v(t, x) + \exp(-r_d(t - t_0)) b_2(u^{2*}) \quad (4.20)$$

Define the function $g_2^*(t, x) = u^{2*}$, $g_2^* : T \times \mathcal{X} \rightarrow \mathbb{R}$. It follows from Assumption (1) that g_2^* is Borel measurable. Consider the infimization problem

$$\inf_{\tilde{\pi} \in \mathbb{R}^k} H_3(t, x, \tilde{\pi}), \quad (4.21)$$

$$H_3(t, x, \tilde{\pi}) \quad (4.22)$$

$$\begin{aligned} &= \frac{1}{2} [\tilde{\pi}_t^T \tilde{C}_t \tilde{\pi}_t] x^2 D_{xx} v(t, x) + \tilde{r}_y(t)^T \tilde{\pi} x D_x v(t, x) \\ &= \frac{1}{2} \left(\tilde{\pi}_t + \left(\frac{x [D_x v(t, x)]}{x^2 D_{xx} v(t, x)} \right) \tilde{C}_t^{-1} \tilde{r}_y(t) \right)^T \times \\ &\quad \times \tilde{C}_t x^2 D_{xx} v(t, x) \left(\left(\tilde{\pi}_t + \left(\frac{x [D_x v(t, x)]}{x^2 D_{xx} v(t, x)} \right) \tilde{C}_t^{-1} \tilde{r}_y(t) \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{[x D_x v(t, x)]^2}{x^2 D_{xx} v(t, x)} \right) \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \right), \\ &\quad \tilde{C}_t x^2 D_{xx} v(t, x) > 0, \quad \forall x \in \mathbb{R} \setminus \{0\}. \end{aligned} \quad (4.23)$$

Hence

$$\tilde{\pi}_k^* = \min \left\{ 1, \max \left\{ 0, -\frac{D_x v(t, x)}{x D_{xx} v(t, x)} \tilde{C}_t^{-1} \tilde{r}_y(t) \right\} \right\}, \quad (4.24)$$

$$g_{3,k}^*(t, x) = \begin{cases} \tilde{\pi}_k^*, & \text{if } \sum_{i=1}^k \tilde{\pi}_i^* \in [0, 1], \\ \frac{\tilde{\pi}_k^*}{\sum_{j=1}^k \tilde{\pi}_j^*}, & \text{if } \sum_{i=1}^k \tilde{\pi}_i^* > 1 \end{cases} \quad \forall k \in \mathbf{Z}_n, \quad (4.25)$$

$$H_3(t, x, \tilde{\pi}^*) = -\frac{[D_x v(t, x)]^2}{2 D_{xx} v(t, x)} \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t). \quad (4.26)$$

If for a $k \in \mathbf{Z}_n$, $\tilde{\pi}_k^* \notin [0, 1]$, then the PDE (4.10) has to be modified with the difference of the terms obtained with and without the constraint. This is not included in this presentation.

(3) Using the infima obtained in Step (2) the PDE for v , Equation (4.10), can now be rewritten. The resulting equation is

$$\begin{aligned} 0 &= D_t v(t, x) + \inf_{u^2 \in \mathbb{R}, \tilde{\pi} \in [0, 1]} \left\{ \frac{1}{2} [(\sigma^e)^2 (1-x)^2 + (\sigma^i)^2 x^2 + x^2 (\tilde{\pi}^T \tilde{C}_t \tilde{\pi})] D_{xx} v(t, x) \right. \\ &\quad + [r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2 + \tilde{r}_y(t)^T \tilde{\pi}] x \\ &\quad + [u_t^1 + u^2 - r_e(t) - (\sigma^e)^2] D_x v(t, x) \\ &\quad \left. + \exp(-r_d(t - t_0)) [b_2(u^2) + b_3(x)] \right\}, \end{aligned} \quad (4.27)$$

$$v(t_1, x) = \exp(-r_d(t_1 - t_0)) b_1(x). \quad (4.28)$$

Thus v , whose existence is assumed by Condition (2), is a solution of the dynamic programming equation. It follows then from a theorem on optimal stochastic control that g_2^* and g_3^* are the optimal control laws and that the value is given by $J^* = J(g^*) = \mathbf{E}[v(t_0, x_0)]$. \square

It is of interest to choose particular cost functions for which an analytic solution can be obtained for the value function and for the control laws. The following theorem provides the optimal control laws for a particular choice of cost functions.

Theorem 4.1.2 (Optimal Stochastic Control of a Simplified Auditing Model with Quadratic Cost Functions)

Consider the optimal stochastic control problem for the simplified auditing system (2.28). Consider the cost function

$$\begin{aligned}
 J(g) = \mathbf{E} & \left[\int_{t_0}^{t_1} \exp(-r_d(s - t_0)) \left[\frac{1}{2} c_2 ((u_s^2)^2) + \frac{1}{2} c_3 (x_t - adr_1)^2 \right] ds \right. \\
 & \left. + \frac{1}{2} c_1 (x_{t_1} - adr_1)^2 \exp(-r_d(t_1 - t_0)) \right]. \tag{4.29}
 \end{aligned}$$

It is assumed that the cost functions satisfy,

$$\begin{aligned}
 b_1(x) &= \frac{1}{2} c_1 (x - adr_1)^2, \quad c_1 \in (0, \infty); \\
 b_2(u^2) &= \frac{1}{2} c_2 (u^2)^2, \quad c_2 \in (0, \infty); \\
 b_3(x) &= \frac{1}{2} c_3 (x - adr_1)^2, \quad c_3 \in (0, \infty), \\
 & \quad adr_1 \in \mathbb{R}, \text{ called the reference value of the asset-debt-ratio} \\
 & \quad \text{which for example may take the value 2.}
 \end{aligned}$$

Define the ordinary differential equations,

$$\begin{aligned}
 -\dot{q}(t) &= -q(t)^2/c_2 + c_3 \\
 &+ q(t)2(r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2) \\
 &+ q(t)[-r_d - \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) + (\sigma^e)^2 + (\sigma^i)^2], \quad q(t_1) = c_1; \quad (4.30)
 \end{aligned}$$

$$\begin{aligned}
 -\dot{x}_r(t) &= -c_3(x_r(t) - adr_1)/q(t) - x_r(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] \\
 &- [u_1(t) - r_e(t) - (\sigma^e)^2] - (x_r(t) - 1)((\sigma^e)^2 + (\sigma^i)^2) - (\sigma^i)^2, \\
 x_r(t_1) &= adr_1; \quad (4.31)
 \end{aligned}$$

$$\begin{aligned}
 -\dot{s}(t) &= -r_d s(t) + c_3(x_r(t) - adr_1)^2 - q(t)(\sigma^e)^2(x_r(t) - 1)^2 - q(t)(\sigma^i)^2 x_r(t)^2, \\
 s(t_1) &= 0. \quad (4.32)
 \end{aligned}$$

The function $x_r : T \rightarrow \mathbb{R}$ will be called the asset-to-debt ratio (*adr*) reference function.

Then:

(a) There exists solutions to the ordinary differential equations (4.30), (4.31), (4.32). Moreover, for all $t \in T$, $q(t) > 0$.

(b) The optimal control laws are ,

$$u_t^{2*} = -(x - x_r(t))q(t)/c_2, \quad (4.33)$$

$$g_2^*(t, x) = u^{2*}, \quad g_2^* : T \times \mathcal{X} \rightarrow \mathbb{R}_+, \quad (4.34)$$

$$\tilde{\pi}^*(t) = -\frac{(x - x_r(t))}{x} \tilde{C}_t^{-1} \tilde{r}_y(t), \quad g_3^* : T \times \mathcal{X} \rightarrow \mathbb{R}^k, \quad (4.35)$$

$$g_{3,k}^*(t, x) = \begin{cases} \tilde{\pi}_k^*, & \text{if } \tilde{\pi}_k^* \in [0, 1], \\ \min\{1, \max\{0, \tilde{\pi}_k^*(t)\}\}, & \text{else,} \end{cases} \quad \forall k \in \mathbf{Z}_n. \quad (4.36)$$

(c) The value function and the value of the problem are

$$v(t, x) = \exp(-r_d(t_1 - t_0)) \left[\frac{1}{2}(x - x_r(t))^2 q(t) + \frac{1}{2}s(t) \right], \quad (4.37)$$

$$J^* = J(g^*) = \mathbf{E}[v(t_0, x_0)]. \quad (4.38)$$

Proof. (a) The ordinary differential equation (4.30) is of Riccati type. It follows from, for example, Sontag, 1998 (Th. 37, p. 364) that an unique solution to this equation exists.

The second statement requires a longer argument. Rewrite the Riccati differential equation (4.30) in the form,

$$-\dot{q}(t) = -bq^2(t) + c + 2a(t)q(t), \quad q(t_1) = c_1; \quad (4.39)$$

$$b = 1/c_2, c = c_3 \in (0, \infty), \quad a : T \rightarrow \mathbb{R}. \quad (4.40)$$

Transform the equation according to,

$$\begin{aligned} q_1(t_1) &= q(t_1 - t), \quad a_1(t) = a(t_1 - t), \quad q_1 : [0, t_1 - t_0] \rightarrow \mathbb{R}, \\ \dot{q}_1(t) &= -\dot{q}(t_1 - t) = -bq_1(t)^2 + c + 2a_1(t)q_1(t), \quad q_1(t_0) = c_1, \\ 0 &= \dot{q}_1(t) + bq_1(t)^2 - c - 2a_1(t)q_1(t), \quad q_1(0) = c_1. \end{aligned} \quad (4.41)$$

Consider the second Riccati differential equation,

$$-\dot{q}_2(t) = -\bar{b}q_2^2(t) + \bar{c}, \quad q_2(t_1) = c_1; \quad \bar{b}, \bar{c} \in (0, \infty). \quad (4.42)$$

The latter Riccati differential equation is time-invariant. The associated first-order linear system with as system matrices $(0, \sqrt{\bar{b}}, \sqrt{\bar{c}})$ is both controllable and observable. It then follows from a result for this Riccati differential equation that $q_2(t) > 0$ for all $t \in T = [0, t_1 - t_0]$.

Next a theorem is used from Kratz, 1995 (Lemma 5.1) on the comparison of solutions of two Riccati differential equations. The lemma states that for all $t \in [t_1 - t_0]$, $q_1(t) \geq q_2(t) > 0$ if,

$$\begin{aligned} b &\in (0, \infty), \\ \begin{pmatrix} -\bar{c} & 0 \\ 0 & \bar{b} \end{pmatrix} &\geq \begin{pmatrix} -c & -a_1(t) \\ -a_1(t) & b \end{pmatrix}, \quad \forall t \in [0, t_1 - t_0]. \end{aligned}$$

The matrix inequality is met if and only if

$$\begin{aligned} b &\in (0, \infty), \quad (\bar{b} - b) \geq 0, \quad (c - \bar{c}) \geq 0, \\ a(t)^2 &\leq (c - \bar{c})(\bar{b} - b) = (c_3 - \bar{c}_3)\left(\frac{1}{c_2} - \frac{1}{c_2}\right). \end{aligned}$$

By assumption all functions in a and hence those of $a_1(t) = a(t_1 - t)$ are bounded on the bounded interval $[0, t_1 - t_0]$. Hence there exists a constant $c_4 \in (0, \infty)$ such that $a_1(t)^2 < c_4$. Then one can determine real numbers $\bar{b}, \bar{c} \in (0, \infty)$ such that

$$\begin{aligned} c - \bar{c} &= c_3 - \bar{c}_3 \geq 0, \quad \bar{b} - b = 1/\bar{c}_2 - 1/c_2 \geq 0, \\ a(t)^2 &\leq c_4 \leq (c - \bar{c})(\bar{b} - b) = (c_3 - \bar{c}_3)(1/\bar{c}_2 - 1/c_2), \end{aligned}$$

for example by choosing $\bar{c}_2, \bar{c}_3 \in (0, \infty)$ both very small. Thus the inequalities are satisfied. Thus, for all $t \in [0, t_1 - t_0]$, $q(t_1 - t) = q_1(t) \geq q_2(t) > 0$.

The ordinary differential equation (4.31) is linear hence it follows from a theorem for this type of differential equation that an unique solution exists.

(b, c) The cost function b_2 satisfies the conditions of Theorem 4.1.1. It will be proven that the function (4.37) is a solution of the partial differential equation (4.10,4.11) of Theorem 4.1.1.

