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VARIATIONAL APPROACH FOR A COUPLED ZAKHAROV-KUZNETSOV SYSTEM AND THE (2+1)-DIMENSIONAL BREAKING SOLITON EQUATION

by

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Declaration

I OFENTSE POROGO student number 21984352, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

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This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF B. MUATJETJEJA

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 2

OP Porogo, B Muatjetjeja, AR Adem, Variational approach and exact solutions for a generalized coupled Zakharov-Kuznetsov system. Submitted for publication to *Computers and Mathematics with Applications.*

Chapter 3

B Muatjetjeja, OP Porogo, Reductions and exact solutions of the (2+1)-dimensional breaking soliton equation via conservation laws. Submitted for publication to *Acta Mathematica Sinica, English Series.*

Dedication

To my family and everyone who showed me support throughout my studies.

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Abstract

In this dissertation the generalized coupled Zakharov-Kuznetsov system and the (2+1)-dimensional breaking soliton equation will be studied. Exact solutions for the coupled Zakharov-Kuznetsov equations are obtained using the Kudryashov method and the Jacobi elliptic function method while the exact solutions for the (2+1)-dimensional breaking soliton equation are derived using the double reduction theory.

Furthermore, Noether theorem is employed to construct conservation laws for the above mentioned partial differential equations. Since the coupled system is of third-order, it does not have a Lagrangian. Therefore, we use the transformations $u = U_x$ and $v = V_x$ to increase a third-order system to a fourth-order coupled system in U and V variables and let $\alpha = 1$. Thus, the new system of equations have a Lagrangian. However the (2+1)-dimensional breaking soliton equation has a Lagrangian in its natural form.

Finally, the conservation laws are expressed in u and v variables for the generalized coupled Zakharov-Kuznetsov system. Some local and infinitely many nonlocal conserved quantities are found and the Kudryashov method and Jacobi elliptic function method are used to obtain the exact solutions for the coupled Zakharov-Kuznetsov system. The (2+1)-dimensional breaking soliton equation possesses only local conserved quantities and the double reduction theory is applied to obtain some exact solutions.

Introduction

In many fields of science and engineering, nonlinear partial differential equations (NLPDEs) such as Korteweg-Vries equation, Burgers equation, Schrödinger equation, Boussinesq equation and many others play an important role in the study of nonlinear wave phenomena. For example, wave-like equations can describe earthquake stresses [1]. The wave phenomena can also be observed in fluid mechanics, plasma, elastic media, optical fibres and in many other areas of mathematical physics. NLPDEs of real life problems are difficult to solve either numerically or theoretically. Finding exact solutions of the NLPDEs plays an important role in nonlinear science. There has been recently much attention devoted to search better and more efficient solution methods for determining solutions to NLPDEs [2–11].

In the last few decades, a variety of effective methods for finding exact solutions, such as homogeneous balance method [4], the ansatz method [5, 6], variable separation approach [7], inverse scattering transform method [8], Bäcklund transformation [9], Darboux transformation [10], Hirota's bilinear method [11] were successfully applied to NLPDEs.

There is no doubt that in the study of differential equations, conservation laws play an important role. In fact, conservation laws describe physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion [12,13]. They have been used in investigating the existence, uniqueness and stability of solutions of NLPDEs [14,15]. Also, they have been used in the development and use of numerical methods [16, 17]. Recently, conservation laws were used to obtain exact solutions of some PDEs [18–20]. Thus, it is essential to study conservation laws of PDEs. For variational problems, the Noether's theorem [21] provides an elegant way to construct conservation laws. The knowledge of a Lagrangian is important in finding Noether point symmetries and Noether conserved vectors. However, in the absence of a Lagrangian, there are other methods that can be applied to obtain the conserved vectors. See, for example, [22, 23].

The theory of double reduction of a PDE is well-known for the association of conservation laws with Noether symmetries [24–26]. This association was extended to Lie Bäcklund symmetries [27] and non-local symmetries [28] recently. This opened doors to the extension of the theory of double reductions to partial differential equations (PDEs) that do not have a Lagrangian and therefore do not admit Noether symmetries.

In this dissertation we study the generalized coupled Zakharov-Kuznetsov system and the (2+1)-dimensional breaking soliton equation. Firstly, we study the generalized coupled Zakharov-Kuznetsov (gcZK) system [29]

$$\begin{aligned} u_t + u_{xxx} + u_{yyx} - 6uu_x - v_x &= 0, \\ v_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - 6\mu v v_x - \alpha u_x &= 0, \end{aligned}$$

where $u(t,x,y)$ and $v(t,x,y)$ are real-valued functions, t is time, x and y are the propagation and transverse coordinates, η is a group velocity shift between the coupled models, δ and λ are the relative longitudinal and transverse dispersion coefficients and μ and α are the relative nonlinear and coupled coefficients. The coupled ZK equations are the model describing two interacting weakly nonlinear waves in anisotropic background stratified fluid flows.

Lastly, we consider the (2+1)-dimensional breaking soliton equation [30]

$$u_{xt} + u_{xxxy} - 4u_x u_{xy} - 2u_{xx} u_y = 0,$$

where $u = u(t,x,y)$ denotes the wave profile with t, x and y representing time and space variables respectively. The (2+1)-dimensional breaking soliton equation is a typical so-called breaking soliton equation describing the (2+1)-dimensional interaction of a Riemann wave propagation along the y-axis with a long wave along the x-axis.

The outline of this dissertation is as follows.

In Chapter one, the basic definitions, theorems and corollaries concerning the Noether theorem and the double reduction theory are presented.

In Chapter two, Noether theorem [21] is used to construct conservation laws for a generalized coupled Zakharov-Kuznetsov system. Moreover, exact solutions of the generalized coupled Zakharov-Kuznetsov system are obtained with the aid of the Kudryashov method [31] and the Jacobi elliptic function method.

In Chapter three, the conservation laws for the (2+1)-dimensional breaking soliton equation are obtained using the Noether theorem [21]. Thereafter, we construct the exact solutions for the (2+1)-dimensional breaking soliton equation using the double reduction theory [24–28].

Finally, in Chapter four, a summary of the results of the dissertation is presented.

A bibliography is given at the end of this dissertation.

Chapter 1

Lie symmetry methods for partial differential equations

In this chapter we present the basic Lie group theory of partial differential equations. We discuss the algorithm for the calculation of the Lie point symmetries. We also give some basic definitions and theorem concerning Noether point symmetries and conservation laws. Furthermore, we we also discuss the theory of double reduction of partial differential equations.

1.1 Introduction

Lie group analysis originated in the late nineteenth century by an outstanding mathematician Sophus Lie (1842-1899). He discovered that majority of the methods for solving differential equations could be explained and deduced simply by means of his theory which is based on the invariance of the differential equations under a continuous group of symmetries. The mathematical ideas of Lie's theory are presented in several books, e.g., G.W. Bluman [25], P.J. Olver [26], and S. Kumei [32], Stephani [24] and Cantwell [33]. For more information on the definitions and results presented in this chapter the books mentioned above can be consulted.

1.2 Local continuous one-parameter Lie group

Let us take $x = (x^1, ..., x^n)$ to be the independent variables with coordinates x^i and $h = (h^1, .., h^m)$ to be the dependent variables with coordinates h^α (n and m finite).

Definition 1.1 A set G of transformations

$$T_a : \bar{x^i} = f^i(x, h, a), \ \bar{h^\alpha} = \phi^\alpha(x, h, a), \tag{1.1}$$

where a is a real parameter which continuously takes values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f^i, ϕ^α are differentiable functions, is called a *local continuous one-parameter Lie group of transformations* in the space of variables x and h if:

(i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure);

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity) and

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse).

The associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{h}, b) = f^i(x, h, \phi(a, b)), \\ \bar{\bar{h}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{h}, b) = \phi^\alpha(x, h, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any composition law $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)},$$

where

$$w(s) = \frac{\partial \phi(s, b)}{\partial b}|_{b=0}.$$

1.3 Prolongation formulas and Group generator

The derivatives of h with respect to x are defined as

$$h_i^\alpha = D_i(h^\alpha), \quad h_{ij}^\alpha = D_j D_i(h_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + h_i^\alpha \frac{\partial}{\partial h^\alpha} + h_{ij}^\alpha \frac{\partial}{\partial h_j^\alpha} + \cdots, \quad i = 1, ..., n, \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives h_i^α is denoted by $h_{(1)}$, i.e.,

$$h_{(1)} = \{h_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$h_{(2)} = \{h_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $h_{(3)} = \{h_{ijk}^\alpha\}$ and likewise $h_{(4)}$ etc, since $h_{ij}^\alpha = h_{ji}^\alpha$, $h_{(2)}$ contains only h_{ij}^α for $i \leq j$. In the same manner $h_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $h_{(4)}$,$h_{(5)} \cdots$. In group analysis all variables $x, h, h_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the h_s^α are called differential variables and a pth-order partial differential equation is given as

$$E(x, h, h_{(1)}, ..., h_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, h)$, one-parameter group of transformations G is

$$\begin{aligned} \bar{x^i} &= f^i(x, h, a), \quad f^i|_{a=0} = x^i, \\ \bar{h^\alpha} &= \phi^\alpha(x, h, a), \quad \phi^\alpha|_{a=0} = h^\alpha. \end{aligned} \tag{1.6}$$

According to Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x^i} \approx x^i + a\,\xi^i(x, h), \quad \bar{h^\alpha} \approx h^\alpha + a\,\eta^\alpha(x, h), \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a, about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = h^\alpha.$$

Thus, we have

$$\xi^i(x,h) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x,h) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{h^\alpha} \approx (1 + a\,X)h,$$

where

$$X = \xi^i(x,h)\frac{\partial}{\partial x^i} + \eta^\alpha(x,h)\frac{\partial}{\partial h^\alpha}. \tag{1.9}$$

The differential operator (1.9) is called the infinitesimal operator or generator of the group G.

Here we see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. Therefore

$$\bar{h}_i^\alpha = \bar{D}_j(h^\alpha), \qquad \bar{h}_{ij}^\alpha = \bar{D}_j(\bar{h}_i^\alpha) = \bar{D}_i(\bar{h}_j^\alpha), \cdots,$$

and

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{h}^\alpha) \\ &= D_i(f^j)\bar{h}_j^\alpha. \end{aligned} \tag{1.11}$$

Hence

$$\left(\frac{\partial f^j}{\partial x^i} + h_i^\beta\frac{\partial f^j}{\partial h^\beta}\right)\bar{h}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + h_i^\beta\frac{\partial \phi^\alpha}{\partial h^\beta}. \tag{1.12}$$

The quantities $\bar{h}_j^\alpha$ can be represented as functions of $x, h, h_{(i)}$, a, for small a, ie., (1.12) is locally invertible:

$$\bar{h}_i^\alpha = \psi_i^\alpha(x, h, h_{(1)}, a), \quad \psi^\alpha|_{a=0} = h_i^\alpha. \tag{1.13}$$

The transformations in x, h, $h_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

Let

$$\bar{h}_i^\alpha \approx h_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14). Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get,

$$\begin{aligned} D_i(f^j)(\bar{h}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(h_j^\alpha + a\zeta_j^\alpha) &= D_i(h^\alpha + a\eta^\alpha) \\ h_i^\alpha + a\zeta_i^\alpha + ah_j^\alpha D_i\xi^j &= h_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - h_j^\alpha D_i(\xi^j), \quad \text{(sum on } j). \end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - h_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^\alpha) - h_{i_1,i_2,\ldots,i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

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The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial h_i^\alpha} \quad \text{(sum on } i, \alpha\text{)}, \\ &\cdot \\ &\cdot \\ &\cdot \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial h_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x, h)\frac{\partial}{\partial x^i} + \eta^\alpha(x, h)\frac{\partial}{\partial h^\alpha}.$$

1.4 Group admitted by a partial differential equation

Definition 1.2 The vector field

$$X = \xi^i(x, h)\frac{\partial}{\partial x^i} + \eta^\alpha(x, h)\frac{\partial}{\partial h^\alpha}, \tag{1.18}$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E) = 0 \tag{1.19}$$

whenever $E = 0$. This can also be written as

$$X^{[p]} E\big|_{E=0} = 0, \tag{1.20}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{h}$, i.e.,

$$E(\bar{x}, \bar{h}, \bar{h}_{(1)}, \cdots, \bar{h}_{(p)}) = 0, \tag{1.21}$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, h)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{h}) \equiv F(f^i(x, h, a), \phi^\alpha(x, h, a)) = F(x, h), \tag{1.22}$$

identically in x, h and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, h)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, h)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, h)\frac{\partial F}{\partial h^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has n functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, h) = c_1, \cdots, J_n(x, h) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, h)} = \cdots = \frac{dx^n}{\xi^n(x, h)} = \frac{dh^1}{\eta^1(x, h)} = \cdots = \frac{dh^n}{\eta^n(x, h)}. \tag{1.24}$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{h}), \quad \frac{d\bar{h}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{h}) \tag{1.25}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{h}^\alpha\big|_{a=0} = h\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,h)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,h)\frac{\partial}{\partial h^\alpha}$$

and

$$X_2 = \xi_2^i(x,h)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,h)\frac{\partial}{\partial h^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,h)\frac{\partial}{\partial x^i} + \eta^\alpha(x,h)\frac{\partial}{\partial h}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,h)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,h)\frac{\partial}{\partial h}, \quad X_2 = \xi_2^i(x,h)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,h)\frac{\partial}{\partial h}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is:

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Fundamental relationship concerning the Noether theorem

In this section we briefly present the notation and pertinent results that will be used in this research. For details the reader is referred to [18, 21, 23, 34, 35]. Consider the system of qth order PDEs

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, ..., u_{(q)}) = 0, \quad \alpha = 1, 2, ..., m, \tag{1.26}$$

if there exist a function $L(x, u, u_{(1)}, u_{(2)}, ...u_{(s)}) \in \mathcal{A}$ (space of differential functions), $s < q$ such that system (1.26) is equivalent to,

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, ..., m, \tag{1.27}$$

then L is called a Lagrangian of (1.26) and (1.27) are the corresponding Euler-Lagrange differential equations.

