Application of spectral analysis techniques on the gamma-ray data of Vela X-1 and PKS 2155-304.

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Abstract

Studies of frequency variability can provide an important understanding of physical processes in the central regions of an AGN (Ulrich et al., 1997). For neutron star X-ray sources, variations in the intrinsic pulse frequency are believed to reflect changes in the rotation rate of the stellar crust, produced by torques originating inside and outside the crust (Boynton et al., 1984). Periodicity analysis therefore plays an important role in astrophysics.

For discretely sampled signals, this analysis is often done using the periodogram, modified periodogram, and the Lomb-Scargle periodogram. These techniques, with possibly the exception of the Lomb-Scargle periodogram, are well known in the subject of discrete signal processing. However, their application to atmospheric Cherenkov $\gamma$-ray telescopes have as of yet not been properly studied.

Atmospheric Cherenkov $\gamma$-ray telescopes, particularly H.E.S.S., can be thought of as photon counting devices. This is very different from devices that sample discretely. Only after binning the data can the data be regarded as discretely sampled. Furthermore, the H.E.S.S. telescope usually samples in 28 minute intervals followed by 3 minutes of dead time after each interval.

A signal will be simulated/modelled through the use of Monte Carlo simulations. These simulated signals can consist of white noise, periodic signals, or a mixture of both. Through the use of these simulations an attempt will be made to find an appropriate bin size, as well as determining the effect of dead time on the time series and correcting for that effect. The effect of dead time on the Rayleigh test, when searching low frequency periodicity will also be studied. An attempt will also be made to explain the discrepancy between the binning of photon counting events, and discretely sampled signals in terms of discrete signal processing.

Monte Carlo simulations showed that the dead time causes low frequency power to increase drastically. This increased power could easily be misinterpreted as power-law noise, as well as resulting in the false positive detection of a further signal when applying significance tests. The proposed method for dealing with this increase in power is to subtract the Fourier transform of the dead time from that of the signal to be analyzed. This method yielded satisfactory results.

The flaring event of PKS 2155-304 on the nights of 27/28 July and 29/30 July was analyzed by Aharonian (2007). A reanalysis of this event will be done using the results obtained from the Monte Carlo simulations. An analysis will also be done on Vela X-1 to determine it as a source of TeV $\gamma$-rays as found by Protheroe et al. (1984), North et al. (1987), Raubenheimer et al. (1989), and Raubenheimer et al. (1994).

A reanalysis of the flaring event of PKS 2155-304 found that the low frequency power was very dependent on bin size. For smaller bin sizes the dead time had significant consequences however, for a bin size of 60 seconds, the results were similar to that found by Aharonian (2007). The analysis of Vela X-1 however confirmed this object to not be a source of high energy $\gamma$-rays.
Keywords: periodicity, periodogram, modified periodogram, Lomb-Scargle periodogram, discrete signal processing, atmospheric Cherenkov $\gamma$-ray telescopes, binning, dead time, photon counting.
Studies van veranderlikheid in frekwensie kan belangrike insig gee om die fisiese prosesse binne die sentrale gebiede van ‘n aktiewe galaktiese kern te verstaan (Ulrich et al., 1997).

Vir neutron ster X-straal bronne word variasie in die intrinsieke puls frekwensie geglo om te verwys na verandering in die rotasie tempo van die stellêre kors wat geskep word deur die momente wat hul oorsprong binne en buite die kors het (Boynton et al., 1984). Dus speel periodisiteit ‘n belangrike rol in astrofisika.

Vir diskreet vesamelde seine word hierdie analyses gereeld met die periodogram, aangepaste periodogram, en die Lomb-Scargle periodogram gedoen. Hierdie tegnieke, met die moontelike uitsluiting van die Lomb-Scargle periodogram, is goed bekend in die vakgebied van diskrete sein verwerking. Alhoewel, hulle toepaslikheid op atmosferiese Cherenkov γ-straal teleskope is tot dusver nie goed bestudeer nie.

Atmosferiese Cherenkov γ-straal teleskope, spesifiek H.E.S.S., kan gereken word as fotonteller toestelle. Hierdie toestelle verskil drasties van toestelle wat op ‘n diskrete manier versamel. Alleenlik na die data gekanaliseer word kan dit beskou word as diskreet versamel. Die H.E.S.S. teleskoop versamel gewoonlik in 28 minuut intervalle gevolg deur 3 minute van dooie tyd na elke interval.