Note first that,

$$\begin{aligned} v(t, x) &= \exp(-r_d(t - t_0))\left[\frac{1}{2}(x - x_r(t))^2 q(t) + \frac{1}{2}s(t)\right], \\ D_t v(t, x) &= -r_d v(t, x) \\ &\quad + \exp(-r_d(t - t_0))\left[-(x - x_r(t))\dot{x}_r(t)q(t) + \frac{1}{2}(x - x_r(t))^2 \dot{q}(t) + \frac{1}{2}\dot{s}(t)\right], \\ D_x v(t, x) &= \exp(-r_d(t - t_0))(x - x_r(t))q(t), \\ D_{xx} v(t, x) &= \exp(-r_d(t - t_0))q(t). \end{aligned}$$

The optimal control laws are calculated according to the formulas of the previous theorem.

$$\begin{aligned} 0 &= D_x v(t, x) + \exp(-r_d(t - t_0))D_{u^2} b_2(u^2) \\ &= \exp(-r_d(t - t_0))[(x - x_r(t))q(t) + c_2 u^2], \\ u^{2*} &= -(x - x_r(t))q(t)/c_2, \\ \tilde{\pi}^* &= -\frac{(x - x_r(t))q(t)}{xq(t)}\tilde{C}_t^{-1}\tilde{r}_y(t). \end{aligned}$$

From the partial differential equation (4.10) and the terminal condition (4.11) then follows that,

$$\begin{aligned}
 q(t_1) &= c_1, \quad x_r(t_1) = adr_1, \quad s(t_1) = 0; \\
 v(t_1, x) &= \exp(-r_d(t_1 - t_0)) \left[\frac{1}{2}(x - x_r(t_1))^2 q(t_1) + \frac{1}{2}s(t_1) \right] \\
 &= \exp(-r_d(t_1 - t_0)) \frac{1}{2} c_1 (x - adr_1)^2 = \exp(-r_d(t_1 - t_0)) b_1(x).
 \end{aligned}$$

$$\begin{aligned}
 & \exp(+r_d(t - t_0)) \left[D_t v(t, x) + \frac{1}{2} [(\sigma^e)^2(1 - x)^2 + (\sigma^i)^2 x^2] D_{xx} v(t, x) \right. \\
 & + x[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] D_x v(t, x) \\
 & + [u_t^1 - r_e(t) - (\sigma^e)^2] D_x v(t, x) \\
 & + u^{2*} D_x v(t, x) + \exp(-r_d(t - t_0)) b_2(u^{2*}) \\
 & \left. + \exp(-r_d(t - t_0)) b_3(x) - \frac{1}{2} \frac{(D_x v(t, x))^2}{D_{xx} v(t, x)} \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \right] \\
 = & -r_d \frac{1}{2} (x - x_r(t))^2 q(t) - \frac{1}{2} r_d s(t) - (x - x_r(t)) q(t) \dot{x}_r(t) \\
 & + \frac{1}{2} (x - x_r(t))^2 \dot{q}(t) + \frac{1}{2} \dot{s}(t) \\
 & + \frac{1}{2} [(\sigma^e)^2(1 - x)^2 + (\sigma^i)^2 x^2] q(t) \\
 & + (x - x_r(t)) q(t) [x(r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2) + u_t^1 - r_e(t) - (\sigma^e)^2] \\
 & - \frac{1}{2} (x - x_r(t))^2 q(t)^2 / c_2 \\
 & + \frac{1}{2} c_3 (x - adr_1)^2 - \frac{1}{2} (x - x_r(t))^2 q(t) \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t).
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}x^2 \left[-r_d q(t) + \dot{q}(t) + (\sigma^e)^2 q(t) + (\sigma^i)^2 q(t) \right. \\
 &\quad \left. + 2q(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] - q(t)^2/c_2 + c_3 \right. \\
 &\quad \left. - q(t)\tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \right] \\
 &+ x \left[+r_d x_r(t)q(t) - q(t)\dot{x}_r(t) - x_r(t)\dot{q}(t) - (\sigma^e)^2 q(t) \right. \\
 &\quad \left. - x_r(t)q(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] + q(t)[u_t^1 - r_e(t) - (\sigma^e)^2] \right. \\
 &\quad \left. + x_r(t)q(t)^2/c_2 - c_3 a d r_1 + x_r(t)q(t)\tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \right] \\
 &+ \frac{1}{2}x^0 \left[-r_d x_r(t)^2 q(t) + 2x_r(t)q(t)\dot{x}_r(t) + x_r(t)^2 \dot{q}(t) - r_d s(t) + (\sigma^e)^2 q(t) \right. \\
 &\quad \left. - 2x_r(t)q(t)[u_t^1 - r_e(t) - (\sigma^e)^2] - x_r(t)^2 q(t)^2/c_2 + c_3 (a d r_1)^2 \right. \\
 &\quad \left. - x_r(t)^2 q(t)\tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) + \dot{s}(t) \right].
 \end{aligned}$$

It will be proven that the terms at the powers of the indeterminate x are zero, thus at x^2 , x^1 , and x^0 . The term at x^2 is zero due to the differential equation for q . The term at x^1 equals,

$$\begin{aligned}
 &-q(t)\dot{x}_r(t) - x_r(t)\dot{q}(t) + r_d x_r(t)q(t) - 2(\sigma^e)^2 q(t) \\
 &-x_r(t)q(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] + q(t)[u_t^1 - r_e(t) - (\sigma^e)^2] \\
 &+ x_r(t)q(t)^2/c_2 - c_3 a d r_1 + x_r(t)q(t)\tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \\
 &= -q(t)\dot{x}_r(t) \\
 &+ x_r(t)[-q(t)^2/c_2 + c_3 + 2q(t)(r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2) \\
 &\quad + q(t)((\sigma^e)^2 + (\sigma^i)^2 - r_d - \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t))] \\
 &-x_r(t)q(t)[(r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2) - r_d - \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t)] \\
 &+ q[-(\sigma^e)^2 + (u_t^1 - r_e(t) - (\sigma^e)^2)] + x_r(t)q(t)^2/c_2 - c_3 a d r_1 \\
 &= q(t)[- \dot{x}_r(t) + c_3(x_r(t) - a d r_1)/q(t) \\
 &\quad + x_r(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] \\
 &\quad + (x_r(t) - 1)((\sigma^e)^2 + (\sigma^i)^2) + (\sigma^i)^2 + (u_t^1 - r_e(t) - (\sigma^e)^2) \\
 &= 0.
 \end{aligned}$$

Similarly, the term of x^0 equals,

$$\begin{aligned}
 & \dot{s}(t) - r_d s(t) - r_d x_r(t)^2 q(t) \\
 & + 2x_r(t)[q(t)\dot{x}_r(t) + x_r(t)\dot{q}(t)] - x_r(t)^2 \dot{q}(t) + (\sigma^e)^2 q(t) \\
 & - 2x_r(t)q(t)[u_t^1 - r_e(t) - (\sigma^e)^2] + c_3 a d r_1^2 - x_r^2(t)q(t)^2/c_2 \\
 & - x_r(t)^2 q(t) \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \\
 = & \text{ (using the term with } x^1), \\
 & \dot{s}(t) - r_d s(t) - r_d x_r^2(t)q(t) + (\sigma^e)^2 q(t) - 2x_r(t)q(t)[u_t^1 - r_e(t) - (\sigma^e)^2] + c_3 a d r_1^2 \\
 & - x_r^2(t)q(t)^2/c_2 - x_r(t)^2 q(t) \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \\
 & + 2r_d x_r(t)^2 q(t) - 2x_r(t)q(t)(\sigma^e)^2 - 2x_r(t)^2 q(t)[r_0 + r_e - r_i + (\sigma^e)^2 + (\sigma^i)^2] \\
 & + 2x_r(t)q(t)[u_t^1 - r_e(t) - (\sigma^e)^2] - c_3 a d r_1^2 2x_r(t) \\
 & + 2x_r(t)^2 q(t)^2/c_2 + 2x_r(t)^2 q(t) \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t) \\
 & - x_r(t)^2 q(t)^2/c_2 + c_3 x_r(t)^2 + x_r(t)^2 q(t) 2(r_0 + r_e - r_i + (\sigma^e)^2 + (\sigma^i)^2) \\
 & + x_r(t)^2 q(t)(-r_d + (\sigma^e)^2 + (\sigma^i)^2 - \tilde{r}_y(t)^T \tilde{C}_t^{-1} \tilde{r}_y(t)) \\
 = & \dot{s}(t) - r_d s(t) + c_3(x_r(t) - a d r_1)^2 - (\sigma^e)^2 q(t)(x_r(t) - 1)^2 - (\sigma^i)^2 q(t)x_r(t)^2 \\
 = & 0.
 \end{aligned}$$

It then follows from Theorem 4.1.1 that the indicated control laws are the optimal ones. \square

Chapter 5

NUMERICAL EXAMPLES

5.1 DATA

5.2 NUMERICAL EXAMPLES: BANK REGULATORY CAPITAL

5.2.1 Illustrations of Capital-to-Risk Weighted Assets Ratio and Capital-to-Total Assets Ratio vs Output Gap for Other OECD Countries

5.2.2 Illustrations of Capital-to-Risk Weighted Assets Ratio and Capital-to-Total Assets Ratio vs Output Gap for South Africa

5.3 NUMERICAL EXAMPLES: ASSETS-TO-OUTSTANDING DEBT RATIO

The main problem investigated in this chapter can be stated as follows.

Problem 5.0.3 (Numerical Examples): *Can we provide numerical examples involving auditing and bank regulatory capital ?*

5.1 DATA

The data for the Organization for Economic Co-operation and Development (OECD) countries was obtained from the OECD and International Monetary Fund (IMF) websites. The South African data was obtained from the South African Reserve Bank (SARB) website. We also obtained data for the capital-to-risk-weighted assets ratio and capital-to-total assets ratio for South Africa and OECD countries and calculated the output gap (see Section 9.4 regarding details on output gap).

5.2 NUMERICAL EXAMPLES: BANK REGULATORY CAPITAL

Subsections 5.2.1 and 5.2.2 are based on the results obtained in Chapter 3. Capital adequacy ratios on their own, as calculated in Subsection 2.2.3, convey a partial message. One of the most important issues related to CARs are their effect on financial stability in the banking industry. It is a known fact that cyclicalities are the root cause of financial instability in banking. Under Basel II, capital requirements are likely to increase in recessions. If capital requirements indicate this tendency - when building reserves from decreasing profits is difficult or raising fresh capital is likely to be extremely costly - banks should reduce their loans and the subsequent credit crunch would add to the downturn. This would make the recession deeper, which might have an adverse effect on the stability of the banking system. For this reason capital requirements are said to be procyclical despite actually increasing (decreasing) during a downturn (upturn). The implications of this link between financial stability and macro-economic stability in terms of the soundness of bank's merit being taken into account in the final design of Basel II.

To confirm the validity of the claims in the previous paragraph, we rely on illustrative data from some member countries of the OECD as supplied on the website [77] as well as South Africa. The countries for which simulation parameters for regulatory capital-to-risk-weighted assets ratio and the capital-to-total assets ratio was obtained are Australia, Finland, Italy, Japan, Norway, South Africa, Spain, Sweden, the United Kingdom and the United States of America. The graphs in Subsection 5.2.1, provide evidence to confirm the fact that capital adequacy ratios are generally negatively correlated with the economic cycle. We also see this behavior in the example given in Subsection 5.2.2 of the South African case.

5.2.1 Illustrations of Capital-to-Risk-Weighted Assets Ratio and Capital-to-Total Assets Ratio vs Output Gap for Other OECD Countries

In this subsection, we represent graphically the capital-to-risk-weighted assets ratio and the capital-to-total assets ratio versus output gap for Australia (period 1990-2000), Finland (period 1992-2000), Italy (period 1987-2000), Norway (period 1992-2000), Spain (period 1986-2000), Sweden (period 1992-2000), the United Kingdom (period 1990-2000) and the United States of America (period 1990-2000). We use these graphs to characterize and discuss the cyclicalities of capital adequacy ratios.