In (1.27), $\delta/\delta u^\alpha$ is the Euler-Lagrange operator defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1}...D_{i_s} \frac{\delta}{\delta u^\alpha_{i_1...i_s}}, \quad \alpha = 1, ..., m. \tag{1.28}$$

Definition 1.8 A Lie-Bäcklund operator X is a Noether symmetry generator associated with a Lagrangian L of (1.27) if there exist a vector $A = (A^1, ..., A^n)$, $A^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(A^i). \tag{1.29}$$

If in (1.29) $A^i = 0$, $i = 1, ..., n$ then X is referred to as a strict Noether symmetry generator associated with Lagrangian $L \in \mathcal{A}$.

Theorem 1.4 For each Noether symmetry generator X associated with a given Lagrangian L, there corresponds a vector $T = (T^1, T^2, ..., T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = N^i L - A^i, \quad i = 1, ..., n, \tag{1.30}$$

which is a conserved vector for the Euler-Lagrange equations (1.27) and the Noether operator associated with X is

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}...D_{i_s}(W^\alpha)\frac{\delta}{\delta u_{i i_1...i_s}^\alpha}, \quad i = 1, ..., n, \tag{1.31}$$

in which the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.28) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}...D_{j_s}\frac{\delta}{\delta u_{ij_1...j_s}^\alpha}, \quad i = 1, ..., n, \quad \alpha = 1, ..., m.$$

In (1.31), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha, \quad \alpha = 1, ..., m.$$

The vector (1.30) is a conserved vector for (1.26) if T^i satisfies

$$D_i T^i|_{(1.26)} = 0. \tag{1.32}$$

1.7.1 Generalized double reduction theorem

Theorem 1.5 Suppose that X is any vector field operator of (1.26) and $T = (T^1, T^2, ..., T^n)$, $T^i \epsilon\, v$, $i = 1, 2, ..., n$ are the components of the conserved vector of (1.26) then,

$$\bar{T}^i = X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i), \; i = 1, 2, ..., n, \tag{1.33}$$

establishes the components of a conserved vector of (1.26) and also

$$D_i \bar{T}^i|_{(1.26)} = 0. \tag{1.34}$$

Theorem 1.6 Suppose $D_i T^i = 0$ is a conservation law of (1.26). Then under a contact transformation, there exist functions $\bar{T}^i$ such that $J D_i T^i = \bar{D}_i \bar{T}^i$ where $\bar{T}^i$

is defined by

$$\begin{pmatrix} \bar{T}^1 \\ \bar{T}^2 \\ \vdots \\ \bar{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} \bar{T}^1 \\ \bar{T}^2 \\ \vdots \\ \bar{T}^n \end{pmatrix},$$

$$J \begin{pmatrix} \bar{T}^1 \\ \bar{T}^2 \\ \vdots \\ \bar{T}^n \end{pmatrix} = A^T \begin{pmatrix} \bar{T}^1 \\ \bar{T}^2 \\ \vdots \\ \bar{T}^n \end{pmatrix} \textit{ where} \tag{1.35}$$

$$A = \begin{pmatrix} \bar{D}_1 x_1 & \bar{D}_1 x_2 & \cdots & \bar{D}_1 x_n \\ \bar{D}_2 x_1 & \bar{D}_2 x_2 & \cdots & \bar{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \bar{D}_1 x_1 & \bar{D}_1 x_2 & \cdots & \bar{D}_1 x_n \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_1 \bar{x}_1 & D_1 \bar{x}_2 & \cdots & D_1 \bar{x}_n \\ D_2 \bar{x}_1 & D_2 \bar{x}_2 & \cdots & D_2 \bar{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_1 \bar{x}_1 & D_1 \bar{x}_2 & \cdots & D_1 \bar{x}_n \end{pmatrix} \tag{1.36}$$

$$\textit{and } J = det(A). \tag{1.37}$$

Theorem 1.7 (Fundamental theorem on double reduction [35]).
Suppose that $D_i \bar{T}^i = 0$, is a conservation law of (1.26). Then under a similarity transformation of a symmetry X of (1.26), there exist a functions $\bar{T}^i$ such that X still remains a symmetry for the partial differential equation $\bar{D}_i \bar{T}^i = 0$ and

$$\begin{pmatrix} X\bar{T}^1 \\ X\bar{T}^2 \\ \vdots \\ X\bar{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [\bar{T}^1, X] \\ [\bar{T}^2, X] \\ \vdots \\ [\bar{T}^n, X] \end{pmatrix} \tag{1.38}$$

where

$$A = \begin{pmatrix} \bar{D}_1 x_1 & \bar{D}_1 x_2 & \cdots & \bar{D}_1 x_n \\ \bar{D}_2 x_1 & \bar{D}_2 x_2 & \cdots & \bar{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \bar{D}_1 x_1 & \bar{D}_1 x_2 & \cdots & \bar{D}_1 x_n \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_1 \bar{x}_1 & D_1 \bar{x}_2 & \cdots & D_1 \bar{x}_n \\ D_2 \bar{x}_1 & D_2 \bar{x}_2 & \cdots & D_2 \bar{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_1 \bar{x}_1 & D_1 \bar{x}_2 & \cdots & D_1 \bar{x}_n \end{pmatrix} \quad (1.39)$$

$$and\ J = det(A). \quad (1.40)$$

Corollary 1.1 (The necessary and sufficient condition for reduced conserved form [35]). The conserved form $D_i T^i = 0$ of system(1.26) can be reduced under a similarity transformation of a symmetry X to a reduced conserved form $\bar{D}_i \bar{T}^i = 0$ if and only if X is associated with the conservation law T, that is, $[T, X]_{(1.26)} = 0$.

Corollary 1.2 A nonlinear system of qth-order partial differential equations with n independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction) can be reduced to a $(q-1)$th-order nonlinear system of ordinary differential equations [35].

1.8 Conclusion

In this chapter we have presented briefly some basic definitions and results of the Lie group analysis of PDEs. We also briefly recalled the fundamental relations concerning Noether symmetries and conservation laws. In addition, we concisely discussed the double reduction theory for partial differential equation.

Chapter 2

Variational approach and exact solutions for a generalized coupled Zakharov-Kuznetsov system

In this chapter we study a generalized coupled system of PDEs which describes two interacting weakly nonlinear waves in anisotropic back-ground stratified fluid flows [36] given by

$$\begin{cases} u_t + u_{xxx} + u_{yyx} - 6uu_x - v_x = 0, \\ v_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - 6\mu v v_x - \alpha u_x = 0. \end{cases} \tag{2.1}$$

Gottwald et al. [37], derived the generalized coupled Zakharov-Kuznetsov system (2.1). It is easy to see that if the transverse variation ($u_y = v_y = 0$), the coupled Zakharov-Kuznetsov system reduces to a family of Korteweg-de Vries equations [37], which describes the interaction of the nonlinear long waves in various fluid flows.

In this dissertation, we will work with a slight modification of the generalized coupled Zakharov-Kuznetsov system (2.1), namely,

$$\begin{cases} u_t + u_{xxx} + u_{yyx} - 6uu_x - v_x = 0, \\ v_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - 6\mu v v_x - u_x = 0. \end{cases} \tag{2.2}$$

The Noether's theorem will be used to construct conservation laws for system (2.2).

Thereafter, we focus our investigations on the derivation of exact solutions for the generalized coupled Zakharov-Kuznetsov system (2.2) by invoking the Kudryashov method and the Jacobi elliptic function method.

2.1 Conservation laws for a generalized coupled Zakharov-Kuznetsov system (2.2)

In this section we derive the conservation laws for system (2.2). Here we observe that system (2.2) does not admit any Lagrangian formulation in its present form. In order to apply the Noether theorem we transform system (2.2) to a fourth-order system using transformations $u = U_x$ and $v = V_x$. Then system (2.2) becomes

$$\begin{cases} U_{tx} + U_{xxxx} + U_{yyxx} - 6U_xU_{xx} - V_{xx} = 0, \\ V_{tx} + \delta V_{xxxx} + \lambda V_{yyxx} + \eta V_{xx} - 6\mu V_xV_{xx} - U_{xx} = 0. \end{cases} \tag{2.3}$$

Here we observe that system (2.3) posses a second-order Lagrangian given by

$$\begin{aligned} L &= \frac{1}{2}U_{xx}^2 + \frac{1}{2}U_{xy}^2 + U_x^3 - \frac{1}{2}U_xU_t + \frac{\delta}{2}V_{xx}^2 + \frac{\lambda}{2}V_{xy}^2 - \frac{\eta}{2}V_x^2 + \mu V_x^3 \\ &\quad -\frac{1}{2}V_xV_t + V_xU_x. \end{aligned} \tag{2.4}$$

It can be verified that the second-order Lagrangian (2.4) satisfies the Euler-Lagrange equations. Thus

$$\frac{\delta L}{\delta U} = 0 \quad \text{and} \quad \frac{\delta L}{\delta V} = 0, \tag{2.5}$$

where $\delta/\delta U$ and $\delta/\delta V$ are defined by

$$\begin{aligned} \frac{\delta}{\delta U} &= \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} - D_y\frac{\partial}{\partial U_y} + D_t^2\frac{\partial}{\partial U_{tt}} + D_x^2\frac{\partial}{\partial U_{xx}} + D_y^2\frac{\partial}{\partial U_{yy}} \\ &\quad +D_xD_t\frac{\partial}{\partial U_{xt}} + D_xD_y\frac{\partial}{\partial U_{xy}} + \cdots, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{\delta}{\delta V} &= \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} - D_y\frac{\partial}{\partial V_y} + D_t^2\frac{\partial}{\partial V_{tt}} + D_x^2\frac{\partial}{\partial V_{xx}} + D_y^2\frac{\partial}{\partial V_{yy}} \\ &\quad +D_xD_t\frac{\partial}{\partial V_{xt}} + D_xD_y\frac{\partial}{\partial V_{xy}} + \cdots, \end{aligned} \tag{2.7}$$

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and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + V_t\frac{\partial}{\partial V} + U_{tt}\frac{\partial}{\partial U_t} + V_{tt}\frac{\partial}{\partial V_t} + U_{tx}\frac{\partial}{\partial U_x} + V_{tx}\frac{\partial}{\partial V_x} + U_{ty}\frac{\partial}{\partial U_y} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + V_x\frac{\partial}{\partial V} + U_{xx}\frac{\partial}{\partial U_x} + V_{xx}\frac{\partial}{\partial V_x} + U_{tx}\frac{\partial}{\partial U_t} + V_{tx}\frac{\partial}{\partial V_t} + U_{xy}\frac{\partial}{\partial U_y} + \cdots,$$

$$D_y = \frac{\partial}{\partial y} + U_y\frac{\partial}{\partial U} + V_y\frac{\partial}{\partial V} + U_{yy}\frac{\partial}{\partial U_y} + V_{yy}\frac{\partial}{\partial V_y} + U_{ty}\frac{\partial}{\partial U_t} + V_{ty}\frac{\partial}{\partial V_t} + U_{xy}\frac{\partial}{\partial U_x} + \cdots.$$

We now show the calculations which verify that the Lagrangian (2.4) satisfies system (2.5)

$$\begin{aligned}
\frac{\delta L}{\delta U} &= D_t\Big(-\frac{1}{2}U_x\Big) - D_x\Big(3U_x^2 - \frac{1}{2}U_t + V_x\Big) + D_x^2\Big(U_{xx}\Big) + D_xD_y(U_{xy}) \\
&= \frac{1}{2}U_{tx} - 6U_xU_{xx} + \frac{1}{2}U_{tx} - V_{xx} + U_{xxxx} + U_{yyxx} \\
&= U_{tx} + U_{xxxx} + U_{yyxx} - 6U_xU_{xx} - V_{xx} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\delta L}{\delta V} &= -D_t\Big(-\frac{1}{2}V_x\Big) - D_x\Big(-\eta V_x + 3\mu V_x^2 - \frac{1}{2}V_t + U_x\Big) + D_x^2\Big(\delta V_{xx}\Big) \\
&\quad + D_xD_y\Big(\lambda V_{xy}\Big) \\
&= \frac{1}{2}V_{tx} + \eta V_{xx} - 6\mu V_xV_{xx} + \frac{1}{2}V_{tx} - U_{xx} + \delta V_{xxxx} + \lambda V_{yyxx} \\
&= V_{tx} + \delta V_{xxxx} + \lambda V_{yyxx} + \eta V_{xx} - 6\mu V_xV_{xx} - U_{xx} \\
&= 0.
\end{aligned}$$

Hence the Lagrangian (2.4) is a Lagrangian for system (2.3).