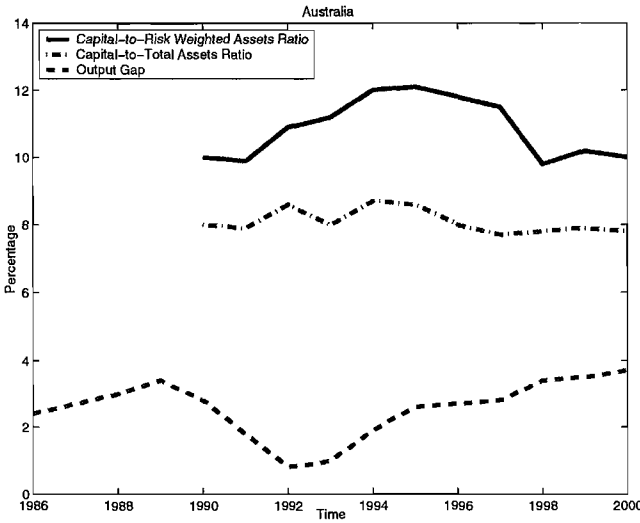


Figure 5.1: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for Australia

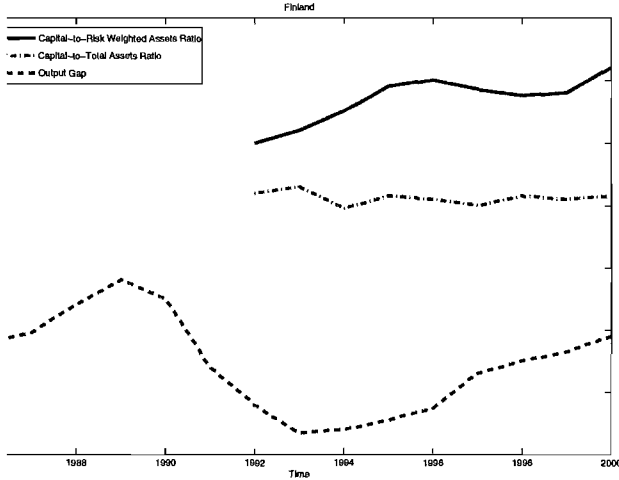


Figure 5.2: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for Finland

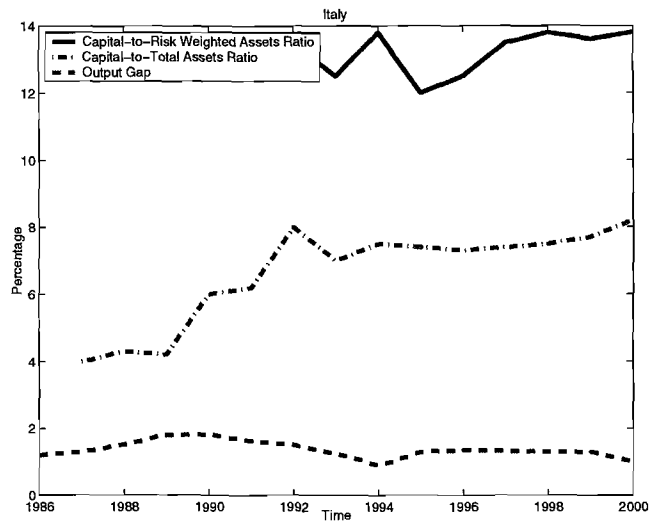


Figure 5.3: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for Italy

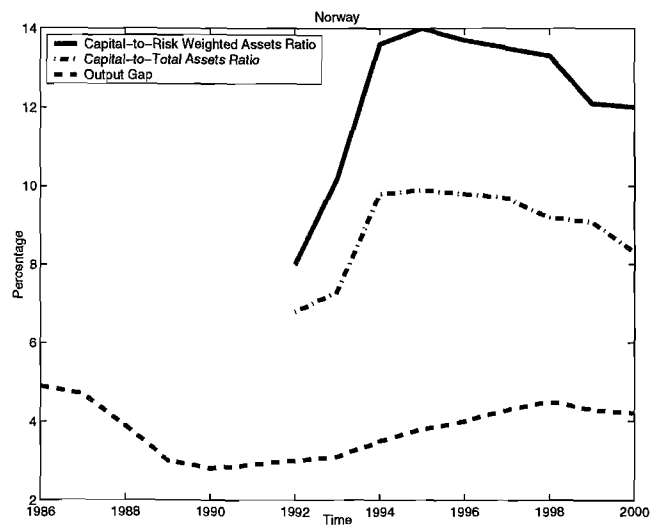


Figure 5.4: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for Norway

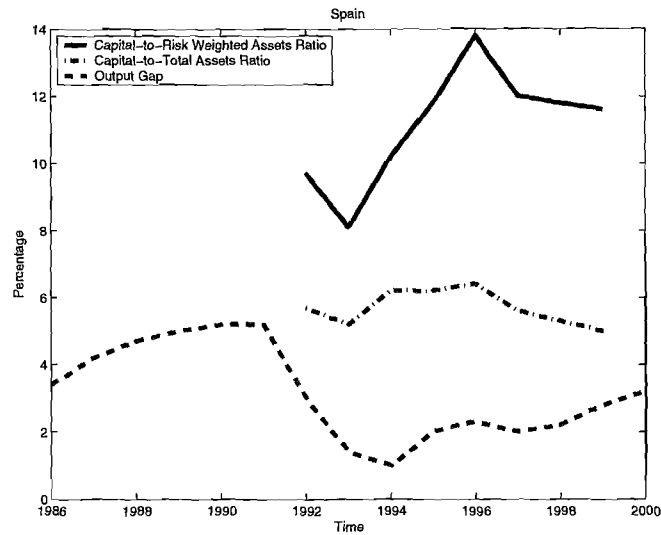


Figure 5.5: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for Spain

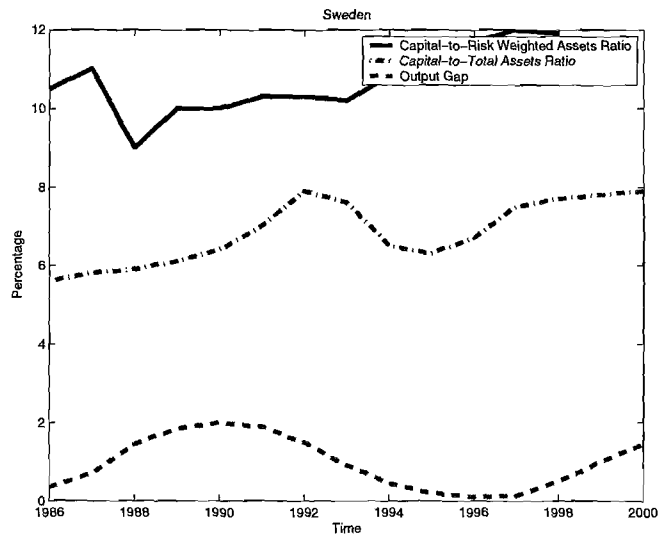


Figure 5.6: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for Sweden

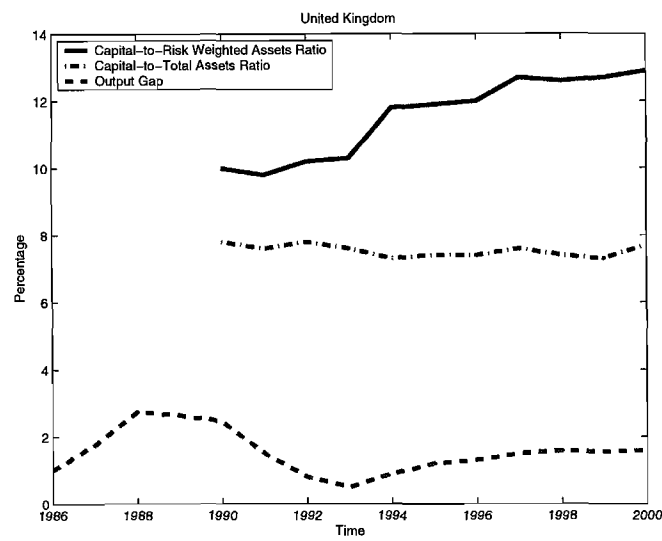


Figure 5.7: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for the UK

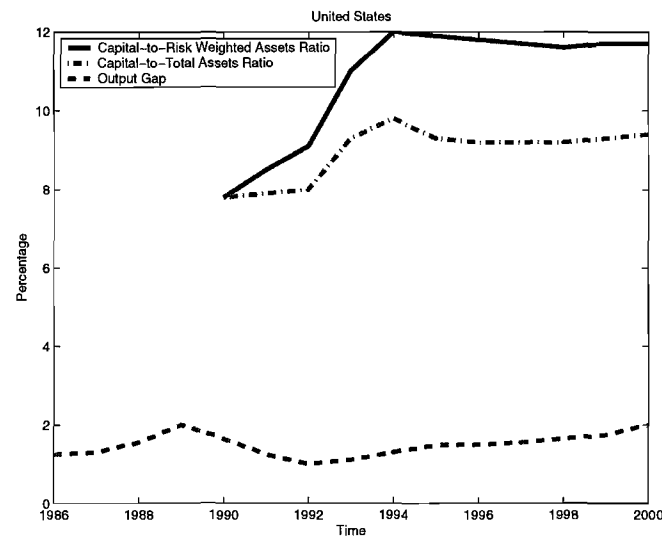


Figure 5.8: Capital-to-Risk-Weighted Assets Ratio vs Capital-to-Total Assets Ratio for the USA

5.2.2 Illustration of Capital-to-Risk-Weighted Assets Ratio and Capital-to-Total Assets Ratio vs Output Gap for South Africa

In this subsection we represent graphically the capital-to-risk weighted assets ratio and the capital-to-total assets ratio versus output gap for the South African situation, which is an example of a non-OECD country. In Section 9.4, we sketched the actual GDP output versus the potential GDP output for South Africa from 1970 to 2006. The minimum capital adequacy changed to 10 % in 2000.

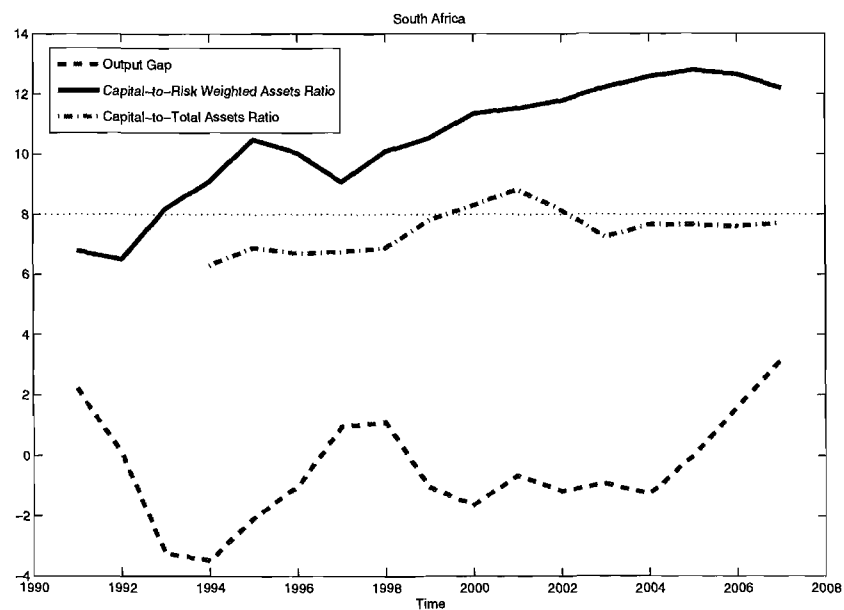


Figure 5.9: Capital-to-Risk-Weighted Assets Ratio and Capital-to-Total Assets Ratio vs Output Gap for SA

5.3 NUMERICAL EXAMPLES: ASSETS-TO-OUTSTANDING DEBT RATIO

Below we represent a table with data about assets and outstanding debt from a composite of South African banks for the period 1994 to 2007. Here we calculate the *total assets-to-outstanding debt ratio* of the banks and represent the data graphically. The total assets-to-outstanding debt ratio measures the liquidity of the bank, in

particular its ability to meet current obligations with existing liquid assets. The higher the total assets-to-outstanding debt ratio, the better the bank's ability to meet current obligations. Some debt agreements require that the borrower should maintain a minimum total assets-to-outstanding debt ratio such as 2 to 1.

| Year | Total Assets | Outstanding Debt | Ratio |
|------|--------------|------------------|-------|
| 1994 | 301096394 | 148323347 | 2.03 |
| 1995 | 356272295 | 169653474 | 2.10 |
| 1996 | 419882409 | 191727127 | 2.19 |
| 1997 | 493207386 | 219203283 | 2.25 |
| 1998 | 602886824 | 257643942 | 2.34 |
| 1999 | 694412855 | 290549312 | 2.39 |
| 2000 | 766714485 | 311672555 | 2.46 |
| 2001 | 919599933 | 362047218 | 2.54 |
| 2002 | 1061864997 | 406844826 | 2.61 |
| 2003 | 1314558464 | 490506889 | 2.68 |
| 2004 | 1390825007 | 509459710 | 2.73 |
| 2005 | 1586275216 | 566526863 | 2.80 |
| 2006 | 1924968094 | 661501063 | 2.91 |
| 2007 | 2172260125 | 702996804 | 3.09 |

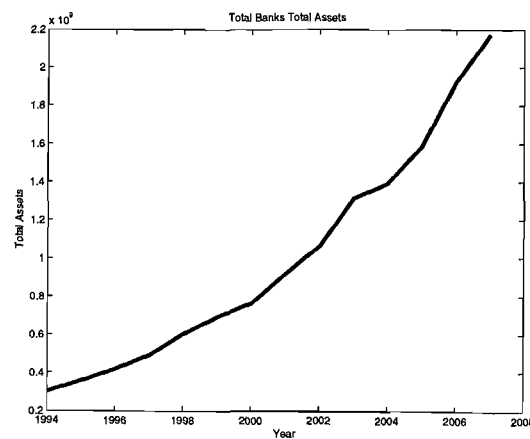


Figure 5.10: Total Assets in South Africa for 1990-2007

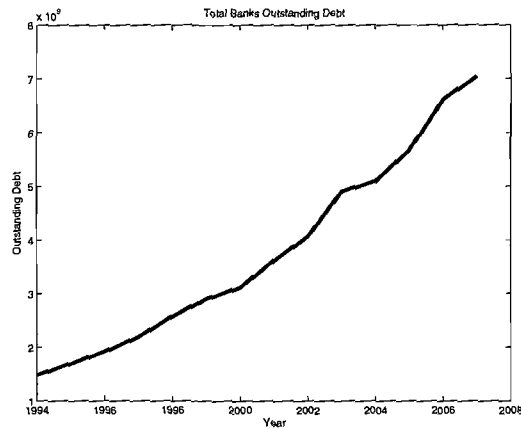


Figure 5.11: Outstanding Debt in South Africa for 1990-2007

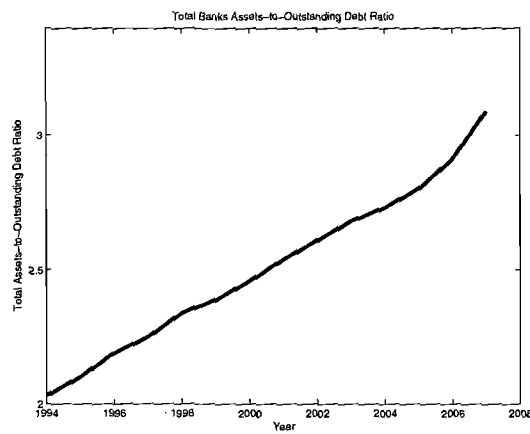


Figure 5.12: Total Assets-to-Outstanding Debt Ratio in South Africa for 1990-2007

Thus we conclude that the composite of South African banks have maintained a total assets-to-outstanding debt ratio greater than 2.