Consider the vector field

$$\begin{aligned}
X &= \xi^1(t,x,y,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,y,U,V)\frac{\partial}{\partial x} + \xi^3(t,x,y,U,V)\frac{\partial}{\partial y} \\
&\quad + \eta^1(t,x,y,U,V)\frac{\partial}{\partial U} + \eta^2(t,x,y,U,V)\frac{\partial}{\partial V},
\end{aligned} \tag{2.8}$$

which has the second-order prolongation defined by

$$\begin{aligned}
X^{[2]} &= \xi^1(t,x,y,U,V)\frac{\partial}{\partial t} + \xi^2(t,x,y,U,V)\frac{\partial}{\partial x} + \xi^3(t,x,y,U,V)\frac{\partial}{\partial y} \\
&+\eta^1(t,x,y,U,V)\frac{\partial}{\partial U} + \eta^2(t,x,y,U,V)\frac{\partial}{\partial V} + \zeta^1_t\frac{\partial}{\partial U_t} + \zeta^1_x\frac{\partial}{\partial U_x} \\
&+\zeta^2_t\frac{\partial}{\partial V_t} + \zeta^2_x\frac{\partial}{\partial V_x} + \zeta^1_{xx}\frac{\partial}{\partial U_{xx}} + \zeta^2_{xx}\frac{\partial}{\partial V_{xx}} + \zeta^1_{xy}\frac{\partial}{\partial U_{xy}} \\
&+\zeta^2_{xy}\frac{\partial}{\partial V_{xy}} + \cdots, \qquad (2.9)
\end{aligned}$$

where

$$\zeta^1_t = D_t(\eta^1) - U_tD_t(\xi^1) - U_xD_t(\xi^2) - U_yD_t(\xi^3), \qquad (2.10)$$

$$\zeta^1_x = D_x(\eta^1) - U_tD_x(\xi^1) - U_xD_x(\xi^2) - U_yD_x(\xi^3), \qquad (2.11)$$

$$\zeta^2_t = D_t(\eta^2) - V_tD_t(\xi^1) - V_xD_t(\xi^2) - V_yD_t(\xi^3), \qquad (2.12)$$

$$\zeta^2_x = D_x(\eta^2) - V_tD_x(\xi^1) - V_xD_x(\xi^2) - V_yD_x(\xi^3), \qquad (2.13)$$

$$\zeta^1_{xx} = D_x(\zeta^1_x) - U_{tx}D_x(\xi^1) - U_{xx}D_x(\xi^2) - U_{xy}D_x(\xi^3), \qquad (2.14)$$

$$\zeta^2_{xx} = D_x(\zeta^2_x) - V_{tx}D_x(\xi^1) - V_{xx}D_x(\xi^2) - V_{xy}D_x(\xi^3), \qquad (2.15)$$

$$\zeta^1_{xy} = D_y(\zeta^1_x) - U_{tx}D_y(\xi^1) - U_{xx}D_y(\xi^2) - U_{xy}D_y(\xi^3), \qquad (2.16)$$

$$\zeta^2_{xy} = D_y(\zeta^2_x) - V_{tx}D_y(\xi^1) - V_{xx}D_y(\xi^2) - V_{xy}D_y(\xi^3). \qquad (2.17)$$

The Lie-Bäcklund operator X defined in (2.9) is a Noether operator corresponding to the Lagrangian L if it satisfies

$$X^{[2]}(L) + L[D_t(\xi^1) + D_x(\xi^2) + D_y(\xi^3)] = D_t(A^1) + D_x(A^2) + D_y(A^3), \qquad (2.18)$$

where $A^1(t,x,y,U,V)$, $A^2(t,x,y,U,V)$ and $A^3(t,x,y,U,V)$ are the gauge terms. The expansion of (2.18) together with the Lagrangian (2.4) results in an overdetermined system of linear PDEs given as,

$$\xi^1_V = 0, \tag{2.19}$$
$$\xi^1_U = 0, \tag{2.20}$$
$$\xi^1_y = 0, \tag{2.21}$$
$$\xi^1_x = 0, \tag{2.22}$$
$$\xi^2_V = 0, \tag{2.23}$$
$$\xi^2_U = 0, \tag{2.24}$$
$$\xi^2_y = 0, \tag{2.25}$$
$$\xi^3_V = 0, \tag{2.26}$$
$$\xi^3_U = 0, \tag{2.27}$$
$$\xi^3_x = 0, \tag{2.28}$$
$$\xi^3_t = 0, \tag{2.29}$$
$$A^3_V = 0, \tag{2.30}$$
$$A^3_U = 0, \tag{2.31}$$
$$\xi^2_{xx} = 0, \tag{2.32}$$
$$\eta^1_V = 0, \tag{2.33}$$
$$\eta^1_{UU} = 0, \tag{2.34}$$
$$\eta^1_{xy} = 0, \tag{2.35}$$
$$\eta^1_{yU} = 0, \tag{2.36}$$
$$\eta^1_{xU} = 0, \tag{2.37}$$
$$\eta^1_{xx} = 0, \tag{2.38}$$
$$\eta^2_U = 0, \tag{2.39}$$
$$\eta^2_{VV} = 0, \tag{2.40}$$
$$\eta^2_{xy} = 0, \tag{2.41}$$
$$\eta^2_{yV} = 0, \tag{2.42}$$
$$\eta^2_{xV} = 0, \tag{2.43}$$
$$\eta^2_{xx} = 0, \tag{2.44}$$
$$\eta^2_V + \eta^1_U + \xi^3_y - \xi^2_x + \xi^1_t = 0, \tag{2.45}$$

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$$2\eta^1_U = -\xi^3_y, \tag{2.46}$$
$$-2\eta^2_V - \xi^3_y = 0, \tag{2.47}$$
$$-2A^1_U - \eta^1_x = 0, \tag{2.48}$$
$$-2A^1_V - \eta^2_x = 0, \tag{2.49}$$
$$-2A^2_V + 2\eta^1_x - 2\eta\eta^2_x - \eta^2_t = 0, \tag{2.50}$$
$$2\eta^2_V + \xi^3_y - 3\xi^2_x + \xi^1_t = 0, \tag{2.51}$$
$$2\eta^1_U + \xi^3_y - 3\xi^2_x + \xi^1_t = 0, \tag{2.52}$$
$$3\eta^1_U + \xi^3_y - 2\xi^2_x + \xi^1_t = 0, \tag{2.53}$$
$$2\eta^2_V - \xi^3_y - \xi^2_x + \xi^1_t = 0, \tag{2.54}$$
$$2\eta^1_U - \xi^3_y - \xi^2_x + \xi^1_t = 0, \tag{2.55}$$
$$3\eta^2_V + \xi^3_y - 2\xi^2_x + \xi^1_t = 0, \tag{2.56}$$
$$6\eta^1_x + \xi^2_t = 0, \tag{2.57}$$
$$-2A^2_U + 2\eta^2_x - \eta^1_t = 0, \tag{2.58}$$
$$2\eta\eta^2_V + \eta\xi^3_y = 6\mu\eta^2_x + \eta\xi^2_x - \eta\xi^1_t + \xi^2_t, \tag{2.59}$$
$$-A^1_t - A^2_x - A^3_y = 0. \tag{2.60}$$

We now solve the above system of linear PDEs for ξ^1, ξ^2, ξ^3, η^1, η^2, A^1, A^2 and A^3. From equations (2.19)-(2.22) we obtain

$$\xi^1(t,x,y,U,V) = a(t), \tag{2.61}$$

where $a(t)$ is an arbitrary function. Solving equations (2.23)-(2.25) we obtain

$$\xi^2(t,x,y,U,V) = b(t,x), \tag{2.62}$$

where $b(t,x)$ is an arbitrary function. From equations (2.26)-(2.29) we attain

$$\xi^3(t,x,y,U,V) = d(y), \tag{2.63}$$

where $d(y)$ is an arbitrary function. Solving equations (2.30) and (2.31) we get

$$A^3(t,x,y,U,V) = S(t,x,y), \tag{2.64}$$

where $S(t,x,y)$ is an arbitrary function.

Using equations (2.33)-(2.38) we obtain

$$\eta^1(t,x,y,U,V) = k(t)U + n(t)x + f(t,y), \tag{2.65}$$

where $k(t)$, $n(t)$ and $f(t,y)$ are arbitrary functions.

Solving equations (2.39)-(2.44) we obtain

$$\eta^2(t,x,y,U,V) = E(t)V + H(t)x + g(t,y), \tag{2.66}$$

where $E(t)$, $H(t)$ and $g(t,y)$ are arbitrary functions. Differentiating (2.62) twice with respect to x and substituting the results into (2.32) we obtain

$$b_{xx} = 0.$$

Integrating the above equation twice with respect to x we attain

$$b(t,x) = p(t)x + q(t), \tag{2.67}$$

where $p(t)$ and $q(t)$ are arbitrary functions of their arguments. Substituting (2.67) into (2.62) implies

$$\xi^2(t,x,y,U,V) = p(t)x + q(t). \tag{2.68}$$

Differentiating (2.65) and (2.66) with respect to t and x respectively and substituting the results into (2.58) we attain

$$A^2_U = H(t) - \frac{1}{2}k'(t)U - \frac{1}{2}n'(t)x - \frac{1}{2}f_t. \tag{2.69}$$

The integration of (2.69) with respect to U gives

$$A^2(t,x,y,U,V) = H(t)U - \frac{1}{4}k'(t)U^2 - \frac{1}{2}n'(t)Ux - \frac{1}{2}Uf_t + r(t,x,y,V), \tag{2.70}$$

where $r(t,x,y,V)$ is an arbitrary function. Differentiating (2.65), (2.66) and (2.70) with respect to x, t and V respectively and substituting the results into (2.50) we obtain

$$r_V = n(t) - \eta H(t) - \frac{1}{2}E'(t)V - \frac{1}{2}H'(t)x - \frac{1}{2}g_t. \tag{2.71}$$

Integrating (2.71) with respect to V we obtain

$$\begin{aligned} r(t,x,y,V) &= n(t)V - \eta H(t)V - \frac{1}{4}E'(t)V^2 - \frac{1}{2}H'(t)Vx - \frac{1}{2}Vg_t \\ &\quad + R(t,x,y), \end{aligned} \tag{2.72}$$

where $R(t,x,y)$ is an arbitrary function. Substituting (2.72) into (2.70) we attain

$$\begin{aligned} A^2(t,x,y,U,V) &= H(t)U - \frac{1}{4}k'(t)U^2 - \frac{1}{2}n'(t)Ux - \frac{1}{2}Uf_t + n(t)V \\ &\quad - \eta H(t)V - \frac{1}{4}E'(t)V^2 - \frac{1}{2}H'(t)Vx - \frac{1}{2}Vg_t \\ &\quad + R(t,x,y). \end{aligned} \tag{2.73}$$

Differentiating (2.65) with respect to x and substituting the results into (2.48) we obtain

$$A^1_U = -\frac{1}{2}n(t). \tag{2.74}$$

The integration of (2.74) with respect to U yields

$$A^1 = -\frac{1}{2}n(t)U + W(t,x,y,V), \tag{2.75}$$

where $W(t,x,y,V)$ is an arbitrary function. Differentiating (2.66) and (2.74) with respect to x and V respectively and substituting the results into (2.49) we obtain

$$W_V = -\frac{1}{2}H(t). \tag{2.76}$$

Integrating (2.76) with respect V we obtain

$$W(t,x,y,V) = -\frac{1}{2}H(t)V + Q(t,x,y), \tag{2.77}$$

where $Q(t,x,y)$ is an arbitrary function. The insertion of (2.77) into (2.74) yields

$$A^1 = -\frac{1}{2}n(t)U - \frac{1}{2}H(t)V + Q(t,x,y). \tag{2.78}$$

By differentiating (2.64), (2.73) and (2.78) with respect to y, x and t respectively and substituting the results into (2.60) we obtain

$$n'(t)U + H'(t)V - Q_t - R_x - S_y = 0. \tag{2.79}$$

Splitting equation (2.79) with respect to U and V we get

$$n'(t) = 0, \tag{2.80}$$

$$H'(t) = 0, \tag{2.81}$$

$$Q_t + R_x + S_y = 0. \tag{2.82}$$

Integrating equations (2.80) and (2.81) yields

$$n(t) = c_1, \qquad H(t) = c_2, \tag{2.83}$$

where c_1 and c_2 are arbitrary constants. The substitution of (2.83) into (2.65), (2.66), (2.73) and (2.78) gives

$$\eta^1(t,x,y,U,V) = k(t)U + c_1 x + f(t,y), \tag{2.84}$$

$$\eta^2(t,x,y,U,V) = E(t)V + c_2 x + g(t,y), \tag{2.85}$$

$$A^1(t,x,y,U,V) = -\frac{1}{2}c_1 U - \frac{1}{2}c_2 V + Q(t,x,y), \tag{2.86}$$

$$\begin{aligned} A^2(t,x,y,U,V) = {} & c_2 U - \frac{1}{4}k'(t)U^2 - \frac{1}{2}U f_t + c_1 V - \eta c_2 V \\ & -\frac{1}{4}E'(t)V^2 - \frac{1}{2}V g_t + R(t,x,y). \end{aligned} \tag{2.87}$$

Differentiating (2.68) and (2.84) with respect to t and x respectively and substituting the results into (2.57) we obtain

$$6c_1 + p'(t)x + q'(t) = 0. \tag{2.88}$$

Splitting equation (2.88) with respect to x we get

$$p'(t) = 0, \tag{2.89}$$

$$6c_1 + q'(t) = 0. \tag{2.90}$$

Integrating equations (2.89) and (2.90) we obtain

$$p(t) = c_3, \qquad q(t) = -6c_1 t + c_4, \tag{2.91}$$

where c_3 and c_4 are arbitrary constants. The insertion of (2.91) into (2.68) gives

$$\xi^2(t,x,y,U,V) = c_3 x - 6c_1 t + c_4. \tag{2.92}$$

Differentiating (2.63) and (2.84) with respect to y and U respectively and inserting the results into (2.46) respectively, yields

$$d'(y) = -2k(t). \tag{2.93}$$

Differentiating (2.63) and (2.85) with respect to y and V respectively and substituting the results into (2.47) yields

$$2E(t) + d'(y) = 0, \tag{2.94}$$

therefore

$$d'(y) = -2E(t) \tag{2.95}$$

and this makes

$$E(t) = k(t).$$

Differentiating (2.61), (2.63) (2.85) and (2.92) with respect to t, y, V and x respectively and substituting the results into (2.51) we obtain

$$2E(t) + d'(y) - 3c_3 + a'(t) = 0. \tag{2.96}$$

The insertion of (2.94) into (2.96) yields

$$a'(t) = 3c_3. \tag{2.97}$$

Integrating (2.97) we obtain

$$a(t) = 3c_3 t + c_5, \tag{2.98}$$

where c_5 is an arbitrary constant. The substitution of (2.98) into (2.61) yields

$$\xi^1(t, x, y, U, V) = 3c_3 t + c_5. \tag{2.99}$$

Differentiating (2.63), (2.85), (2.92) and (2.99) with respect to y, V, x and t respectively and substituting the results into (2.54) we obtain

$$2E(t) - d'(y) = -2c_3. \tag{2.100}$$

The sum of (2.94) and (2.100) gives

$$4E(t) = -2c_3,$$

thus

$$E(t) = -\frac{1}{2}c_3. \qquad (2.101)$$

Since $E(t) = k(t)$ then

$$k(t) = -\frac{1}{2}c_3. \qquad (2.102)$$

Substituting (2.101) into (2.95) we obtain

$$d'(y) = -2\Big(-\frac{1}{2}c_3\Big),$$

then

$$d'(y) = c_3. \qquad (2.103)$$

The integration of (2.103) yields

$$d(y) = c_3 y + c_6, \qquad (2.104)$$

where c_6 is an arbitrary constant. Inserting (2.104) into (2.63) we obtain

$$\xi^3(t,x,y,U,V) = c_3 y + c_6. \qquad (2.105)$$

Substituting (2.101) and (2.102) into (2.84), (2.85) and (2.87) yields

$$\eta^1(t,x,y,U,V) = -\frac{1}{2}c_3 U + c_1 x + f(t,y), \qquad (2.106)$$

$$\eta^2(t,x,y,U,V) = -\frac{1}{2}c_3 V + c_2 x + g(t,y), \qquad (2.107)$$

$$A^2(t,x,y,U,V) = c_2 U - \frac{1}{2}U f_t + c_1 V - \eta c_2 V - \frac{1}{2}V g_t + R(t,x,y). \qquad (2.108)$$

Differentiating equations (2.92), (2.99), (2.105) and (2.106) with respect to x, t, y and U respectively and substituting the results into (2.53) we obtain

$$-\frac{3}{2}c_3 + c_3 - 2c_3 + 3c_3 = 0.$$

Thus $c_3 = 0$. Therefore equations (2.45), (2.52), (2.54) and (2.56) are satisfied.

The differentiation of equations (2.92), (2.99), (2.105), (2.106) and (2.107) with respect to x, t, y, U and V and substituting their derivatives into (2.59) yields

$$6\mu c_2 - 6c_1 = 0,$$

thus we have

$$c_2 = \frac{c_1}{\mu}. \tag{2.109}$$

The substitution of (2.109) into (2.86), (2.92) (2.99), (2.105), (2.106), (2.107) and (2.108) gives,

$$\begin{aligned}
\xi^1(t,x,y,U,V) &= c_5,\\
\xi^2(t,x,y,U,V) &= -6c_1 t + c_4,\\
\xi^3(t,x,y,U,V) &= c_6,\\
\eta^1(t,x,y,U,V) &= c_1 x + f(t,y),\\
\eta^2(t,x,y,U,V) &= \frac{c_1 x}{\mu} + g(t,y),\\
A^1(t,x,y,U,V) &= -\frac{c_1}{2}U - \frac{c_1}{2\mu}V + Q(t,x,y),\\
A^2(t,x,y,U,V) &= \frac{c_1}{\mu}U - \frac{1}{2}Uf_t + c_1 V - \frac{\eta c_1}{\mu}V - \frac{1}{2}Vg_t + R(t,x,y).
\end{aligned}$$

Thus the general solutions of system (2.19)-(2.60) are:

$$\begin{aligned}
&\xi^1 = c_5, \quad \xi^2 = -6c_1 t + c_4, \quad \xi^3 = c_6, \quad \eta^1 = c_1 x + f(t,y), \quad \eta^2 = \frac{c_1 x}{\mu} + g(t,y),\\
&A^1 = -\frac{c_1}{2}U - \frac{c_1}{2\mu}V + Q(t,x,y),\\
&A^2 = \frac{c_1}{\mu}U - \frac{1}{2}Uf_t + c_1 V - \frac{\eta c_1}{\mu}V - \frac{1}{2}Vg_t + R(t,x,y),\\
&A^3 = S(t,x,y), \quad Q_t + R_x + S_y = 0.
\end{aligned} \tag{2.110}$$

We can choose $Q(t,x,y) = R(t,x,y) = S(t,x,y) = 0$ as they contribute to the trivial part of the conserved vectors. Hence the Noether symmetries and gauge functions are

$$\begin{aligned}
X_1 &= 6\mu t\frac{\partial}{\partial x} - \mu x\frac{\partial}{\partial U} - x\frac{\partial}{\partial V}, \quad A^1 = -\frac{U}{2} - \frac{V}{2\mu}, \quad A^2 = \frac{U}{\mu} + V - \frac{\eta}{\mu}V, \quad A^3 = 0,\\
X_2 &= \frac{\partial}{\partial x}, \qquad A^1 = 0, \qquad A^2 = 0, \qquad A^3 = 0,\\
X_3 &= \frac{\partial}{\partial t}, \qquad A^1 = 0, \qquad A^2 = 0, \qquad A^3 = 0,\\
X_4 &= \frac{\partial}{\partial y}, \qquad A^1 = 0, \qquad A^2 = 0, \qquad A^3 = 0,\\
X_f &= f(t,y)\frac{\partial}{\partial U}, \qquad A^1 = 0, \qquad A^2 = -\frac{1}{2}Uf_t, \qquad A^3 = 0,\\
X_g &= g(t,y)\frac{\partial}{\partial V}, \qquad A^1 = 0, \qquad A^2 = -\frac{1}{2}Vg_t, \qquad A^3 = 0.
\end{aligned}$$

The above results will now be used to find the components of the conserved vectors. Applying Theorem 1.4, [21, 38] and reverting back into the original variables we obtain the following nontrivial conserved vectors associated with the above Noether point symmetries:

$$T_1^1 = \frac{1}{2\mu}\Big\{ -6\mu t u^2 - \mu x u + \mu\int u\,dx - 6\mu t v^2 - xv + \int v\,dx\Big\}, \tag{2.111}$$

$$\begin{aligned}
T_1^2 = \frac{1}{2\mu}\Big\{ &-6\mu t u_{yy}u - 12\mu t u_{xx}u - 12\delta\mu t v_{xx}v - 6\lambda\mu t v_{yy}v + 12\mu t uv\\
&+24\mu t u^3 + 6\mu x u^2 + 2xu - 2\int u\,dx - 6\eta\mu t v^2 - 2\eta x v + 2\eta\int v\,dx\\
&+24\mu^2 t v^3 + 6\mu x v^2 + 2\mu x v - 2\mu\int v\,dx + 6\mu t {u_x}^2 - \mu x\int u_t\,dx\\
&+6\delta\mu t {v_x}^2 - x\int v_t\,dx + 2\mu u_x - 2\mu x u_{xx} - \mu x u_{yy} + 2\delta v_x\\
&-2\delta x v_{xx} - \lambda x v_{yy}\Big\},
\end{aligned} \tag{2.112}$$

$$\begin{aligned}
T_1^3 = \frac{1}{2\mu}\Big\{ &-6\mu t u_{xy}u - 6\lambda\mu t v_{xy}v + 6\mu t u_x u_y + 6\lambda\mu t v_x v_y - \mu x u_{xy}\\
&+\mu u_y - \lambda x v_{xy} + \lambda v_y\Big\};
\end{aligned} \tag{2.113}$$

$$T_2^1 = \frac{1}{2}\Big\{u^2+v^2\Big\}, \tag{2.114}$$

$$T_2^2 = \frac{1}{2}\Big\{u_{yy}u+2u_{xx}u+2\delta v_{xx}v+\lambda v_{yy}v-2uv-4u^3+\eta v^2-4\mu v^3$$
$$-{u_x}^2-\delta {v_x}^2\Big\}, \tag{2.115}$$

$$T_2^3 = \frac{1}{2}\Big\{u_{xy}u+\lambda v_{xy}v-u_xu_y-\lambda v_xv_y\Big\}; \tag{2.116}$$

$$T_3^1 = \frac{1}{2}\Big\{2uv+2u^3-\eta v^2+2\mu v^3+{u_x}^2+{u_y}^2+\delta {v_x}^2+\lambda {v_y}^2\Big\}, \tag{2.117}$$