Chapter 6

ANALYSIS OF THE MAIN ECONOMIC ISSUES

6.1 STOCHASTIC MODEL FOR BANKS

6.1.1 Assets

6.1.2 Bank Capital

6.1.3 Bank Assets-to-Outstanding Debt Model

6.1.4 Alternative Stochastic Model for Banks

6.1.4.1 Assets

6.1.4.2 Liabilities

6.1.4.3 Bank Capital

6.1.4.4 Comparison Between the Models

6.2 OPTIMAL AUDITING IN THE BANKING INDUSTRY

6.2.1 Random Auditing: Reserve Requirements

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In accordance with the dictates of the Basel II, the models of bank items constructed in this thesis are related to the methods being used to assess the riskiness of bank portfolios and their minimum capital requirement (see [11] and [16]).

6.1 STOCHASTIC MODEL FOR BANKS

In this subsection, we analyse aspects of the bank items such as assets, capital and liabilities that are introduced in Chapter 2.

6.1.1 Assets

The opportunity for bank investment in loans is discussed in Subsection 2.1.2, where the loan demand is Hicksian and described in terms of macroeconomic activity in the loan market. Banks experience shocks that affect the value of the loan demand, Λ , when the minimum capital requirements for Basel II are calculated by using risk-weighted assets. In the Hicksian case typified by (2.1), these responses are usually sensitive to macroeconomic conditions that are related to the term $l_2 M_t$. The discussion in Subsection 2.1.2 leads to a definition of the *elasticity of demand*, d^e , of the form

$$d^e = \frac{l_1 r_t^\Lambda}{\lambda_t}.$$

It is well-known that if the bank is perfectly competitive, d^e will tend to ∞ . This, of course, is not the case in our imperfectly competitive paradigm.

The bank's investment in loans may yield substantial returns but may also result in loan losses. Loan defaults are generally independent of the capital adequacy paradigm that is chosen. In this regard, empirical evidence suggests that better macroeconomic conditions reduce the loan default rate and thus the loan marginal cost (see the discussion on the procyclicality of loan losses in, for instance, [20], [22] and [29]). In line with this reality, our dynamic banking loan model in (2.2) allows for loan losses and their partial provisioning. The related default risk may be modeled as a compound Poisson process where N is a Poisson process with a deterministic frequency parameter, $\phi(t)$. Here N is stochastically independent of the Brownian motion, Z , given in (2.2). Furthermore, we introduce the value of *loan losses* as

$$L(M_t, t) = r^d(M_t)\lambda_t, \tag{6.1}$$

where L is independent of N . Also, we may assume that the default or loan loss rate, $r^d \in [0, 1]$, increases when macroeconomic conditions deteriorate according to

$$0 \leq r^d(M_t) \leq 1, \quad \frac{\partial r^d(M_t)}{\partial M_t} < 0.$$

As is the case with the relationship between profit and macroeconomic activity, the above description of the loan loss rate is consistent with empirical evidence that suggests that bank losses on loan portfolios are correlated with the business cycle under any capital adequacy regime (see, for instance, [20], [22], [29] and [62]). Furthermore, it may be that the provision made by the bank for loan losses takes the form of a continuous contribution that can be expressed as

$$\left[1 + \theta(s)\right] \phi(s) \mathbf{E}[P_s(l)],$$

where θ is a loading term dependent on the level of credit risk, $\theta(t) \geq 0$ and P_t is the actual provision for loan losses. This means that if the bank suffers a loan loss of $\lambda = l$ at time t , the provisions, $P_t(l)$, covers these losses.

Several interesting contributions have led to the choice of representation (2.5) for the dynamics of the sum of the Treasuries and reserves. Amongst these is a paper by Chan (see [31]) that treats the case where the Lévy process decomposition of general assets is a lot finer than is the case here.

6.1.2 Bank Capital

Despite the analysis in Section 2.2, bank capital is notoriously difficult to define, monitor and measure. In this regard, for instance, the measurement of equity depends on how all of a bank's financial instruments and other assets are valued. In general, the modeling of the shareholder equity component of bank capital, E , is underpinned by the following two observations. In the first place, it is meant to reflect the nature of the book value of equity and, secondly, to recognize that the book and market value of equity is highly correlated.

Under Basel II, bank capital requirements have replaced reserve requirements (see Subsection 2.1.3) as the main constraint on the behavior of banks. A first motivation for this is that bank capital has a major role to play in overcoming the moral hazard problem arising from asymmetric information between banks, creditors and debtors. Also, bank regulators require capital to be held to protect themselves against the costs of financial distress, agency problems and the reduction of market discipline

caused by the safety net.

Subsection 2.2.2 suggests that a close relationship exists between bank capital holding and macroeconomic activity in the loan market. As was mentioned before, Basel II dictates that a macroeconomic shock will affect the loan risk-weights in the CAR. In general, a negative (positive) shock results in the tightening (loosening) of the capital constraint given by (6.6). As a consequence, in terms of a possible binding capital constraint, banks are free to increase (decrease) the loan supply when macroeconomic conditions improve (deteriorate). On the other hand, if the risk-weights are constant, a shock does not affect the loan supply but rather results in a change in the loan rate when the capital constraint binds. It is not always true that Basel II risk-sensitive weights lead to an increase (decrease) in bank capital when macroeconomic activity in the loan market increases (decreases). A simple explanation for this is that macroeconomic conditions do not necessarily only affect loan demand but also influences the total capital constraint from (6.6). Furthermore, banks do not necessarily need to raise new capital to expand their loan supply, since a positive macroeconomic shock may result in a decrease in the RWAs with a corresponding increase in CARs (compare (1.1)). Similarly, banks are not compelled to decrease their capital when the loan demand decreases since the capital constraint usually tightens in response to a negative macroeconomic shock. A further complication is that an improvement in economic conditions may result in an increase in the loan demand and, as a consequence, an increase in the probability that the capital constraint will be binding. Banks may react to this situation by increasing capital to maximize profits (compare the definition of the return on equity (ROE)). Our main conclusion is that bank capital is procyclical because it is dependent on fluctuations in loan demand which, in turn, is reliant on macroeconomic activity.

6.1.3 Banks Assets-to-Outstanding Debt Model

The *total assets-to-outstanding debt ratio* measures the liquidity of the bank, in particular its ability to meet current obligations with existing liquid assets. The *total assets-to-outstanding debt ratio* are used in auditing and as an indicator of solvency.

6.1.4 Alternative Stochastic Model for Banks

In this subsection, our main objective is to construct alternative Lévy process-driven stochastic dynamic models for bank assets and capital (see the submitted paper [27]). We discuss a bank with a planning horizon equal to a number of back-testing periods which, in turn, is divided into n non-intersecting reporting periods of equal length.

6.1.4.1 Assets

In this subsection, we discuss bank asset price processes and unweighted and risk-weighted assets. In order to model the uncertainty associated with these items we consider the filtered probability space $(\Omega_1, \mathbf{G}, (\mathcal{G}_t)_{0 \leq t \leq T}, \mathbf{P}_1)$.

The bank's investment portfolio is constituted by $m + 1$ assets that includes loans, shares, intangible assets and Treasuries. We pick the first asset to be the riskless Treasuries, \mathbf{T} , that earns a constant, continuously-compounded interest rate of $r^{\mathbf{T}}$. Profit maximizing banks consider their rates of return on assets to be the sum of the risk-free Treasuries rate, $r^{\mathbf{T}}\bar{\mathbf{1}}$ and the drift, μ . Here the unitary vector and drift are given by $\bar{\mathbf{1}} = (1, 1, \dots, 1)^T$ and

$$\mu = (\mu_1, \mu_2, \dots, \mu_m)^T, \text{ where } \mu_i = \begin{cases} \mathbf{r}_i + \mathbf{E}(d_i), & i\text{-th asset is a loan,} \\ r_i^a, & i\text{-th asset is not a loan.} \end{cases}$$

respectively. Also, \mathbf{r}_i is the risk premium for the i -th asset class, r_i^a is the auxiliary rate of return of the non-loan asset i and the expected loan loss ratio is given by $\mathbf{E}(d) = (\mathbf{E}(d_1), \mathbf{E}(d_2), \dots, \mathbf{E}(d_m))^T$. The sum $r_i^a\bar{\mathbf{1}} + \mathbf{r}_i$ covers, for instance, the cost of monitoring and screening of loans in the i -th asset class and cost of capital for that class. The $\mathbf{E}(d_i)$ component corresponds to the amount of provisioning that is needed to match the average expected losses faced by the loans in the i -th asset class. The m assets besides Treasuries are risky and their price process, S (taking the reinvestment of dividends into account), is driven by a Lévy process with drift vector, $r^{\mathbf{T}}\bar{\mathbf{1}} + \mu$ and diffusion matrix, σ , as in

$$S_t = S_0 + \int_0^t I_s^S \left(r^{\mathbf{T}}\bar{\mathbf{1}} + \mu + \sigma a \bar{\mathbf{1}} \right) ds + c \int_0^t I_s^S \sigma dZ_s + \int_0^t I_s^S \sigma dM_s, \quad (6.2)$$

where I_t^S denotes the $m \times m$ diagonal matrix with entries S_t and L is an m -dimensional Lévy process. Furthermore, we assume that $\text{rank}(\sigma) = m$ and the bank is allowed to engage in continuous frictionless trading over the planning horizon.

In the sequel, we suppose that ρ is the m -dimensional stochastic process that denotes the *current value of risky assets*. In this case, the dynamics of the *current value of the bank's entire asset portfolio*, A , over any reporting period may be given by

$$dA_t = A_t \left\{ r^T + \rho_t^T (\mu + \sigma a) \right\} dt + c A_t \rho_t^T \sigma dZ_t + A_t \rho_t^T \sigma dM_t - r^T D_t dt, \quad (6.3)$$

where the face value of the deposits, D , is as described in Subsubsection 6.1.4.2 below, and $r^T D_t dt$ represents the interest paid to depositors.

The charge to cover credit risk equals the sum of the bank's long and short trading positions multiplied by risk weights for specific assets. As a result, if we let $\omega \in [0, 1]^m$ denote the $m \times 1$ vector of asset risk weights, then the *capital charge to cover credit risk* at time t equals

$$\omega^T \left(\rho_t^+ + \rho_t^- \right), \quad (6.4)$$

where for any ρ we denote by ρ^+ the $m \times 1$ vector with components $\rho_i^+ = \max[0, \rho_i]$ (long trading positions) and by ρ^- the $m \times 1$ vector with components $\rho_i^- = \max[0, -\rho_i]$ (short trading positions).

6.1.4.2 Liabilities

The bank's liabilities are solely constituted by deposits. For simplicity, we assume that the face value (or outstanding value) of deposits, D , is fixed over the planning horizon and that complications related to equity issues and dividend payments are removed over this period. Furthermore, we suppose that deposit insurance covers deposits completely, the deposit rate equals the risk-free Treasuries rate, $r^T \geq 0$, and depositors are continuously being compensated at this rate.

6.1.4.3 Bank Capital

In this subsection, we discuss bank regulatory capital, K (see, for instance [38]), and its stochastic dynamics as well as capital adequacy. Let us define the bank's regulatory capital, K , as

$$K_t = A_t - D_t, \quad (6.5)$$

where A is the current value of the total assets and D is the face value of the deposits. According to Basel II regulation, the bank is required to maintain regulatory capital

above a minimum level equal to the sum of the charges to cover general market risk, credit risk and operational risk (see, for instance, [13]). As far as the charge to cover market risk is concerned, we suppose that the bank divulges its current VaR at the start of each reporting period as well as the value of its profits and losses from the preceding reporting period. In fact, the market risk charge equals the VaR reported at the beginning of the current reporting period times a *capital reserve multiple* k . As a consequence of the above, if $VaR \geq 0$ is the VaR reported to regulators at the beginning of the current reporting period and k is the multiple that currently applies, the bank must satisfy the constraint

$$K_t \geq kVaR + \omega^T \left(\rho_t^+ + \rho_t^- \right) + \mathfrak{O} \quad (6.6)$$

throughout the duration of the reporting period. In essence, the reported VaR can differ from the true VaR because of the fact that the bank's future trading strategy, and hence the bank's true VaR, cannot be observed by regulators. Moreover, Basel II prescribes that \mathfrak{O} in inequality (6.6) may be written as

$$\mathfrak{O} = \max \left[\sum_{k=1}^8 \beta_k g_k, 0 \right]$$

and constitutes the capital charge to cover operational risk under the standardized approach.

From this point forward we do not consider operational risk (compare the last term in (6.6)), since it may be considered to be constant over all reporting periods. It may happen that the bank incurs a cost, c , at the termination of each reporting period in which the actual loss exceeds the reported VaR. This cost does not take the increase in the capital reserve multiple k into account and is meant to relate to further regulatory interventions that can occur as a result of exceptions or reputation losses. For simplicity, we refer to these costs simply as *reputation costs* and assume that they are proportional to the amount by which the actual loss exceeds the reported VaR. This implies that

$$c = \lambda(K_b - K_e - VaR)^+,$$

where $\lambda \geq 0$ is the proportionality cost constant. Here, K_b and K_e is the value of the

bank's regulatory capital at the onset and termination of the particular reporting period, respectively. At the end of each back-testing period, the number $i = 0, 1, \dots, n$ of reporting periods in which the actual loss exceeded the reported VaR is determined. In this case, the reserve multiple, k , for the next back-testing period is set equal to $k(i)$ such that

$$k(0) \leq k(1) \leq k(2) \leq \dots \leq k(n).$$

It is clear that the sceptre of reputation costs and the revision of the value of k at the end of each back-testing period removes the incentive for under-reporting the true VaR. On the other hand, capital requirements provide an incentive to not over-report.

Besides the market risk emanating from volatility in the value of its assets (as given in (6.3)), the bank experiences unhedgeable credit risk. In this regard, at the termination of each reporting period, a small probability, p , may exist that an anomaly will occur that will lead to the loss of an amount of qK_b , where $q \in [0, 1]$. These anomalies can result in bank failure if the value of the bank's capital becomes negative so that debts cannot be paid. While these shocks are unhedgeable, the bank can manipulate the default probability by controlling the probability of losses in the market value of its assets exceeding $(1 - q)K_b$ in any given period. In essence, this means that the bank avoids excessively risky investment strategies. Since the market value of deposits is kept constant and issues related to equity does not play a role, it follows from (6.3) that the dynamics of the bank's regulatory capital may be represented by

$$dK_t = K_t \left\{ r^T + \rho_t^T (\mu + \sigma a) \right\} dt + cK_t \rho_t^T \sigma dZ_t + K_t \rho_t^T \sigma dM_t \quad (6.7)$$

in the absence of anomalies.

In the sequel, we deduce the dynamics of the capital-to-total assets ratio and the capital-to-risk weighted ratio. We start by recalling that the current value of total assets, A , and bank regulatory capital, K , are given by equations (6.3) and (6.7), respectively. In this regard, we obtain a SDE for the dynamics of the capital-to-total assets ratio in the following result.

Theorem 6.1.1 (Computation of the Capital-to-Total Assets Ratio): *Let (6.3) and (6.7) represent the total assets and bank capital, respectively. Then we have that the dynamics of the capital-to-total assets ratio, $\tilde{\kappa}$, has the form*

$$d\tilde{\kappa}_t = \tilde{\kappa}_t \left\{ \left(-2 \left[r^T + \rho^T (\mu + \sigma a) \right] - 2(\rho^T)^2 \sigma^4 - A_t^{-1} r^T D_t \right) dt + 2c\rho_t^T \sigma dZ_t + 2\rho_t^T \sigma dM_t \right\} \quad (6.8)$$

Proof. The proof is a straightforward application of Ito's formula. \square

Next, we comment on a simplified version of the capital-to-risk weighted assets ratio, κ , of the form

$$\kappa = \frac{K}{A^r}. \quad (6.9)$$

In this calculation of the capital-to-risk weighted assets ratio, the total risk charge is only constituted by the credit risk charge with the capital charges for market and operational risk not being considered (see, for instance, [13]). By way of justifying this simplification, we may consider the capital charges for market and operational risk to be invariant over the reporting period and hence of lesser importance dynamically. Next, we derive an expression for the capital-to-risk weighted assets ratio by considering the SDE for bank regulatory capital given by (6.7). This result is proved in the same way as Theorem 6.1.1.

Theorem 6.1.2 (Computation of the Capital-to-Risk Weighted Assets Ratio): *Let the dynamics of bank regulatory capital be as in (6.7). Then the capital-to-risk weighted assets ratio, κ , may be represented by*

$$d\kappa_t^{-1} = A_t^r \left\{ r^T + \rho_t^T (\mu + \sigma a) \right\} dt + cA_t^r \rho_t^T \sigma dZ_t + A_t^r \rho_t^T \sigma dM_t. \quad (6.10)$$

6.1.4.4 Comparison Between the Models

In this section, we compare the model in the optimal auditing paper [26] by Bosch *et al.* (denoted by BM) with the alternative model (denoted by AM) presented in Subsection 6.1.4.

Both the BM and AM describe the bank assets, liabilities and capital in a stochastic dynamic paradigm. In this regard, for instance, deposits are defined in the same way in both contributions. Also, the loan rate, r^Λ , is expressed as the sum of the risk-free Treasuries rate, r^T , expected loss ratio, $\mathbf{E}(d)$, and risk premium, \mathbf{r} , in both the BM and AM. The capital adequacy ratio plays a major role in the BM and AM. In this regard, the capital constraint from the Basel II capital accord is of common interest.

In the AM the underlying stochastic process is a Lévy process that is more general than the jump diffusion process considered in the BM. The investigation in the AM takes place during time intervals involving the planning, back-testing and reporting periods. This clear distinction between time periods is not made in the BM. The BM considers loans, Λ , as the sum of loans from all credit types while the AM differentiates between credit types on the grounds of asset classes. Also, the AM includes all possible bank asset classes while the BM does not. More specifically, in the latter model, the only asset classes considered are reserves, Treasuries and loans. The BM includes a discussion on macroeconomic factors and procyclicality of assets, while the AM does not. Risk-weighted assets in the BM are given for specific credit types like loans and Treasuries, while such assets are described more generally in the AM. In the latter, the charge to cover credit risk equals the sum of the bank's long and short trading positions multiplied by risk weights for specific assets. Here the market risk charge involves VaR considerations. As a result of the discussion in this paragraph, we may conclude that the AM has a closer connection with issues related to banking risk than the BM. The AM provides a wider definition of bank capital than the BM does. In the latter, we are only concerned with equity capital and subordinate debt. The capital constraint given in (2.9) reflects that fact. As was mentioned before, the AM considers capital charges for market, credit and operational risk while the BM only considers credit risk. The capital adequacy ratio in the AM is calculated from its constituent financial variables, viz., assets and capital. This is not the case for the BM.

6.2 OPTIMAL AUDITING IN THE BANKING INDUSTRY

In this section, we discuss some of the issues related to the optimal auditing problem presented in Chapter 3. We note that the paper [26] also contains a discussion on asset requirements for a random audit which coincides with the paper [19].

6.2.1 Random Auditing: Reserve Requirements

As was mentioned before, we define the rate of depository consumption as the rate at which Treasuries and reserves are consumed by the taking, holding and anticipated withdrawal of deposits. Here, a typical bank owner has to make decisions about deposit taking via the fixing of costs related to cheque clearing and bookkeeping, the holding of deposits by means of the choice of the deposit and coupon rate and

anticipated withdrawals via the provisioning provided by Treasuries and reserves. We note that we are dealing with the bank owner's utility in (3.1) where the optimal rate of depository consumption in Theorem 3.1.2 is expressed almost exclusively in terms of the components of the Treasuries and reserves. For mutual banks, of course, the aforementioned utility corresponds with the depositor's utility because they are also the owners of the bank.

The formulation of Theorem 3.1.2 of Section 3.1 suggests that [1] and Theorem 3.5 of [78] can be used to derive the associated HJB equation for optimal control of jump diffusions. In addition, certain verification theorems claim that if the objective function, V , is smooth and the related HJB equation has a smooth solution, \tilde{V} , then under certain regularity conditions, $V = \tilde{V}$. We are able to appeal to the theory of viscosity solutions in those cases for which a smooth solution does not exist.

The optimal consumption rate from (3.7) in Theorem 3.1.2 suggests that k^* is directly proportional to the sum of the Treasuries and reserves, $w = T_0 + R_0$ at $t = 0$. Here the proportionality constant $(\alpha K)^{1/\alpha-1}$, $0 < \alpha < 1$, is related to the discounting exponent, δ , the bank reserve allocation, π^* , the dynamics of the sum of Treasuries and reserves, dW , as well as, by definition, the loans, Λ . As a consequence, the associated control law would seem to suggest that as the initial value of the sum of Treasuries and reserves, w , increases (decreases) the optimal consumption rate, k^* , will also increase (decrease). In essence, this increase may be due to an increase (decrease) in the deposit rate and resulting depository activity.

A strong correlation has been established between reserve requirements and bank default and closure. In order to monitor the fluctuations in the reserves, in practice, a regular liquid reserve requirement report compiled by the individual bank may be audited along with its financial statements and relevant information. This process may result in a consolidated liquid reserve requirement report that is authored by the auditors. Effective legal reserve requirements may hamper the private capital market's ability to price bank deposits. In the model developed in [49], the market has less information about bank assets than the banks have, and a bank can therefore signal its superior information through its choice of excess reserves. Mandatory reserves can inhibit such signaling and therefore result in inefficient deposit pricing.

6.2.2 Non-Random Auditing: Capital Requirements

The solution to the optimal auditing time problem presented in Theorem 3.2.2 of Section 3.2 has a few interesting ramifications for capital adequacy regulation. From (3.10), we note for the optimality exponent $\kappa > 1$ and $\rho^r > 0$ that z^* must have

a positive value that is greater than the value of ρ^r by a specific proportionality constant. At optimality, (3.10) and (3.11) allow a direct comparison between κ on the one hand and the discounting exponent, δ , and the coefficient, ϱ , of the second term of the generator, G^z , on the other. More specifically, for an optimal CAR, z^* , we have that

$$\frac{\kappa}{\kappa - 1} \geq \frac{\delta}{\delta - \varrho}.$$

In addition, (3.11) and (3.12) seem to imply, for $(t, z) \in \mathcal{D}$, that z^κ is an important index for auditing purposes. This may suggest that banks with a suboptimal CAR value should be more concerned with the CAR value itself rather than its relationship with the regulatory ratio, ρ^r . By contrast, for $(t, z) \notin \mathcal{D}$, the value of ρ^r now plays a significant role. This may have something to do with the function of bank capital holdings as both a buffer against insolvency and an indicator of profitability via the return on equity (ROE) indices. Finally, (3.12) intimates that the optimal CAR, z^* , is bounded away from the regulatory threshold, ρ^r , by a constant factor that depends on the discounting exponent, δ , and the coefficient of the second term, ϱ , of the generator, G^z .

The optimization problem in Theorem 3.2.2 of Section 3.2 only makes sense if the bank owner can decide when to be audited. In particular, the formulas in (3.10) only yield the correct value function, $V^{(1)}(s, z)$, providing auditing actually happens at $z = z^*$. In particular, the value function has the power form z^κ before auditing occurs and the linear form $z^* - \rho^r$ after auditing occurs.

6.3 STOCHASTIC CONTROL OF BANKING SYSTEMS

In this section, we discuss the cost function and optimal control law for the simplified auditing model.

6.3.1 Optimal Stochastic Control for the Simplified Auditing Model

6.3.1.1 The Cost Function

The control objectives are to meet the financial obligations of the bank's members and to ask the members as little extra contributions as possible. The first control objective, to meet the financial obligations, is formulated as a cost on the asset-liability ratio, x in the simplified model. The second control objective is formulated as a cost on the extra contribution of the members of the bank, u^2 .

As to the mathematical form for the cost functions, the authors have considered several options, discussed below. Of course, one can formulate any cost function. The question then is whether the resulting dynamic programming equation can be solved analytically? The authors have obtained an analytic solution so far only for the case of quadratic cost functions, see Theorem 4.1.2.

The cost on the extra contributions by members, the function $b_2(u^2)$ is considered. If $u^2 > 0$ then the members need to pay additional contributions to the bank. These payments may be made by the members themselves or by their employers. The cost function should be such that extra payments are penalized, hence $u^2 > 0$ should imply that $b_2(u^2) > 0$.

If amounts are paid back to the bank's members, $u^2 < 0$ this is of course good but goes against the intentions of the bank. It may also be good to save a temporary surplus for payments to members later or for a rainy day. In general it seems best not to penalize payments to bank's members. Another option is to restrict the input variable u^2 to the set \mathbb{R}_+ .

For Theorem 4.1.2 the authors have selected the cost function $b_2(u^2) = \frac{1}{2}c_2(u^2)^2$. This penalizes both positive and negative values of u^2 in equal ways. The only reason for doing so is that then an analytic solution of the value function can be obtained.

The cost on meeting the bank payments will be encoded in a cost on the asset-debt-ratio. If the ADR $x < adr_1$ is strictly smaller than a set value adr_1 then there should be a strictly positive cost. If $x > adr_1$ then there may be a cost though most banks will be satisfied and not impose a cost. A major problem of bank's is what to do with surplus assets, assets which are not needed for meeting the obligations by contract. To which members do the surpluses belong?

The authors have selected in Theorem 4.1.2 the cost function $b_3(x) = \frac{1}{2}c_3(x - adr_1)^2$. This is done also to obtain an analytic solution of the value function and that case

by itself is interesting. Another cost function considered is

$$b_3(x) = c_3[\exp(adr_1 - x) + (x - adr_1) - 1],$$

which is strictly convex and asymmetric in x with respect to the value adr_1 . For this cost function costs with $x < adr_1$ are penalized higher than those with $x > adr_1$. This seems realistic. Another cost function considered is to keep $b_3(x) = 0$ for $x > adr_1$.

A future investigation will address other cost functions.