$$T_3^2 = \frac{1}{2}\Big\{-2v\int u_t\,dx-2u\left(\int v_t\,dx\right)-6u^2\int u_t\,dx+2\eta v\left(\int v_t\,dx\right)$$
$$-6\mu v^2\left(\int v_t\,dx\right)+u_{yy}\int u_t\,dx-u_y\left(\int u_{ty}\,dx\right)-2u_tu_x+\left(\int u_t\,dx\right)^2$$
$$+2u_{xx}\int u_t\,dx-2\delta v_tv_x+2\delta v_{xx}\left(\int v_t\,dx\right)-\lambda v_y\left(\int v_{ty}\,dx\right)$$
$$+\lambda v_{yy}\left(\int v_t\,dx\right)+\left(\int v_t\,dx\right)^2\Big\}, \tag{2.118}$$

$$T_3^3 = \frac{1}{2}\Big\{u_{xy}\left(\int u_t\,dx\right)-u_tu_y+\lambda v_{xy}\left(\int v_t\,dx\right)-\lambda v_tv_y\Big\}; \tag{2.119}$$

$$T_4^1 = \frac{1}{2}\Big\{u\left(\int u_y\,dx\right)+v\left(\int v_y\,dx\right)\Big\}, \tag{2.120}$$

$$T_4^2 = \frac{1}{2}\Big\{-2u\left(\int v_y\,dx\right)-2v\left(\int u_y\,dx\right)-6u^2\left(\int u_y\,dx\right)$$
$$+2\eta v\left(\int v_y\,dx\right)-6\mu v^2\left(\int v_y\,dx\right)+\int u_t\,dx\left(\int u_y\,dx\right)$$
$$+\int v_t\,dx\left(\int v_y\,dx\right)-2u_xu_y-u_y\left(\int u_{yy}\,dx\right)+u_{yy}\left(\int u_y\,dx\right)$$
$$+2u_{xx}\left(\int u_y\,dx\right)-2\delta v_xv_y+2\delta v_{xx}\left(\int v_y\,dx\right)$$
$$-\lambda v_y\left(\int v_{yy}\,dx\right)+\lambda v_{yy}\left(\int v_y\,dx\right)\Big\}, \tag{2.121}$$

$$T_4^3 = \frac{1}{2}\Big\{-u\left(\int u_t\,dx\right)-v\left(\int v_t\,dx\right)+2uv+2u^3-\eta v^2+2\mu v^3$$
$$+u_{xy}\left(\int u_y\,dx\right)+{u_x}^2+\delta {v_x}^2+\lambda v_{xy}\left(\int v_y\,dx\right)\Big\}; \tag{2.122}$$

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$$
\begin{aligned}
T_f^1 &= -\frac{1}{2}f(t,y)u, &(2.123)\\
T_f^2 &= \frac{1}{2}\Big\{-2u_{xx}f(t,y) - f(t,y)\int u_t\,dx + f_t(\int u\,dx) - u_{yy}f(t,y)\\
&\quad +6f(t,y)u^2 + 2f(t,y)v + f_y u_y\Big\}, &(2.124)\\
T_f^3 &= -\frac{1}{2}f(t,y)u_{xy}; &(2.125)\\
T_g^1 &= -\frac{1}{2}g(t,y)v, &(2.126)\\
T_g^2 &= \frac{1}{2}\Big\{-2\delta v_{xx}g(t,y) - g(t,y)\int v_t\,dx + g_t(\int v\,dx) - \lambda v_{yy}g(t,y)\\
&\quad +2g(t,y)u - 2\eta g(t,y)v + 6\mu g(t,y)v^2 + \lambda g_y v_y\Big\}, &(2.127)\\
T_g^3 &= -\frac{1}{2}\lambda g(t,y)v_{xy}. &(2.128)
\end{aligned}
$$

The conservation law (2.114)-(2.116) is a local conservation law whereas the remaining ones are nonlocal conservation laws. We note that for arbitrary values of $f(t,y)$ and $g(t,y)$ infinitely many nonlocal conservation laws exist for system (2.2).

2.2 Exact solutions of (2.2) using the Kudryashov method

This section aims to show the algorithm of the Kudryashov method for computing exact solutions of systems of nonlinear evolution equations. The Kudryashov method was one of the earliest methods for finding exact solutions of nonlinear partial differential equations [39–41]. It should be emphasized that due to the lack of popularity of computer algebra systems such as Maple and Mathematica in the late 1980s, the Kudryashov method was not well-known [41].

Let us shortly revisit the basic steps of the Kudryashov method. Consider the system nonlinear partial differential equation of the form

$$E_1[u_t, u_x, u_y, v_t, v_x, v_y, \cdots] = 0. \qquad (2.129)$$

We use the following ansatz

$$u(x,y,t) = F(z), \quad u(x,y,t) = G(z), \quad z = k_1 x + k_2 y - ct. \tag{2.130}$$

From (2.129) we obtain the system of ordinary differential equations

$$\begin{aligned} &E_2[k_1F'(z), k_2F'(z), cF'(z), k_1^2F''(z), k_2^2F''(z), c^2F''(z), k_1G'(z), k_2G'(z), \\ &cG'(z), k_1^2G''(z), k_2^2G''(z), c^2G''(z), \cdots] = 0, \end{aligned} \tag{2.131}$$

which has a solution of the form

$$F(z) = \sum_{i=0}^{M} A_i(H(z))^i, \quad G(z) = \sum_{i=0}^{M} B_i(H(z))^i, \tag{2.132}$$

where

$$H(z) = \frac{1}{1 + \cosh(z) + \sinh(z)},$$

satisfies the Riccati equation

$$H'(z) = H(z)^2 - H(z) \tag{2.133}$$

and M is a positive integer that can be determined by balancing technique as in [41] and $A_0, \cdots, A_M, B_0, \cdots, B_M$ are parameters to be determined.

2.2.1 Application of the Kudryashov method

Employing anstaz (2.130), we obtain the following nonlinear ordinary differential equation

$$k_1{}^3F_{zzz} + k_2{}^2k_1F_{zzz} - 6k_1FF_z - cF_z - k_1G_z = 0, \tag{2.134a}$$

$$k_1{}^3\delta G_{zzz} + \lambda k_2{}^2k_1G_{zzz} - 6Gk_1\mu G_z + \eta k_1G_z - cG_z - k_1F_z = 0. \tag{2.134b}$$

The balancing technique [41] gives $M = 2$ so the solutions of (2.134) are of the form

$$F(z) = A_0 + A_1H + A_2H^2, \tag{2.135a}$$

$$G(z) = B_0 + B_1H + B_2H^2. \tag{2.135b}$$

Replacing (2.135a) into (2.134) and making use of (2.133) and then equating all coefficients of the functions H^i to zero, we obtain an overdetermined system of algebraic equations. Solving this system of algebraic equations with the aid of Maple, one obtains,

$$
\begin{aligned}
c = & \frac{1}{\delta k_1^2 + \lambda k_2^2}\Big\{k_1(\delta^2 k_1^4 + 2\,\delta\,\lambda\, k_1^2 k_2^2 + \lambda^2 k_2^4 \\
& -6\,\delta\,\mu\, B_0 k_1^2 - 6\,\lambda\,\mu\, B_0 k_2^2 + \delta\,\eta\, k_1^2 + \eta\,\lambda\, k_2^2 - \mu\, k_1^2 - \mu\, k_2^2)\Big\}, \\
A_0 = & -\frac{1}{6\left(\delta k_1^4 + \delta k_1^2 k_2^2 + \lambda k_1^2 k_2^2 + \lambda k_2^4\right)\mu}\Big\{\delta^2 \mu\, k_1^6 + \delta^2 \mu\, k_1^4 k_2^2 \\
& +2\,\delta\,\lambda\,\mu\, k_1^4 k_2^2 + 2\,\delta\,\lambda\,\mu\, k_1^2 k_2^4 + \lambda^2 \mu\, k_1^2 k_2^4 + \lambda^2 \mu\, k_2^6 - 6\,\delta\,\mu^2 B_0 k_1^4 \\
& -6\,\delta\,\mu^2 B_0 k_1^2 k_2^2 - \delta\,\mu\, k_1^6 - 2\,\delta\,\mu\, k_1^4 k_2^2 - \delta\,\mu\, k_1^2 k_2^4 - 6\,\lambda\,\mu^2 B_0 k_1^2 k_2^2 \\
& -6\,\lambda\,\mu^2 B_0 k_2^4 - \lambda\,\mu\, k_1^4 k_2^2 - 2\,\lambda\,\mu\, k_1^2 k_2^4 - \lambda\,\mu\, k_2^6 + \delta\,\eta\,\mu\, k_1^4 + \delta\,\eta\,\mu\, k_1^2 k_2^2 \\
& +\eta\,\lambda\,\mu\, k_1^2 k_2^2 + \eta\,\lambda\,\mu\, k_2^4 + \delta^2 k_1^4 + 2\,\delta\,\lambda\, k_1^2 k_2^2 + \lambda^2 k_2^4 - \mu^2 k_1^4 - 2\,\mu^2 k_1^2 k_2^2 \\
& -\mu^2 k_2^4\Big\}, \\
A_1 = & -2\, k_1^2 - 2\, k_2^2, \\
A_2 = & \, 2\, k_1^2 + 2\, k_2^2, \\
B_1 = & -2\,\frac{\delta k_1^2 + \lambda k_2^2}{\mu}, \\
B_2 = & \, 2\,\frac{\delta k_1^2 + \lambda k_2^2}{\mu}.
\end{aligned}
$$

Consequently a solution of (2.2) is,

$$
\begin{aligned}
u(t,x,y) &= A_0 + A_1\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\} \\
&+ A_2\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}^2, \qquad (2.136a)
\end{aligned}
$$

$$
\begin{aligned}
v(t,x,y) &= B_0 + B_1\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\} \\
&+ B_2\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}^2, \qquad (2.136b)
\end{aligned}
$$

where $z = k_1 x + k_2 y - ct$.

A profile solution of (2.136) is given in Figure 2.1.

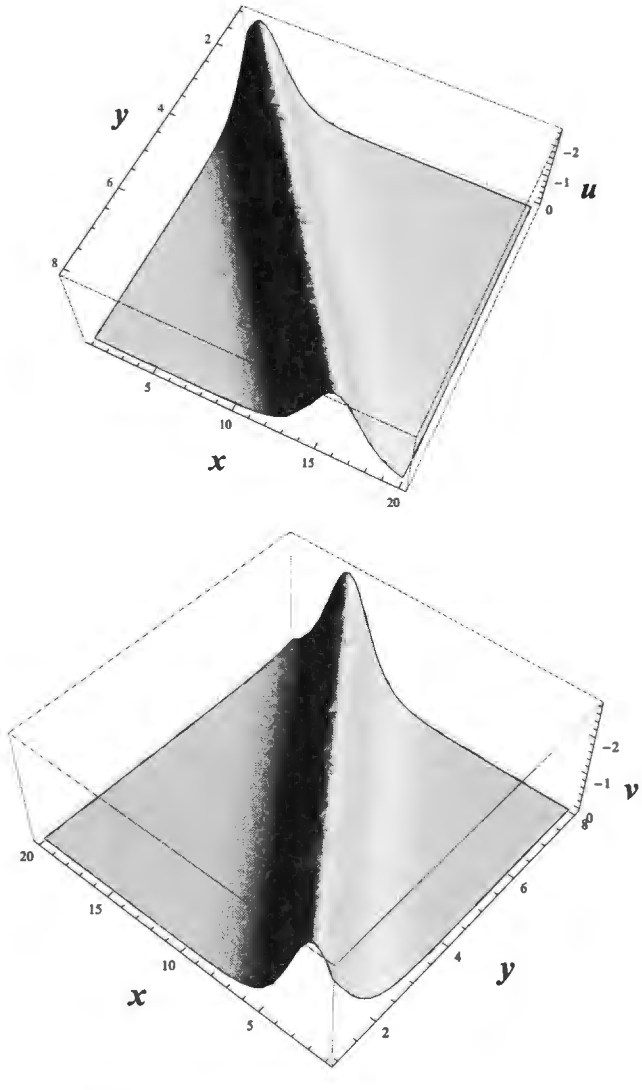


Figure 2.1: Profile of solitary waves (2.136)

2.3 Solutions of (2.2) using Jacobi elliptic function method

Periodic exact solutions of (2.2) in terms of Jacobi elliptic function are shown in this section. The cosine-amplitude function, $\mathrm{cn}(z|\omega)$, and the sine-amplitude function,