6.3.1.2 The Optimal Control Law

The optimal control law for the simplified model and with a quadratic cost function is stated in Theorem 4.1.2. The formulas for the optimal control law are:

$$u^{2*} = g_2^*(t, x) = -(x - x_r(t))q(t)/c_2, \quad (6.11)$$

$$\tilde{\pi}^* = g_1^*(x) = -\frac{x - x_r(t)}{x} \tilde{C}_t^{-1} \tilde{r}_y(t). \quad (6.12)$$

An interpretation of this control law follows. The additional contribution u^{2*} or the pay back of assets to the bank members is proportional to the difference between the asset-liability ratio, x , and the reference process for this ratio, $x_r(t)$ at time $t \in T$. The proportionality factor is $q(t)/c_2$ which depends on the relative ratio of the cost function on u^2 and on the deviation from the reference ratio, $(x - x_r)$. The property that the control law is symmetric in x with respect to the reference process x_r is a direct consequence of the cost function $b_r(x) = \frac{1}{2}c_3(x - x_r)^2$ being symmetric with respect to $(x - x_r)$.

The optimal portfolio distribution is proportional to the relative difference between the asset-debt-ratio and its reference process, $(x - x_r(t))/x$. This seems natural. The proportionality factor is $\tilde{C}_t^{-1} \tilde{r}_y(t)$ which represents the relative rates of increase of the assets multiplied with the inverse of the corresponding variances.

It is surprising that the control law has this structure. Apparently the optimal control law is not to sell first all assets with the highest variance or the lower rate, then the assets with the next highest variance or the next to lowest rate, etc. The proportion of all assets depends on the relative weighting in $\tilde{C}_t^{-1} \tilde{r}_y(t)$ and not on the deviation $(x - x_r(t))$.

The novel structure of the optimal control law is the *reference process for the asset-debt-ratio*, $x_r : T \rightarrow \mathbb{R}$. The differential equation for this reference function is

$$\begin{aligned}
-\dot{x}_r(t) &= -c_3(x_r - adr_1)/q(t) \\
&\quad -x_r(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2] \\
&\quad -[u_t^1 - r_e(t) - (\sigma^e)^2] - (x_r(t) - 1)((\sigma^e)^2 + (\sigma^i)^2) - (\sigma^i)^2, \\
x_r(t_1) &= adr_r.
\end{aligned} \tag{6.13}$$

This differential equation is new for the area of banking control and therefore deserves a discussion.

The differential equation has several terms on its right-hand side which will be discussed separately. Consider the term,

$$[u_t^1 - r_e(t) - (\sigma^e)^2].$$

This represents the difference between primary inflow and outflow of the assets and outstanding debt. Note that u_t^1 is the rate of contributions due to bank contracts and $r_e(t)$ is the rate of the decrease of the assets due to bank payments. Note that if $[u_t^1 - r_e(t) - (\sigma^e)^2] > 0$ then the reference ADR function can be increasing in time due to this inequality so that, for $t > t_1$, $x(t) < adr_1$.

The term $c_3(x_r(t) - adr_1)/q(t)$ models that if the reference ADR function is smaller than adr_1 then the function has to increase with time. The quotient $c_3/q(t)$ is a weighting term which accounts for the running costs and for the effect of the solution of the Riccati differential equation.

The term

$$x_r(t)[r_0(t) + r_e(t) - r_i(t) + (\sigma^e)^2 + (\sigma^i)^2],$$

accounts for two effects. The difference $r_e(t) - r_i(t)$ is the net effect of the rate of the bank payments, r_e , and that of the increase in outstanding debt. The term $r_0(t) + (\sigma^e)^2 + (\sigma^i)^2$ is the effect of the increase in the outstanding debt due to the riskless assets and the variance of the risky assets.

The last term,

$$(x_r(t) - 1)((\sigma^e)^2 + (\sigma^i)^2) - (\sigma^i)^2,$$

accounts for the effect that the bank payments have an effect both on the assets and on the outstanding debt.

More information is obtained by streamlining the ordinary differential equation for x_r . Assume that the parameters of the problem are all time-invariant and also that q has become constant with value q_0 . Then the differential equation for x_r can be rewritten as

$$-\dot{x}_r(t) = -k(x_r(t) - m), \quad x_r(t_1) = adr_1; \quad (6.14)$$

$$k = (r_o + r_e - r_1 + 2((\sigma^e)^2 + (\sigma^i)^2)) + c_3/q_0; \quad (6.15)$$

$$m = \frac{adr_1c_3/q_0 - (u_1 - r_e - (\sigma^e)^2) + (\sigma^e)^2}{(r_o + r_e - r_i + 2((\sigma^e)^2 + (\sigma^i)^2)) + c_3/q_0}. \quad (6.16)$$

Because the finite horizon is an artificial phenomenon to make the optimal stochastic control problem tractable, it is of interest to consider the long term behavior of the asset-debt-ratio-reference trajectory, x_r . If the values of the parameters are such that $k > 0$ then the differential equation with the terminal condition is stable. If this condition holds then $\lim_{t \downarrow 0} q(t) = q_0$ and $\lim_{t \downarrow 0} x_r(t) = m$ where the down arrow prescribes to start at t_1 and to let t decrease to 0. Depending on the value of m , the control law for at a time very far away from the terminal time becomes then:

$$u_t^{2*} = -(x(t) - m)q_0/c_2 = \begin{cases} > 0, & \text{if } x(t) < m, \\ < 0, & \text{if } x(t) > m, \end{cases}$$

$$\pi^*(t) = -\frac{(x(t) - m)}{x(t)}\tilde{C}\tilde{r}_j = \begin{cases} > 0, & \text{if } x(t) < m, \\ < 0, & \text{if } x(t) > m, \end{cases} \text{ if } \pi^* < 0 \text{ then set } \pi^* = 0,$$

The interpretation for the two case is: Case 1 ($x(t) > m$): Then the ADR x is too high. This is penalized by the cost function hence the control law prescribes not to invest in the risky asset and to hand back any surplus to the contributors. The payback advice is due to the quadratic cost function which was selected to make the solution analytically tractable.

Case 2 ($x(t) < m$): The ADR x is too low. The cost function penalizes and the control law prescribes to invest more in the risky asset and to request additional contributions from contributors because both will lead in the long run to a higher ADR.

An interesting question is: What is the value of the variable m compared with the variable adr_1 which is used in the running cost and in the terminal cost. Is m higher or lower than adr_1 ? Note that from the formula of m follows that

$$m = \frac{adr_1c_3/q_0 - (u_1 - r_e - (\sigma^e)^2) + (\sigma^e)^2}{(r_o + r_e - r_i + 2((\sigma^e)^2 + (\sigma^i)^2)) + c_3/q_0}.$$

An expression for the difference $m - adr_1$ is not obvious and depends on many parameters of the problem as it should.

Chapter 7

CONCLUDING REMARKS AND FUTURE DIRECTIONS

7.1 CONCLUDING REMARKS

7.1.1 Conclusions about Chapter 2

7.1.2 Conclusions about Chapter 3

7.1.3 Conclusions about Chapter 4

7.1.4 Conclusions about Chapter 5

7.1.5 Conclusions about Chapter 6

7.1.6 Conclusions about Chapter 7

7.1.7 Conclusions about Chapter 8

7.1.8 Conclusions about Chapter 9

7.2 FUTURE DIRECTIONS

In this chapter, we make a few concluding remarks and comment about possible future research projects.

7.1 CONCLUDING REMARKS

The current section provides conclusions about the main issues discussed in the thesis.

7.1.1 Conclusions about Chapter 2

Chapter 2 described a stochastic model for banks. In Section 2.1, we discussed related items such as cash, bonds, shares, loans, Treasuries, reserves, intangible assets, total unweighted assets and risk-weighted assets. Next, in Section 2.2 we discussed total bank capital, binding capital constraints and capital adequacy ratios. Section 2.3, the only liability that we considered was deposits as it pertains to outstanding debt. In the final section of Chapter 2 we described the bank assets-to-outstanding debt model that was important for the ensuing analysis.

7.1.2 Conclusions about Chapter 3

Chapter 3 described the optimal auditing in the banking industry. In Section 3.1, we discussed the random auditing reserve requirements. The main optimization result for bank reserve allocation and depository consumption was stated in Theorem 3.1.2. Here we can switch between Treasuries and reserves at will. The aforementioned allocation strategy is instrumental in establishing guidelines that inform the optimal volume of (Treasuries and) reserves required during a random audit. In Section 3.2 we discussed the non-random auditing capital requirements. Theorem 3.2.2 established an optimal auditing time that was related to an optimal CAR level. As a consequence of this, we were able to add to the debate about one of the major shortcomings of Basel II, viz., the characterization of regulatory thresholds like closure and corrective action thresholds. In the main, banking items that constituted the optimal reserve allocation and auditing time problems were described by jump diffusion models.

7.1.3 Conclusions about Chapter 4

In Chapter 4 we discussed stochastic control of banking systems. Section 4.1 described the optimal stochastic control for the simplified auditing model.

7.1.4 Conclusions about Chapter 5

Chapter 5 we considered numerical and illustrative examples for the asset-to-outstanding debt ratio.

7.1.5 Conclusions about Chapter 6

In Chapter 6 we discussed some of the economic issues arising from the analysis of stochastic dynamic models of banking items and their optimization. Section 6.2 contained the analysis of the auditing process with emphasis on reserve, asset and capital requirements. In this regard, one of the main novelties of the thesis was encapsulated by Theorem 3.1.2 in Section 3.1, where the optimal proportion of bank reserves and rate of depository consumption was determined via Lévy processes. In this case, the *rate of depository consumption* was defined as the rate at which Treasuries and reserves were consumed by the taking, holding and anticipated withdrawal of deposits. Typically the bank owner has to make decisions about deposit taking via the fixing of costs related to cheque clearing and bookkeeping, the holding of deposits by means of the choice of the deposit and coupon rate and anticipated withdrawals via the provisioning provided by Treasuries and reserves. Examples of anticipated withdrawals are stop orders, anticipated living expenses and certain payments. Also, the specific choice of a power utility function for the bank owner was made in order to obtain an analytic solution. Theorem 3.2.2 in Subsection 3.2.2 solved an optimal (non-random) auditing problem in terms of the CAR in a Lévy process setting. This result provided information about the optimal timing of an audit by the regulator when the ambient value of the CAR was taken into account and the bank was able to choose the time at which the audit occurred. In Section 6.1, we analyzed the main economic issues arising from the stochastic banking model and the auditing process.

7.1.6 Conclusions about Chapter 7

Chapter 7 contains the conclusion that we can draw from the study. We also point out what further research problems may be addressed by future students.

7.1.7 Conclusions about Chapter 8

The bibliography in Chapter 8, amongst other references, contained most of the contributions made by our research group in the last few years.

7.1.8 Conclusions about Chapter 9

In the appendix in Chapter 9 results that were central to the analysis in this thesis was discussed.

7.2 FUTURE DIRECTIONS

In general, we would like to extend and adapt the existing semi-martingale theory to produce mathematical models that have relevance for institutional finance. From a practical point of view, we would like to apply this knowledge to specific situations in real-world financial systems.

Several interesting questions related to auditing in the banking industry remain open. Amongst these is the development of multi-criteria decision making (MCDM) tool for auditors. Also, in this thesis, the regulatory strategy is strongly related to incentive compatibility. Also, we require that risk shifting incentives are completely eliminated by the closure rules. In a more general framework than the one used in this thesis, regulators should be able to take into account their monitoring costs and deadweight costs of deposit insurance and bank reorganization. Furthermore, future modeling should bear in mind that changes to the riskiness of a bank's operations also influence other parameters such as its asset value. If this effect is very strong, then eliminating risk-shifting incentives to deviate from the overall value-maximizing investment choice would be most desirable. In general, the implementation of any regulatory strategy will require that it is incentive compatible for the banks shareholders not to alter the investment strategies on which regulation is based.

Another specific activity will involve developing a better understanding of the dynamics of banking items such as loans, reserves, capital and regulatory ratios. Also, we would like to use Lévy processes to analyze financial time series at time scales of varying length and to optimize banking behavior with regard to loan issuing, profitability, auditing, deposit withdrawal, provisioning and other essential operations. Future research will also involve ongoing investigations into the implications for provisioning of the implicit subsidy enjoyed by borrowers in the form of limited liability; stable assessment of the provisioning needed to prevent a negative outcome and expected costs of administering insolvency in the event of borrower default. Also, we will strive to generalize our provisioning model in terms of the dynamics of the Treasuries, risk premium and expected loss rates; incorporate banking business lines for operational risk; understand the liquidity of the provisioning portfolio; optimize dynamic provisioning portfolios with a large number of loan positions and establish the dependence between the provisioning portfolio and the total loan loss process. Another improvement would be to validate our findings against some data, thereby showing that using our model would yield, at least, a proved good practice.

One of many specific uses of the rigorous mathematical models, that will emanate from the theoretical component of the research, can shortly be described as follows.

Sharp increases in output and asset (especially property and shares) prices, rapid credit expansion and the loosening of capital requirements during economic booms tend to result in the overextension of the banking system and subsequent financial instability once the movements reverse. These tendencies have led to concerns that the financial system is inherently and excessively procyclical with the corresponding exacerbation of fluctuations in the real economy. In line with the above, we know that existing empirical evidence is generally consistent with the view that the procyclicality of asset prices and credit could be a major cause of financial instability in the banking industry worldwide. In applying the economic models developed earlier, we intend to understand the interaction between asset prices, credit and capital and cyclicity in the global financial system. In particular, we would like to identify the possible financial stability implications of the procyclicality of asset prices and credit and to determine the affect these implications should have on macro-prudential policy and prudential behavior. In addition, we would like to highlight the link between cyclicity, financial stability and regulation as it pertains to the global financial system. The implementation of the practical considerations listed above are largely dependent on the tractability of the mathematical models obtained during the financial analysis. The main results from this project should have a seminal impact on the way global financial system instability is viewed from both a theoretical and practical perspective.