$\mathrm{sn}(z|\omega)$ satisfy the following first-order differential equations [42]:

$$H'(z) = -\left\{ \left(1 - H^2(z)\right)\left(1 - \omega + \omega H^2(z)\right) \right\}^{\frac{1}{2}} \tag{2.137}$$

and

$$H'(z) = \left\{ \left(1 - H^2(z)\right)\left(1 - \omega H^2(z)\right) \right\}^{\frac{1}{2}}. \tag{2.138}$$

By following the same technique as in the Kudryashov method we obtain the following cnoidal and snoidal wave solutions:

$$u(x, y, t) = A_0 + A_1 \mathrm{cn}(z|\omega) + A_2 \mathrm{cn}^2(z|\omega), \tag{2.139a}$$

$$v(x, y, t) = B_0 + B_1 \mathrm{cn}(z|\omega) + B_2 \mathrm{cn}^2(z|\omega), \tag{2.139b}$$

$$\begin{aligned}
c = {} & \frac{1}{\delta\,{k_1}^2 + \lambda\,{k_2}^2} \Big\{ k_1 (8\,\delta^2 \omega\,{k_1}^4 + 16\,\delta\,\lambda\,\omega\,{k_1}^2{k_2}^2 + 8\,\lambda^2\omega\,{k_2}^4 - 4\,\delta^2{k_1}^4 \\
& -8\,\delta\,\lambda\,{k_1}^2{k_2}^2 - 4\,\lambda^2{k_2}^4 - 6\,\delta\,\mu\,B_0{k_1}^2 - 6\,\lambda\,\mu\,B_0{k_2}^2 + \delta\,\eta\,{k_1}^2 + \eta\,\lambda\,{k_2}^2 \\
& -\mu\,{k_1}^2 - \mu\,{k_2}^2) \Big\}, \\
A_0 = {} & -\frac{1}{6\mu\,\left(\delta\,{k_1}^4 + \delta\,{k_1}^2{k_2}^2 + \lambda\,{k_1}^2{k_2}^2 + \lambda\,{k_2}^4\right)} \Big\{ 8\,\delta^2\mu\,\omega\,{k_1}^6 + 8\,\delta^2\mu\,\omega\,{k_1}^4{k_2}^2 \\
& +16\,\delta\,\lambda\,\mu\,\omega\,{k_1}^4{k_2}^2 + 16\,\delta\,\lambda\,\mu\,\omega\,{k_1}^2{k_2}^4 + 8\,\lambda^2\mu\,\omega\,{k_1}^2{k_2}^4 + 8\,\lambda^2\mu\,\omega\,{k_2}^6 \\
& -4\,\delta^2\mu\,{k_1}^6 - 4\,\delta^2\mu\,{k_1}^4{k_2}^2 - 8\,\delta\,\lambda\,\mu\,{k_1}^4{k_2}^2 - 8\,\delta\,\lambda\,\mu\,{k_1}^2{k_2}^4 - 8\,\delta\,\mu\,\omega\,{k_1}^6 \\
& -16\,\delta\,\mu\,\omega\,{k_1}^4{k_2}^2 - 8\,\delta\,\mu\,\omega\,{k_1}^2{k_2}^4 - 4\,\lambda^2\mu\,{k_1}^2{k_2}^4 - 4\,\lambda^2\mu\,{k_2}^6 - 8\,\lambda\,\mu\,\omega\,{k_1}^4{k_2}^2 \\
& -16\,\lambda\,\mu\,\omega\,{k_1}^2{k_2}^4 - 8\,\lambda\,\mu\,\omega\,{k_2}^6 - 6\,\delta\,\mu^2B_0{k_1}^4 - 6\,\delta\,\mu^2B_0{k_1}^2{k_2}^2 + 4\,\delta\,\mu\,{k_1}^6 \\
& +8\,\delta\,\mu\,{k_1}^4{k_2}^2 + 4\,\delta\,\mu\,{k_1}^2{k_2}^4 - 6\,\lambda\,\mu^2B_0{k_1}^2{k_2}^2 - 6\,\lambda\,\mu^2B_0{k_2}^4 + 4\,\lambda\,\mu\,{k_1}^4{k_2}^2 \\
& +8\,\lambda\,\mu\,{k_1}^2{k_2}^4 + 4\,\lambda\,\mu\,{k_2}^6 + \delta\,\eta\,\mu\,{k_1}^4 + \delta\,\eta\,\mu\,{k_1}^2{k_2}^2 + \eta\,\lambda\,\mu\,{k_1}^2{k_2}^2 + \eta\,\lambda\,\mu\,{k_2}^4 \\
& +\delta^2{k_1}^4 + 2\,\delta\,\lambda\,{k_1}^2{k_2}^2 + \lambda^2{k_2}^4 - \mu^2{k_1}^4 - 2\,\mu^2{k_1}^2{k_2}^2 - \mu^2{k_2}^4 \Big\}, \\
A_1 = {} & 0, \\
A_2 = {} & -2\,\omega\,{k_1}^2 - 2\,\omega\,{k_2}^2, \\
B_1 = {} & 0, \\
B_2 = {} & -2\,\frac{\omega\,\left(\delta\,{k_1}^2 + \lambda\,{k_2}^2\right)}{\mu}
\end{aligned}$$

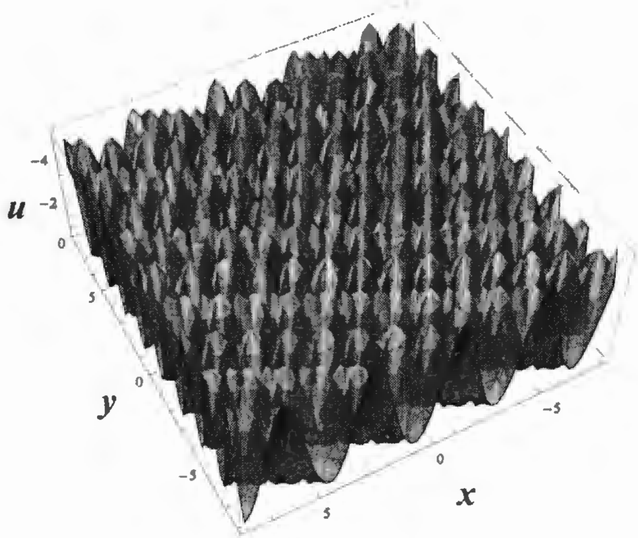
$$u(x, y, t) = A_0 + A_1 \mathrm{sn}(z|\omega) + A_2 \mathrm{sn}^2(z|\omega),$$

$$v(x, y, t) = B_0 + B_1 \mathrm{sn}(z|\omega) + B_2 \mathrm{sn}^2(z|\omega),$$

$$c = -\frac{1}{\delta k_1{}^2 + \lambda k_2{}^2 k_1}\Big\{4\delta^2\omega k_1{}^4 + 8\delta\lambda\omega k_1{}^2 k_2{}^2 + 4\lambda^2\omega k_2{}^4 + 4\delta^2 k_1{}^4$$
$$+8\delta\lambda k_1{}^2 k_2{}^2 + 4\lambda^2 k_2{}^4 + 6\delta\mu B_0 k_1{}^2 + 6\lambda\mu B_0 k_2{}^2 - \delta\eta k_1{}^2 - \eta\lambda k_2{}^2$$
$$+\mu k_1{}^2 + \mu k_2{}^2\Big\},$$
$$A_0 = \frac{1}{6\mu\,(\delta k_1{}^4 + \delta k_1{}^2 k_2{}^2 + \lambda k_1{}^2 k_2{}^2 + \lambda k_2{}^4)}\Big\{4\delta^2\mu\omega k_1{}^6 + 4\delta^2\mu\omega k_1{}^4 k_2{}^2$$
$$+8\delta\lambda\mu\omega k_1{}^4 k_2{}^2 + 8\delta\lambda\mu\omega k_1{}^2 k_2{}^4 + 4\lambda^2\mu\omega k_1{}^2 k_2{}^4 + 4\lambda^2\mu\omega k_2{}^6 + 4\delta^2\mu k_1{}^6$$
$$+4\delta^2\mu k_1{}^4 k_2{}^2 + 8\delta\lambda\mu k_1{}^4 k_2{}^2 + 8\delta\lambda\mu k_1{}^2 k_2{}^4 - 4\delta\mu\omega k_1{}^6 - 8\delta\mu\omega k_1{}^4 k_2{}^2$$
$$-4\delta\mu\omega k_1{}^2 k_2{}^4 + 4\lambda^2\mu k_1{}^2 k_2{}^4 + 4\lambda^2\mu k_2{}^6 - 4\lambda\mu\omega k_1{}^4 k_2{}^2 - 8\lambda\mu\omega k_1{}^2 k_2{}^4$$
$$-4\lambda\mu\omega k_2{}^6 + 6\delta\mu^2 B_0 k_1{}^4 + 6\delta\mu^2 B_0 k_1{}^2 k_2{}^2 - 4\delta\mu k_1{}^6 - 8\delta\mu k_1{}^4 k_2{}^2 - 4\delta\mu k_1{}^2 k_2{}^4$$
$$+6\lambda\mu^2 B_0 k_1{}^2 k_2{}^2 + 6\lambda\mu^2 B_0 k_2{}^4 - 4\lambda\mu k_1{}^4 k_2{}^2 - 8\lambda\mu k_1{}^2 k_2{}^4 - 4\lambda\mu k_2{}^6$$
$$-\delta\eta\mu k_1{}^4 - \delta\eta\mu k_1{}^2 k_2{}^2 - \eta\lambda\mu k_1{}^2 k_2{}^2 - \eta\lambda\mu k_2{}^4 - \delta^2 k_1{}^4 - 2\delta\lambda k_1{}^2 k_2{}^2 - \lambda^2 k_2{}^4$$
$$+\mu^2 k_1{}^4 + 2\mu^2 k_1{}^2 k_2{}^2 + \mu^2 k_2{}^4\Big\},$$
$$A_1 = 0,$$
$$A_2 = 2\omega k_1{}^2 + 2\omega k_2{}^2,$$
$$B_1 = 0,$$
$$B_2 = 2\frac{\omega\,(\delta k_1{}^2 + \lambda k_2{}^2)}{\mu}$$

$$z = k_1 x + k_2 y - ct.$$

A profile solution of (2.139) is given in Figure 2.2.



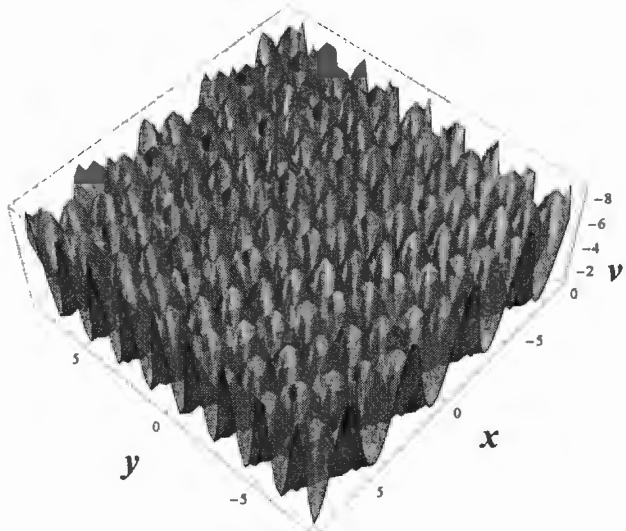


Figure 2.2: Profile of cnoidal wave (2.139)

Remark 2: Note that the Kudryashov method yields a solitary wave solution whereas the Jacobi elliptic function method gives periodic solutions.

2.4 Conclusion

In this chapter we have studied the generalized coupled Zakharov-Kuznetsov system (2.2) which did not have a variational edifice. By letting $u = U_x, v = V_x$, the generalized coupled Zakharov-Kuznetsov system (2.2) was converted to the fourth order partial differential equation that had a variational structure. Subsequently, Noether's theorem was used to acquire infinitely many conservation laws. Furthermore the Kudryashov and the Jacobi elliptic function methods were employed to construct exact solutions for the coupled Zakharov-Kuznetsov system (2.2). The solutions attained were solitary, cnoidal and snoidal waves.

Chapter 3

Reductions and exact solutions of the (2+1)-dimensional breaking soliton equation via conservation laws

In this chapter we study the (2+1)-dimensional breaking soliton equation in the form

$$u_{xt} - 2u_{xx}u_y - 4u_xu_{xy} + u_{xxxy} = 0. \tag{3.1}$$

Equation (3.1) was first presented by Calogero and Degasperis [43, 44] and is used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. Due to the importance of equation (3.1), there has recently been much attention devoted to studying solutions of equation (3.1). In [45], the author employed the homogeneous method and some soliton-like solutions were obtained. The classical Lie symmetry method was employed in [46] and some new non-traveling wave explicit solutions of Jacobian elliptic function were derived. Although a great deal of research work has been devoted to finding different methods of solving nonlinear evolution equations, there is no unique method for finding exact solutions of nonlinear partial differential equations.

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Here we use the Noether theorem [21] to construct conservation laws for equation (3.1). Thereafter, we employ the definition of the association of symmetries with conservation laws to obtain exact solutions for the (2+1)-dimensional breaking soliton equation via the generalized double reduction theorem.

3.1 Construction of conservation laws for (2+1)-dimensional breaking soliton equation (3.1)

Consider the (2+1)-dimensional breaking soliton equation (3.1), viz.,

$$u_{xt} - 2u_{xx}u_y - 4u_xu_{xy} + u_{xxxy} = 0.$$

It can be verified that the corresponding second-order Lagrangian for equation (3.1) is

$$L = u_x^2u_y - \frac{1}{2}u_xu_t + \frac{1}{2}u_{xx}u_{xy}. \tag{3.2}$$

The insertion of L from (3.2) into equation (1.29) and splitting with respect to the derivatives of $u(t, x, y)$ yields an overdetermined system of PDEs: These are:

$$\xi^1_u = 0, \tag{3.3}$$

$$\xi^1_y = 0, \tag{3.4}$$

$$\xi^1_x = 0, \tag{3.5}$$

$$\xi^2_u = 0, \tag{3.6}$$

$$\xi^2_y = 0, \tag{3.7}$$

$$A^3_u = 0, \tag{3.8}$$

$$\xi^3_u = 0, \tag{3.9}$$

$$\xi^3_x = 0, \tag{3.10}$$

$$\xi^2_{xx} = 0, \tag{3.11}$$

$$\eta_{uu} = 0, \tag{3.12}$$

$$\eta_{xy} = 0, \tag{3.13}$$

$$\eta_{yu} = 0, \tag{3.14}$$

$$\eta_{xu} = 0, \tag{3.15}$$

$$\eta_{xx} = 0, \tag{3.16}$$

$$2\eta_u + \xi^3_y = 0, \tag{3.17}$$

$$2\eta_y + \xi^2_t = 0, \tag{3.18}$$

$$4\eta_x + \xi^3_t = 0, \tag{3.19}$$

$$2\eta_u - 2\xi^2_x + \xi^1_t = 0, \tag{3.20}$$

$$3\eta_u - \xi^2_x + \xi^1_t = 0, \tag{3.21}$$

$$2A^1_u + \eta_x = 0, \tag{3.22}$$

$$2A^2_u + \eta_t = 0, \tag{3.23}$$

$$A^3_y + A^2_x + A^1_t = 0. \tag{3.24}$$

After some very much computations, the above system of PDEs yields,

$$\begin{aligned}
\xi^1 &= -4c_1 t + c_2, \\
\xi^2 &= -c_1 x + c_4 t + c_5, \\
\xi^3 &= -2c_1 y - 4c_3 t + c_7, \\
\eta &= c_1 u + c_3 x - \frac{1}{2} c_4 y + c_6, \\
A^1 &= -\frac{1}{2} c_3 u + E(t,x,y), \\
A^2 &= D(t,x,y), \\
A^3 &= R(t,x,y), \\
E_t + D_x + R_y &= 0.
\end{aligned}$$

Here we choose $D(t,x,y) = E(t,x,y) = R(t,x,y) = 0$ as they lead to trivial part of the conserved vectors. The invocation of theorem 1.4, results in the following nontrivial conserved vectors corresponding to the seven Noether point symmetries respectively:

$$
\begin{aligned}
T_1^1 &= -4tu_x^2u_y - 2tu_{xx}u_{xy} - \frac{1}{2}uu_x - \frac{1}{2}xu_x^2 - yu_xu_y,\\
T_1^2 &= xu_x^2u_y + 2uu_xu_y + 8tu_tu_xu_y + 4yu_xu_y^2 - \frac{1}{2}uu_t - 2tu_t^2 - yu_tu_y\\
&\quad -\frac{3}{4}u_{xxy}(u + 4tu_t + xu_x + 2yu_y) + \frac{1}{2}u_{xy}(2u_x + 4tu_{xt} + 2yu_{xy})\\
&\quad +\frac{1}{4}u_{xx}(3u_y + 4tu_{ty} + xu_{xy} + 2yu_{yy}),\\
T_1^3 &= yu_xu_t + uu_x^2 + 4tu_tu_x^2 + xu_x^3 - \frac{1}{4}u_{xxx}(u + 4tu_t + xu_x + 2yu_y)\\
&\quad +\frac{1}{4}u_{xx}(2u_x - 2yu_{xy} + xu_{xx} + 4tu_{xt}); \qquad (3.25)\\
T_2^1 &= u_x^2u_y + \frac{1}{2}u_{xx}u_{xy},\\
T_2^2 &= -2u_tu_xu_y + \frac{1}{2}u_t^2 + \frac{3}{4}u_tu_{xxy} - \frac{1}{4}u_{xx}u_{ty} - \frac{1}{2}u_{xt}u_{xy},\\
T_2^3 &= -u_tu_x^2 + \frac{1}{4}u_tu_{xxx} - \frac{1}{4}u_{tx}u_{xx}; \qquad (3.26)\\
T_3^1 &= \frac{1}{2}u - 2tu_xu_y - \frac{1}{2}xu_x,\\
T_3^2 &= 2xu_xu_y - \frac{1}{2}xu_t + 8tu_xu_y^2 - 2tu_tu_y - \frac{3}{4}u_{xxy}(x + 4tu_y) + \frac{1}{2}u_{xy} + 2tu_{xy}^2\\
&\quad +tu_{xx}u_{yy},\\
T_3^3 &= 2tu_xu_t + xu_x^2 + \frac{1}{4}u_{xx} - tu_{xx}u_{xy} - \frac{1}{4}u_{xxx}(x + 4tu_y); \qquad (3.27)\\
T_4^1 &= \frac{1}{4}yu_x + \frac{1}{2}tu_x^2,\\
T_4^2 &= -tu_x^2u_y - yu_xu_y + \frac{1}{4}yu_t + \frac{3}{4}u_{xxy}\left(\frac{1}{2}y + tu_x\right) - \frac{1}{8}u_{xx} - \frac{1}{4}tu_{xx}u_{xy},\\
T_4^3 &= -\frac{1}{4}tu_{xx}^2 - \frac{1}{2}yu_x^2 - tu_x^3 + \frac{1}{4}u_{xxx}\left(\frac{1}{2}y + tu_x\right); \qquad (3.28)\\
T_5^1 &= \frac{1}{2}u_x^2,\\
T_5^2 &= -u_x^2u_y - \frac{1}{4}u_{xx}u_{xy} + \frac{3}{4}u_xu_{xxy},\\
T_5^3 &= -\frac{1}{4}u_{xx}^2 - u_x^3 + \frac{1}{4}u_xu_{xxx}; \qquad (3.29)\\
T_6^1 &= -\frac{1}{2}u_x,\\
T_6^2 &= 2u_xu_y - \frac{1}{2}u_t - \frac{3}{4}u_{xxy},\\
T_6^3 &= u_x^2 - \frac{1}{4}u_{xxx}; \qquad (3.30)
\end{aligned}
$$

$$\begin{aligned}
T_7^1 &= \frac{1}{2}u_x u_y, \\
T_7^2 &= -2u_x u_y^2 + \frac{1}{2}u_t u_y + \frac{3}{4}u_y u_{xxy} - \frac{1}{2}u_{xy}^2 - \frac{1}{4}u_{xx}u_{yy}, \\
T_7^3 &= -\frac{1}{2}u_x u_t + \frac{1}{4}u_{xx}u_{xy} + \frac{1}{4}u_y u_{xxx}.
\end{aligned} \tag{3.31}$$

3.2 Double reduction of (3.1) via conservation laws

The aim of this section is to employ the double reduction method to the (2+1)-dimensional breaking soliton equation by using the derived conservation laws and the associated symmetries. Employing the classical Lie algorithm for symmetry method [26,47], it can be shown that the (2+1)-dimensional breaking soliton equation admits seven Lie symmetries which are all Noether symmetries. We first look for the possible associations between the symmetries and the conserved vectors. This will be achieved through the following formula,

$$X\begin{bmatrix} T^t \\ T^x \\ T^y \end{bmatrix} - \begin{bmatrix} D_t\xi^1 & D_x\xi^1 & D_y\xi^1 \\ D_t\xi^2 & D_x\xi^2 & D_y\xi^2 \\ D_t\xi^3 & D_x\xi^3 & D_y\xi^3 \end{bmatrix}\begin{bmatrix} T^t \\ T^x \\ T^y \end{bmatrix} + (D_t\xi^1 + D_x\xi^2 + D_y\xi^3)\begin{bmatrix} T^t \\ T^x \\ T^y \end{bmatrix} = 0. \tag{3.32}$$

It can easily be verified that the associated symmetries are X_2, X_5 and X_7, so the combination of these symmetries,

$$X = X_2 + \rho X_5 + \beta X_7 = \frac{\partial}{\partial t} + \rho\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y},$$

can be used to get a reduced conserved vector where ρ, β are constants. The generator X has the canonical form X $= \frac{\partial}{\partial q}$, then from (1.24) we have

$$\frac{dt}{1} = \frac{dx}{\rho} = \frac{dy}{\beta} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dv}{0} \tag{3.33}$$

and thus the canonical variables are:

$$r = y - \beta t, \quad s = x - \rho t, \quad q = t, \quad v(r,s) = u(t,x,y). \tag{3.34}$$

Employing the following formula, we can reach the reduced conserved vector,

$$\begin{bmatrix} T_6^r \\ T_6^s \\ T_6^q \end{bmatrix} = J(A^{-1})^T \begin{bmatrix} T_6^t \\ T_6^x \\ T_6^y \end{bmatrix}, \tag{3.35}$$

where A^{-1} from (3.35) is given by

$$A^{-1} = \begin{bmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{bmatrix}, \quad J = \det(A). \tag{3.36}$$

Equations (3.35) and (3.36) for the conserved vector (3.30) result in

$$\begin{aligned} T_6^r &= -\frac{1}{2}\beta v_s - v_s^2 + \frac{1}{4}v_{sss}, \\ T_6^s &= -\frac{1}{2}\rho v_s - 2v_s v_r - \frac{1}{2}(\rho v_s + \beta v_r) + \frac{3}{4}v_{ssr}, \\ T_6^q &= \frac{1}{2}v_s \end{aligned} \tag{3.37}$$

and the reduced conserved vector is

$$D_r T_6^r + D_s T_6^s = 0. \tag{3.38}$$

We further determine the associated symmetry with the reduced conserved vector (3.38) through the formula,

$$\mathbf{X}\begin{bmatrix} T^r \\ T^s \end{bmatrix} - \begin{bmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{bmatrix}\begin{bmatrix} T^r \\ T^s \end{bmatrix} + (D_r\xi^r + D_s\xi^s)\begin{bmatrix} T^r \\ T^s \end{bmatrix} = 0. \tag{3.39}$$

It can be shown that the associated symmetries are

$$Y_1 = \frac{\partial}{\partial r} \quad \text{and} \quad Y_2 = \frac{\partial}{\partial s}. \tag{3.40}$$

Thus, it is possible to obtain a further reduced conserved vector by the combination of,

$$Y = Y_1 + \gamma Y_2 = \frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s},$$

where γ is a constant and the generator Y has canonical form

$$Y = \frac{\partial}{\partial m},$$

if

$$\frac{dr}{1} = \frac{ds}{\gamma} = \frac{dm}{1} = \frac{dn}{0} = \frac{dv}{0} = \frac{dw}{0} \tag{3.41}$$

with the similarity variables

$$n = \gamma r - s, \quad m = r, \quad w(n) = v(r, s). \tag{3.42}$$

Invoking the following formula, we can reach the reduced conserved vector,

$$\begin{bmatrix} T_6^n \\ T_6^m \end{bmatrix} = J(A^{-1})^T \begin{bmatrix} T_6^r \\ T_6^s \end{bmatrix}, \tag{3.43}$$

with

$$A^{-1} = \begin{bmatrix} D_r n & D_r m \\ D_s n & D_s m \end{bmatrix}, \quad J = \det(A). \tag{3.44}$$

Equations (3.43) and (3.44) for the conserved vector (3.37) yield the following form,

$$\begin{bmatrix} T_6^n \\ T_6^m \end{bmatrix} = \begin{bmatrix} \gamma & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}\beta v_s - v_s^2 + \frac{1}{4}v_{sss} \\ -\frac{1}{2}\rho v_s - 2v_s v_r - \frac{1}{2}(\rho v_s + \beta v_r) + \frac{3}{4}v_{ssr} \end{bmatrix}. \tag{3.45}$$

Equation (3.45) expressed in terms of variable n becomes

$$\begin{aligned} T_4^n &= -(\gamma w''' + \rho w' + 3\gamma w'^2) + \gamma\beta w', \\ T_6^m &= -\frac{1}{2}\beta w' - w'^2 \end{aligned} \tag{3.46}$$

and the reduced conserved form is

$$D_n T_6^n = 0. \tag{3.47}$$

Equation (3.47) gives $T_6^n = -k_1$ and (3.46) can be written as:

$$\gamma w''' + (\rho - \beta\gamma)w' + 3\gamma w'^2 = 0. \tag{3.48}$$

The integration of (3.48) with respect to z three times and taking the constants of integration to be zero leads to a first-order variable separable ordinary differential equation, which can be integrated easily. Reverting back to the original variables and

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taking specific values of the constant of integration, we obtain the following exact solution for the (2+1)-dimensional breaking soliton equation;

$$u(t,x,y) = \frac{\sqrt{\beta\gamma - \rho}\tanh\left(\frac{z\sqrt{\beta\gamma-\rho}}{2\sqrt{\gamma}}\right)}{\sqrt{\gamma}}, \tag{3.49}$$

where $z = \gamma(y - \beta t) - (x - \rho t)$. A profile solution of (3.49) is presented in Figure 3.1.