Chapter 8

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Chapter 9

APPENDICES

9.1 APPENDIX A: OPTIMAL CONTROL OF JUMP DIFFUSIONS

9.1.1 Appendix A1: Hamilton-Jacobi-Bellman for Optimal Control of Jump Diffusions

9.1.2 Appendix A2: Proof of Theorem 3.1.2

9.2 APPENDIX B: OPTIMAL STOPPING OF JUMP DIFFUSIONS

9.2.1 Appendix B1: Integro-Variational Inequalities for Optimal Stopping

9.2.2 Appendix B2: Optimal Stopping Conditions

9.2.3 Appendix B3: Proof of Theorem 3.2.2

9.3 APPENDIX C: OPERATIONAL RISK

9.4 APPENDIX D: OUTPUT GAP

9.4.1 Computing the Output Gap

9.4.2 Actual Output versus Potential Output for South Africa

9.4.3 Output Gap Tables

In this chapter, we discuss operational risk and information about the output gap and how it was determined. We use the output gap data to do graphs of the capital-to-risk weighted assets ratio and the capital-to-total assets ratio for Australia, Finland, Italy, Japan, Norway, Spain, Sweden, the United Kingdom and the United States of America. We also present the background results needed to prove Theorems 3.1.2 and 3.2.2. The proof of the former theorem utilizes information from [1] and Theorem 3.5 of [78]. On the other hand, the proof of Theorem 3.2.2 makes use of Theorems 2.2 and 2.3 of Chapter 2 in [78] which are presented in Appendix B1 and B2, respectively. Furthermore, the proof of Theorem 3.1.2 is reliant on the analysis in [19].

9.1 APPENDIX A: OPTIMAL CONTROL OF JUMP DIFFUSIONS

We consider an open set $\mathcal{S} \subset \mathbb{R}^n$ which is called a *solvency region* and let $Y_t = Y_t^{(u)}$ be a stochastic process in \mathbb{R}^n of the form

$$dY_t = b(Y_t, u(t))dt + \sigma(Y_t, u(t))dZ_t + \int_{\mathbb{R}^n} \zeta(Y_{t-}, u(t-), x)\tilde{N}(dt, dx), \quad Y_0 = y \in \mathbb{R}^n, \quad (9.0)$$

where $b: \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n$, $\sigma: \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^{n \times m}$ and $\zeta: \mathbb{R}^k \times \mathcal{U} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l}$ are functions and $\mathcal{U} \subset \mathbb{R}^k$ is a given set. The process

$$u(t) = u(t, \omega): [0, \infty) \times \Omega \rightarrow \mathcal{U}$$

is the *control process* which is assumed to be cadlag and adapted. In this case, we call $Y_t = Y_t^{(u)}$ a *controlled jump diffusion*. In the sequel, we consider a performance criterion $J = J^{(u)}(y)$ of the form

$$J^{(u)}(y) = \mathbf{E}^y \left[\int_0^{\tau_{\mathcal{S}}} f(Y_t, u(t))dt + g(Y_{\tau_{\mathcal{S}}}) \cdot \mathcal{X}_{\{\tau_{\mathcal{S}} < \infty\}} \right],$$

where

$$\tau_{\mathcal{S}} = \tau_{\mathcal{S}}(z, \omega) = \inf \left\{ t > 0 : Y_t^{(u)} \notin \mathcal{S} \right\}$$

is the *bankruptcy time* and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous functions. We say that the control process u is *admissible* and write $u \in \mathcal{A}$ if (9.0) has a unique strong solution Y_t for all $y \in \mathcal{S}$ and

$$\mathbf{E}^y \left[\int_0^{\tau_S} f^-(Y_t, u(t)) dt + g(Y_{\tau_S}) \cdot \mathcal{X}_{\{\tau_S < \infty\}} \right] < \infty.$$

The *stochastic control problem* is the following:

Find $V(y)$ and an *optimal control* $u^* \in \mathcal{A}$ such that

$$V(y) = \sup_{u \in \mathcal{A}} J^{(u)}(y) = J^{(u^*)}(y).$$

We know that under mild conditions it suffices to consider *Markov control*, i.e., controls $u(t)$ of the form

$$u(t) = u_0(Y_{t-})$$

for some function $u_0 : \mathbb{R}^n \rightarrow \mathcal{U}$. We only consider Markov controls and write $u(t) = u(Y_{t-})$. Note that if $u = u(y)$ is a Markov control then $Y_t = Y_t^{(u)}$ is a Lévy diffusion with generator

$$\begin{aligned} G\varphi(y) = G^u\varphi(y) &= \sum_{j=1}^n b_j(y, u(y)) \frac{\partial \varphi}{\partial y_j}(y) + \frac{1}{2} \sum_{j,k=1}^n (\sigma \sigma^T)_{jk}(y, u(y)) \cdot \frac{\partial^2 \varphi}{\partial y_j \partial y_k}(y) \\ &+ \sum_{k=1}^l \int_{\mathbb{R}} \left\{ \varphi(y + \zeta^{(k)}(y, u(y), x_k)) - \varphi(y) - \nabla \varphi(y) \cdot \zeta^{(k)}(y, u(y), x_k) \right\} \nu_k(dx_k) \end{aligned} \quad (9.1)$$

9.1.1 Appendix A1: Hamilton-Jacobi-Bellman for Optimal Control of Jump Diffusions

Suppose that there exists a function $\varphi : \bar{\mathcal{S}} \rightarrow \mathbb{R}$ such that

$$(A1.1) \quad \varphi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}}), \quad G^v \varphi(y) + f(y, v) \leq 0 \text{ for all } y \in \mathcal{S}, v \in \mathcal{U}$$

$$(A1.2) \quad Y(\tau_S \in \partial \mathcal{S} \text{ a.s. on } \{\tau_S < \infty\} \text{ and for all } u \in \mathcal{A} \text{ we have}$$

$$\lim_{t \rightarrow \tau_S^-} \varphi(Y_t) = g(Y_{\tau_S}) \cdot \mathcal{X}_{\{\tau_S < \infty\}}$$

(A1.3) For all $u \in \mathcal{A}$ and all $\tau \in \mathcal{T}$ we have that

$$\mathbf{E}^y \left[|\varphi(Y_\tau)| + \int_0^{\tau_S} \left\{ |G\varphi(Y_t)| + \sigma^T(Y_t) \nabla \varphi(Y_t)|^2 \right. \right. \\ \left. \left. \sum_{k=1}^l \left[\int_{\mathbb{R}} |\varphi(Y_t) + \zeta^{(k)}(Y_t, u(t), x_k) - \varphi(Y_t)|^2 \nu_k(dx_k) \right] \right\} dt \right] < \infty. \quad (9.2)$$

(A1.4) $\{\varphi(Y_\tau) : \tau \in \mathcal{T}\}$ is uniformly integrable for all $u \in \mathcal{A}$ and $y \in \mathcal{S}$.

If (A1.1) to (A1.4) hold, then, for all $y \in \overline{\mathcal{S}}$, we have that

$$\varphi(y) \geq V(y).$$

Moreover, assume that for each $y \in \overline{\mathcal{S}}$ there exists $v = \widehat{u}(y) \in \mathcal{U}$ such that

$$(A1.5) \quad G^{\widehat{u}(y)}\varphi(y) + f(y, \widehat{u}(y)) = 0 \text{ and}$$

$$(A1.6) \quad \left\{ \varphi(Y_\tau^{(\widehat{u})}) \right\}_{\tau \leq \tau_S} \text{ is uniformly integrable.}$$

Suppose $u^*(t) := \widehat{u}(Y_{t-}) \in \mathcal{A}$. Then u^* is an optimal control and

$$\varphi(y) = V(y) = J^{(u^*)}(y) \text{ for all } y \in \mathcal{S}.$$

9.1.2 Appendix A2: Proof of Theorem 3.1.2

The main thrust of the proof is to show that conditions (A1.1) to (A1.6) in Appendix A1 are satisfied. In order to accomplish this, we employ a Hamilton-Jacobi-Bellman equation (HJBE) approach for optimal control of jump diffusions (see [1] and Theorem 3.1 of Chapter 3 in [78]). For the generator of Q given by (2.6) we choose

$$\varphi(q) = \varphi(s, w) = \exp\{-\delta s\} K w^\alpha,$$

where K is given by (3.6). In this case, the form of (2.6) leads to

$$g(w, \pi) = G^w K w^\alpha + f(w, u), \quad (9.3)$$

where, for a constant θ , we have that

$$g(w, \pi) = -\theta K w^\alpha + \left([r^\top(1 - \pi) + \mu^R \pi] w - k \right) K \alpha w^{\alpha-1} + K \frac{1}{2} (\sigma^R)^2 \pi^2 w^2 \alpha (\alpha - 1) w^{\alpha-2} \\ + K w^\alpha \int_{-1}^{\infty} \left\{ (1 + \pi x^R)^\alpha - 1 - \alpha \pi x^R \right\} \nu(dx^R) + \frac{w^\alpha}{\alpha}.$$

In this case, $g(k, \pi)$ is concave in (k, π) and attains a maximum in the case where

$$\frac{\partial g}{\partial \pi} = (\mu^R - r^\top) K \alpha w^\alpha + K (\sigma^R)^2 \pi \alpha (\alpha - 1) w^\alpha + K w^\alpha \int_{-1}^{\infty} \left\{ \alpha (1 + \pi x^R)^{\alpha-1} x^R - \alpha x^R \right\} \nu(dx^R) = 0 \quad (9.4)$$

and

$$\frac{\partial g}{\partial k} = -K \alpha w^{\alpha-1} + k^{\alpha-1} = 0. \quad (9.5)$$

From (9.4) and (9.5) we obtain (3.5) and (3.7), respectively.

9.2 APPENDIX B: OPTIMAL STOPPING OF JUMP DIFFUSIONS

We consider an open set $\mathcal{S} \subset \mathbb{R}^n$ which is called a *solvency ratio* and let Y_t be a jump diffusion in \mathbb{R}^n of the form

$$dY_t = b(Y_t)dt + \sigma(Y_t)dZ_t + \int_{\mathbb{R}^n} \zeta(Y_{t-}, x) \tilde{N}(dt, dx); \quad Y_0 = y \in \mathbb{R}^n, \quad (9.6)$$

where $b: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and $\zeta: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l}$ are functions such that a unique solution Y_t exists. Let

$$\tau_{\mathcal{S}} = \tau_{\mathcal{S}}(z, \omega) = \inf \left\{ t > 0 : Y_t \notin \mathcal{S} \right\}$$

be the *bankruptcy time* and let \mathcal{T} denote the set of all stopping times $\tau \leq \tau_S$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous functions satisfying the conditions

$$\mathbf{E}^y \left[\int_0^\tau f^-(Y_t) dt \right] < \infty, \text{ for all } y \in \mathbb{R}^n.$$

and the family

$$\left\{ g^-(Y_\tau) \cdot \mathcal{X}_{\{\tau < \infty\}} : \tau \in \mathcal{T} \right\}$$

is uniformly integrable for all $y \in \mathbb{R}^n$. The *general optimal stopping problem* is the following:

Find $V(y)$ and $\tau^* \in \mathcal{T}$ such that

$$V(y) = \sup_{\tau \in \mathcal{T}} J_\tau(y) = J_{\tau^*}(y); \quad y \in \mathbb{R}^n$$

where

$$J_\tau(y) = \mathbf{E}^y \left[\int_0^\tau f(Y_t) + g(Y_\tau) \cdot \mathcal{X}_{\{\tau < \infty\}} \right], \quad \tau \in \mathcal{T}$$

is the *performance criterion*.

The function V is called the *value function* and the stopping time τ^* (if it exists) is called an *optimal stopping time*. In the sequel, we let G be the integro-differential operator which coincides with the generator of Y_t on $C_0^2(\mathbb{R}^n)$ given by

$$\begin{aligned} G\varphi(y) &= \sum_{j=1}^n b_j(y) \frac{\partial \varphi}{\partial y_j}(y) + \frac{1}{2} \sum_{j,k=1}^n (\sigma \sigma^T)_{jk}(y) \frac{\partial^2 \varphi}{\partial y_j \partial y_k}(y) \\ &\quad + \sum_{k=1}^l \int_{\mathbb{R}} \left\{ \varphi(y + \zeta^{(k)}(y, x_k)) - \varphi(y) - \nabla \varphi(y) \cdot \zeta^{(k)}(y, x_k) \right\} \nu_k(dx_k) \end{aligned} \quad (9.7)$$

for all $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $y \in \mathbb{R}^n$ such that (9.7) exists.