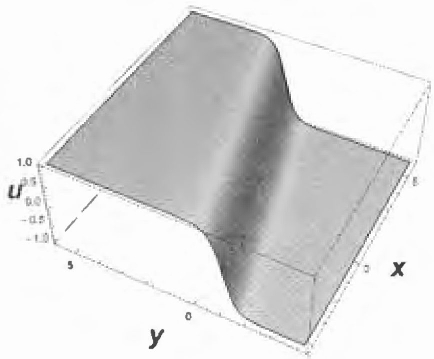

Figure 3.1: Evolution of solitary travelling wave solution (3.49) with parameters $t = 0, \gamma = 4, \beta = 2, \rho = 4$.

It can also be shown that the symmetries X_2, X_5 and X_7 are associated with the conserved vector T_2^1, T_2^2 and T_2^3 through formula (3.32). So we can get a reduced conserved vector by the combination of

$$X = X_2 + \rho X_5 + \beta X_7 = \frac{\partial}{\partial t} + \rho\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y},$$

and the generator X has the canonical form X $= \dfrac{\partial}{\partial q}$ if

$$\frac{dt}{1} = \frac{dx}{\rho} = \frac{dy}{\beta} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dv}{0} \tag{3.50}$$

where the canonical variables are

$$r = y - \beta t, \quad s = x - \rho t, \quad q = t, \quad v(r,s) = u(t,x,y). \tag{3.51}$$

Through the following formula, we can achieve the reduced conserved vector,

$$\begin{bmatrix} T_2^r \\ T_2^s \\ T_2^q \end{bmatrix} = J(A^{-1})^T \begin{bmatrix} T_2^t \\ T_2^x \\ T_2^y \end{bmatrix}, \tag{3.52}$$

where A^{-1} from (3.52) is given by

$$A^{-1} = \begin{bmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{bmatrix}, \quad J = \det(A). \tag{3.53}$$

Equations (3.52) and (3.53) for the conserved vector (3.26) result in

$$\begin{aligned} T_2^r &= \frac{1}{4}\beta v_{ss} v_{rs} - \rho v_s^3 - \frac{1}{4}\rho v_{ss}^2 + \frac{1}{4}(\rho v_s + \beta v_r) v_{sss}, \\ T_2^s &= -\rho v_s^2 v_r - 2\beta v_s v_r^2 - \frac{1}{2}(\rho v_s + \beta v_r)^2 + \frac{3}{4}(\rho v_s + \beta v_r) v_{ssr} - \frac{1}{2}\beta v_{rs}^2 \\ &\quad -\frac{1}{4}\beta v_{ss} v_{rr} - \frac{1}{4}\rho v_{ss} v_{rs}, \\ T_2^q &= -v_s^2 v_r - \frac{1}{2} v_{ss} v_{rs} \end{aligned} \tag{3.54}$$

and the reduced conserved vector is

$$D_r T_2^r + D_s T_2^s = 0. \tag{3.55}$$

Similarly, we can further determine the associated symmetry with the reduced conserved vector (3.55) using the following formula,

$$\mathbf{X} \begin{bmatrix} T^r \\ T^s \end{bmatrix} - \begin{bmatrix} D_r \xi^r & D_s \xi^r \\ D_r \xi^s & D_s \xi^s \end{bmatrix} \begin{bmatrix} T^r \\ T^s \end{bmatrix} + (D_r \xi^r + D_s \xi^s) \begin{bmatrix} T^r \\ T^s \end{bmatrix} = 0. \tag{3.56}$$

One can verify that the associated symmetries are

$$Y_1 = \frac{\partial}{\partial r} \quad \text{and} \quad Y_2 = \frac{\partial}{\partial s} \tag{3.57}$$

so we can further get a reduced conserved vector by the combination of

$$Y = Y_1 + \gamma Y_2 = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s},$$

where γ is an arbitrary constant and the generator Y has canonical form

$$Y = \frac{\partial}{\partial m},$$

if

$$\frac{dr}{1} = \frac{ds}{\gamma} = \frac{dm}{1} = \frac{dn}{0} = \frac{dv}{0} = \frac{dw}{0} \tag{3.58}$$

and

$$n = \gamma r - s, \quad m = r, \quad w(n) = v(r,s). \tag{3.59}$$

Employing the following formula, we can attain the reduced conserved vector,

$$\begin{bmatrix} T_2^n \\ T_2^m \end{bmatrix} = J(A^{-1})^T \begin{bmatrix} T_2^r \\ T_2^s \end{bmatrix}, \tag{3.60}$$

with

$$A^{-1} = \begin{bmatrix} D_r n & D_r m \\ D_s n & D_s m \end{bmatrix}, \quad J = \det(A). \tag{3.61}$$

Equations (3.60) and (3.61) for the conserved vector (3.54) has the following form:

$$\begin{bmatrix} T_2^n \\ T_2^m \end{bmatrix} = \begin{bmatrix} \gamma & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4}\beta v_{ss}v_{rs} - \rho v_s^3 - \frac{1}{4}\rho v_{ss}^2 + \frac{1}{4}(\rho v_s + \beta v_r)v_{sss} \\ -\rho v_s^2 v_r - 2\beta v_s v_r^2 - \frac{1}{2}(\rho v_s + \beta v_r)^2 + \frac{3}{4}(\rho v_s + \beta v_r)v_{ssr} - \\ \frac{1}{2}\beta v_{rs}^2 - \frac{1}{4}(\beta v_{rr} + \rho v_{rs})v_{ss} \end{bmatrix}. \tag{3.62}$$

Expressing equation (3.62) in terms of variable n yields

$$\begin{aligned} T_2^n &= -2\gamma(\gamma\beta - \rho)w'^3 - \gamma(\gamma\beta - \rho)w'w''' + \frac{1}{2}(\gamma\beta - \rho)^2 w'^2 \\ &\quad + \frac{1}{2}\gamma(\gamma\beta - \alpha)w''^2, \\ T_2^m &= -\frac{1}{2}\gamma\beta w''^2 + \rho w'^3 - \frac{1}{2}(\gamma\beta - \rho)w'w''' \end{aligned} \tag{3.63}$$

and the reduced conserved form is

$$D_n T_2^n = 0. \tag{3.64}$$

Equation (3.64) gives $T_2^n = -k_1$ and (3.63) can be written as:

$$4\gamma(\gamma\beta - \rho)w'^3 + 2\gamma(\gamma\beta - \rho)w'w''' - (\gamma\beta - \rho)^2 w'^2 - \gamma(\gamma\beta - \rho)w''^2 = k_3. \tag{3.65}$$

Despite its simplicity, this nonlinear third-order ordinary differential equation is intractable to solve explicitly in general. However, these difficulties will be by passed using the Kudryashov method [39–41].

3.3 Exact solution using Kudryashov method

In this section we utilize the Kudryashov method to construct exact solution of the nonlinear evolution equation. The third-order ordinary differential equation (3.65) has a solution of the form

$$w(z) = \sum_{i=0}^{M} A_i (H(z))^i, \tag{3.66}$$

where

$$H(z) = \frac{1}{1 + \cosh(z) + \sinh(z)},$$

satisfies the equation

$$H'(z) = H(z)^2 - H(z) \tag{3.67}$$

and M is a positive integer that can be determined by balancing technique as in [41] and $A_0, \cdots, A_M$ are parameters to be determined.

3.3.1 Solution of (3.1) via Kudryashov method

$$4\gamma(\beta\gamma - \rho)w'^3 + 2\gamma(\beta\gamma - \rho)w'w''' - (\beta\gamma - \rho)^2 w'^2 - \gamma(\beta\gamma - \rho)w''^2 = 0. \tag{3.68}$$

The balancing procedure [41] yields $M = 1$, so the solutions of (3.68) are of the form

$$w(z) = A_0 + A_1 H. \tag{3.69}$$

Inserting (3.69) into (3.68) and making use of (3.67) followed by equating all coefficients of the functions H^i to zero, we obtain an overdetermined system of algebraic equations in terms of A_0, A_1. Solving the system of algebraic equations with the aid of Maple, we obtain the following cases:

CASE 1

$$\rho = \beta\gamma,$$

A_1, A_0 arbitrary.

CASE 2

$$\rho = \beta\gamma - \gamma,$$
$$A_1 = -2 \text{ and } A_0 \text{ arbitrary.}$$

Consequently a solution of (3.1) is

$$u(t,x,y) = A_0 + A_1\left\{\frac{1}{1+\cosh(z)+\sinh(z)}\right\}, \tag{3.70}$$

where $z = \gamma(y - \beta t) - (x - \rho t)$.

A profile solution of (3.70) is given in Figure 3.2.

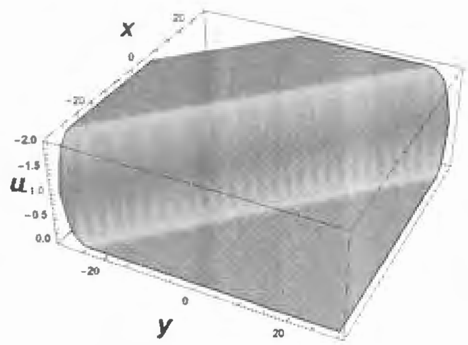

Figure 3.2: Evolution of solitary travelling wave solution (3.70) with parameters $t = 0$, $\gamma = 1$, $A_0 = 0$.

3.4 Concluding remarks

We have constructed the associated conservation laws for the admitted Noether point symmetries for the (2+1)-dimensional breaking soliton equation. Thereafter, two solutions have been obtained by making use of the double reduction theorem. This has been attained after establishing the association between the conserved vectors and the symmetries. The correctness of these solutions has been verified by back substitution into equation (3.1).

Chapter 4

Conclusion and Discussions

In this dissertation we first recalled some important definitions and results concerning Lie, Noether theorem and the double reduction theorem, which were later used in the dissertation.

In Chapter two we derived the conservation laws for the third-order coupled system (2.2) through a very interesting method of increasing the order of the third-order coupled system (2.2) to a forth-order coupled system (2.3) using $u = U_x$ and $v = V_x$. Thereafter we applied the Noether theorem to derive conservation laws in U and V variables. We then reverted back to the original variables u and v and obtained the conservation laws for the third-order coupled system (2.2) which did not admit a variational structure in its present form. Moreover, we constructed exact solutions for the generalized coupled Zakharov-Kuznetsov system (2.2) using the Kudryashov method and Jacobi elliptic function method.

Lastly in Chapter three the Noether theorem was employed to construct the conservation laws for the (2+1)-dimensional breaking soliton equation (3.1). Furthermore, the double reduction theorem was used to find exact solutions for equation (3.1). This was attained after establishing the association between the conserved vectors and the symmetries.

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