9.2.1 Appendix B1: Integro-Variational Inequalities for Optimal Stopping

Suppose that there exists a function $\varphi : \bar{\mathcal{S}} \rightarrow \mathbb{R}$ such that

$$(B1.1) \quad \varphi \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}}), \quad \varphi \geq g \text{ on } \mathcal{S}$$

Define the *continuation region* by

$$\mathcal{D} = \left\{ y \in \mathcal{S} : \varphi(y) > g(y) \right\}$$

and suppose that

$$(B1.2) \quad \mathbf{E}^y \left[\int_0^{\tau_{\mathcal{S}}} \mathcal{X}_{\partial\mathcal{D}} Y_t dt \right] = 0, \quad \partial\mathcal{D} \text{ is a Lipschitz surface}$$

$$(B1.3) \quad \varphi \in C^2(\mathcal{S} \setminus \partial\mathcal{D}) \text{ with locally bounded derivatives near } \partial\mathcal{D}, \quad G\varphi + f \leq 0 \text{ on } \mathcal{S} \setminus \partial\mathcal{D}$$

$$(B1.4) \quad Y_{\tau_{\mathcal{S}}} \in \partial\mathcal{S} \text{ a.s. on } \{\tau_{\mathcal{S}} < \infty\} \text{ and}$$

$$\lim_{t \rightarrow \tau_{\bar{\mathcal{S}}}} \varphi(Y_t) = g(Y_{\tau_{\mathcal{S}}}) \cdot \mathcal{X}_{\{\tau_{\mathcal{S}} < \infty\}}$$

and

$$(B1.5) \quad \text{for all } \tau \in \mathcal{T} \text{ we have that}$$

$$\begin{aligned} & \mathbf{E} \left[|\varphi(Y_{\tau})| + \int_0^{\tau_{\mathcal{S}}} \left\{ |G\varphi(Y_t)| + \sigma^T(Y_t) \nabla \varphi(Y_t) \right\}^2 \right. \\ & \left. \sum_{k=1}^l \left[\int_{\mathbb{R}} |\varphi(Y_t) + \zeta^{(k)}(Y_t, x) - \varphi(Y_t)|^2 \nu_k(dx_k) \right] dt \right] < \infty. \end{aligned} \quad (9.8)$$

If (B1.1) to (B1.5) hold, then, for all $y \in \bar{\mathcal{S}}$, we have that

$$\varphi(y) \geq V(y).$$

Moreover, assume that

$$(B1.6) \quad G\varphi + f = 0 \text{ on } \mathcal{D}, \quad \tau_{\mathcal{D}} := \inf\{t > 0 : Y_t \notin \mathcal{D}\} < \infty \text{ a.s for all } y$$

(B1.7) $\{\varphi(Y_\tau) : \tau \in \mathcal{T}\}$ is uniformly integrable for all y .

Then we have that

$$\varphi(y) = V(y) \text{ and } \tau^* = \tau_{\mathcal{D}}$$

is an optimal stopping time.

9.2.2 Appendix B2: Optimal Stopping Conditions

Suppose that the conditions of Appendix A1 hold. Suppose $g \in C^2(\mathbb{R}^n)$ and that $\varphi = g$ satisfies (9.8). Define

$$\mathcal{U} = \left\{ y \in \mathcal{S} : Gg(y) + f(y) > 0 \right\}.$$

Suppose that for all $y \in \mathcal{U}$ there exists a neighbourhood \mathcal{N}_y of y such that

$$\tau_{\mathcal{N}_y} := \inf \left\{ t > 0 : Y_t \notin \mathcal{N}_y \right\} < \infty \text{ a.s.}$$

Then

$$\mathcal{U} \subset \left\{ y \in \mathcal{S} : V(y) > g(y) \right\} = \mathcal{D}.$$

Hence it is never optimal to stop while $Y_t \in \mathcal{U}$.

9.2.3 Appendix B3: Proof of Theorem 3.2.2

We can show that for the choices of z^* and $K^{(1)}$ made in (3.10) and φ given by (3.11) all the conditions given in Appendices B1 (conditions (B1.1) to (B1.7) particularly) and B2 are satisfied.

9.3 APPENDIX C: OPERATIONAL RISK

In this section, we discuss how to determine the charge for operational risk. We note that the charge for credit risk and market risk were covered in the introduction. The charge to cover operational risk equals the sum of the charges for each of the eight business categories (corporate finance, trading and sales, retail banking, commercial banking, payment and settlement, agency services, asset management and retail brokerage). The capital charge for operational risk, under the standardized approach outlined in Basel II, may be expressed as

$$\max \left[\sum_{k=1}^8 \beta_k g_k, 0 \right],$$

where

- g_{1-8} : Three-Year Average of Gross Income for Each of the Eight Business Categories;
- β_{1-8} : Fixed Percentage Relating Level of Required Capital to Level of Gross Income for Each of the Eight Business Categories.

The β -values for operational risk are provided in [13].

9.4 APPENDIX D: OUTPUT GAP

In this section, we calculate the output gap and represent graphically the potential output and the actual output in a South African paradigm. We show an algorithm that can be used to calculate the output gap.

9.4.1 Computing the Output Gap

The output gap is measured as the percentage difference between actual GDP and estimated potential GDP. Symbolically this means that

$$\text{Output Gap} = \frac{\text{Actual Output} - \text{Potential Output}}{\text{Potential Output}} \times 100.$$

The output gap involves measuring the position of output in relation to potential. Potential outputs are difficult to estimate and subjected to margins of substantial error. Potential output is measured to capture the level of output that an economy can produce based on the available production factors (labour and capital) and the efficiency with which they are combined (total factor productivity). There are various methods of estimating potential output. In the literature, usually a choice from four methods is made. These methods may be listed as

Smoothing Split Time Trend Method; Smoothing Actual GDP via the Hodrick-Prescott Filter Method; Potential Output Method Using a Production Function Approach and Cyclically Adjusted Budget Balances.

We used the Hodrick-Prescott (HP) filter in our analysis. The HP filter derives a trend output such that it minimizes a weighted average of the gap between actual output, Y_t and trend output Y_t^* , and the rate of change in trend output, or its smoothness, over the whole period.

$$\min \frac{1}{T} \sum_{t=1}^T (\ln Y_t - \ln Y_t^*)^2 + \frac{\lambda}{T} \sum_{t=2}^{T-1} [(\ln Y_{t+1}^* - \ln Y_t^*) - (\ln Y_t^* - \ln Y_{t-1}^*)]^2$$

where T is the number of observations, and λ is the factor that determines the smoothness of the trend. A major disadvantage of the HP filter is that, since it is a two-sided symmetric filter, the estimated trend output series suffers from end-point biases. The method also fails to take account of structural breaks in the output series. The computation of potential output is usually based on a production function approach, taking into account the capital stock, changes in labour supply, factor productivities and underlying “non-accelerating wage rate of unemployment” (NAWRU) for South Africa. As regards the latter, the particular idea of potential output (from a supply perspective) considered in the sequel refers to the maximum level of output that is consistent with stable inflation which incorporates the role of NAWRU. The aforementioned approach coincides with the emphasis on the labour market and the control of inflation as a key medium term priority.

From the viewpoint of macroeconomic analysis, a limitation of the smoothing methods are that they are largely mechanistic and bring to bear no information about the structural constraints and limitations on production through the availability of factors of production or other endogenous influences. Thus, trend output growth projected by time series methods may be inconsistent (too high or too low) with what is known or being assumed about the growth in capital, labour supply or factor productivity or maybe unsustainable because of inflationary pressures. The preferred

potential output method attempts to overcome these shortcomings whilst adjusting for the limiting influence of demand pressure on employment and inflation. This is accomplished within a structural framework in which consistent judgement can also be exercised on some of the key elements. For the sake of implementability, this thesis relies on the Hodrick-Prescott (HP) filter, a common univariate filtering technique to decompose a time series into a trend and cyclical part. The simplicity and transparency of the HP filter come at a cost as regard the endpoint biases.

9.4.2 Actual Output versus Potential Output for South Africa

In this subsection, we consider historical data for the actual and potential GDP in South Africa for the period 2000-2006. We use this data to calculate the output gap via the method of smoothing actual GDP by means of the Hodrick-Prescott (HP) filter. Figure 9.1 represents the actual output versus the potential output for South Africa.

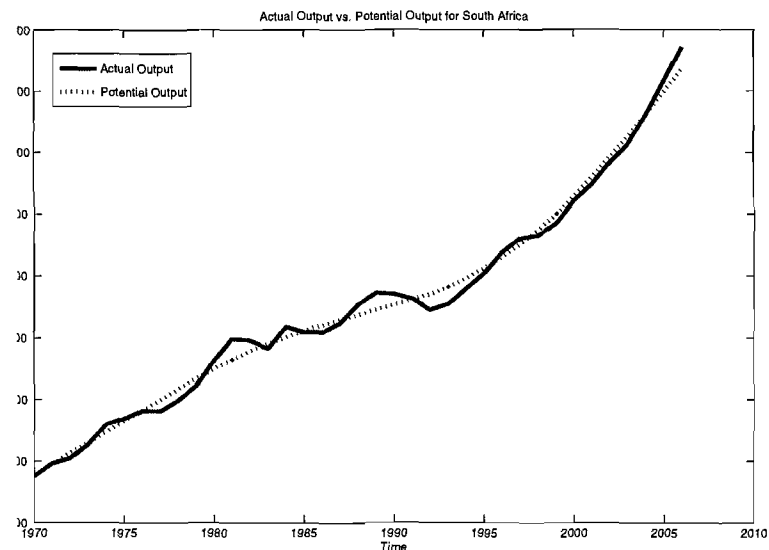


Figure 9.1: Actual Output vs Potential Output for South Africa
Data Source: South African Reserve Bank

From Figure 9.1, we conclude that the actual GDP and potential GDP increased substantially for the period 2000 to 2006.

9.4.3 Output Gap Tables

This subsection contains the observed and calculated output gaps of the OECD countries and South Africa.

| Year | United States | Japan | Italy | Australia | Finland |
|------|---------------|-------|-------|-----------|---------|
| 1986 | 1.25 | 1.01 | 1.20 | 2.40 | 3.60 |
| 1987 | 1.30 | 0.88 | 1.30 | 2.70 | 3.90 |
| 1988 | 1.55 | 1.20 | 1.52 | 3.00 | 4.80 |
| 1989 | 2.00 | 1.52 | 1.80 | 3.40 | 5.60 |
| 1990 | 1.65 | 2.10 | 1.82 | 2.80 | 5.00 |
| 1991 | 1.25 | 2.25 | 1.62 | 1.80 | 2.80 |
| 1992 | 1.00 | 1.75 | 1.50 | 0.80 | 1.60 |
| 1993 | 1.10 | 1.50 | 1.25 | 1.00 | 0.70 |
| 1994 | 1.30 | 1.30 | 0.89 | 1.90 | 0.80 |
| 1995 | 1.49 | 1.10 | 1.30 | 2.60 | 1.10 |
| 1996 | 1.50 | 1.54 | 1.35 | 2.70 | 1.50 |
| 1997 | 1.55 | 1.80 | 1.33 | 2.80 | 2.60 |
| 1998 | 1.65 | 1.20 | 1.31 | 3.40 | 3.00 |
| 1999 | 1.75 | 0.60 | 1.30 | 3.50 | 3.30 |
| 2000 | 2.00 | 0.55 | 1.00 | 3.70 | 3.80 |

Table 9.1: Output Gap for United States, Japan, Italy, Australia and Finland

| Year | Norway | Spain | Sweden | United Kingdom |
|-------------|---------------|--------------|---------------|-----------------------|
| 1986 | 4.90 | 3.40 | 0.35 | 1.00 |
| 1987 | 4.70 | 4.20 | 0.70 | 1.75 |
| 1988 | 3.90 | 4.70 | 1.45 | 2.75 |
| 1989 | 3.00 | 5.00 | 1.85 | 2.65 |
| 1990 | 2.80 | 5.20 | 2.00 | 2.45 |
| 1991 | 2.90 | 5.20 | 1.90 | 1.50 |
| 1992 | 3.00 | 3.00 | 1.50 | 0.80 |
| 1993 | 3.10 | 1.40 | 0.90 | 0.50 |
| 1994 | 3.50 | 1.00 | 0.45 | 0.90 |
| 1995 | 3.80 | 2.00 | 0.20 | 1.20 |
| 1996 | 4.00 | 2.30 | 0.10 | 1.30 |
| 1997 | 4.30 | 2.00 | 0.13 | 1.50 |
| 1998 | 4.50 | 2.20 | 0.50 | 1.58 |
| 1999 | 4.30 | 2.80 | 1.00 | 1.55 |
| 2000 | 4.20 | 3.20 | 1.45 | 1.59 |

Table 9.2: Output Gap for Norway, Sweden, Spain and the United Kingdom

| Year | Output Gap | Year | Output Gap |
|------|------------|------|------------|
| 1970 | -0.26 | 1989 | 3.62 |
| 1971 | 0.26 | 1990 | 2.22 |
| 1972 | -1.61 | 1991 | 0.07 |
| 1973 | -0.56 | 1992 | -3.26 |
| 1974 | 2.11 | 1993 | -3.49 |
| 1975 | 0.64 | 1994 | -2.06 |
| 1976 | -0.15 | 1995 | -1.00 |
| 1977 | -3.08 | 1996 | 0.97 |
| 1978 | -2.91 | 1997 | 1.11 |
| 1979 | -1.88 | 1998 | -1.06 |
| 1980 | 2.03 | 1999 | -1.62 |
| 1981 | 5.07 | 2000 | -0.64 |
| 1982 | 2.57 | 2001 | -1.17 |
| 1983 | -1.10 | 2002 | -0.92 |
| 1984 | 2.30 | 2003 | -1.26 |
| 1985 | -0.35 | 2004 | 0.01 |
| 1986 | -1.61 | 2005 | 1.59 |
| 1987 | -0.73 | 2006 | 3.15 |
| 1988 | 2.28 | - | - |

Table 9.3: Output Gap for South Africa