’n Sein sal gesimmuleer/modduleer word deur gebruik te maak van Monte Carlo simmulasies. Hierdie gesimmuleerde seine kan bestaan uit wit ruis, periodiese seine, of ‘n mengsel van albei. Deur gebruik te maak van hierdie simmulasies sal ‘n poging aangewend word om ‘n toepaslike bin grootte vas te stel, asook om die effek van dooie tyd op die tydreek te bestudeer en daarvoor te korrigeer. Die effek van dooie tyd op die Rayleigh toets, wanneer gesoek word vir lae frekwensie periodisiteit, sal ook ondersoek word. ’n Poging sal ook aangewend word om die verskil tussen gekanaliseerde fotonteller gebeurtenisse, en diskreet versamelde seine te verduidelik in terme van diskrete sein prosesserings.

Monte Carlo simmulasies het daarop gewys dat dooie tyd veroorsaak dat die lae frekwensie seinsterkte drasties vermeerder. Hierdie toename in seinsterkte kan maklik misinterpretie word as eksponentiële ruis, en kan lei tot die vals positiewe deteksie van ‘n sein wanneer statistiese toets toegepas word. Die voorgestelde metode om ontslae te raak van hierdie toename in seinsterkte is om die Fourier transform van die dooie tyd af te trek van die Fourier transform van die sein wat geanaliseer moet word. Hierdie metode het positiewe resultate vertoon.

Die uitbarsting gebeurtenis van PKS 2155-304 het plaasgevind op die aande van 27/28 Julie en 29/30 Julie en was geanaliseer deur Aharonian (2007). ’n Heranalise van die gebeurtenis sal gedoen word deur gebruik te maak van resultate wat verkry is deur Monte Carlo simmulasies. ’n Analise sal ook gedoen word op Vela X-1 om vas te stel of dit ‘n bron is van TeV γ-strale soos wat gevind was deur Protheroe et al. (1984), North et al. (1987), Raubenheimer et al. (1989), en
Raubenheimer et al. (1994)

’n Heranalise van die uitbarsting gebeurtenis van PKS 2155-304 het bevind dat lae frekwensie seinsterkte baie afhanklik is van bin grootte. Vir kleiner bin groottes het die binning beduiende gevolge, alhoewel vir ’n kanalisasie grootte van 60 sekondes, was die resultate baie dieselfde soos die van Aharonian (2007). ’n Analise van Vela X-1 het egter bevestig dat dit nie ’n bron van hoë energie γ-strale is nie.

Sleutelwoorde: periodisiteit, periodogram, modified periodogram, Lomb-Scargle periodogram, diskrete sein verwerking, atmosferiese Cherenkov γ-straal teleskope, gekanaliseer, dooie tyd, foton teller.
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Chapter 1

Introduction

1.1 The development of γ-ray astronomy

Čerenkov (1937) found that when a particle moves through a medium at a constant velocity, such that the velocity of the particle is faster than the velocity of light in that medium, it results in the polarization of the medium along the trajectory of the particle. This polarization is the result of the incident particle’s electromagnetic field interacting with the fields of the atoms in the medium and causing perturbing fields (Jackson, 1975). When the medium is spontaneously depolarized the so called Cherenkov emission is emitted. The atmospheric Cherenkov emission lasts only for a few nanoseconds and is seen when ultrahigh γ-rays enter the top of the atmosphere.

High energy γ-rays entering the atmosphere initiate a cascade of electrons and photons by means of pair production and bremsstrahlung. The electron-positron pairs created have velocities greater than the speed of light at the edge of the atmosphere and will therefore radiate optical Cherenkov radiation. The first Cherenkov signal of atmospheric origin was detected by Galbraith & Jelley (1953), who found its duration to be $0.2\mu$s.

It was realized at an early stage that the above phenomenon could make it possible to detect astrophysical point sources of γ-ray showers with high efficiency, since primary cosmic ray particles are rendered isotropic by interstellar magnetic fields. This implied the possibility of the detection of point sources of neutral quanta, like γ-ray photons or perhaps neutrons. The lateral spread of the Cherenkov shower as it strikes the ground is such that it can easily be detected by even a simple light receiver with modest dimensions, which would still have an effective collection area of thousands to tens of thousands of square meters. The development of the Cherenkov air shower will be discussed in more detail in a later chapter. As will also be discussed later, the light pulse preserves much of the information as to the original direction of the primary particle while the intensity of the Cherenkov light is proportional to the total number of secondary particles, and hence the energy of the primary particle, all of which makes such a detection technique very powerful.

However, a similar signal can be produced by primary protons of cosmic ray origin, and al-
though the first generation of telescopes could easily detect $\gamma$-rays they did not have means of discriminating between $\gamma$-ray and proton induced showers. It was only when the Crab nebula was detected that it was possible to unambiguously detect $\gamma$-rays from a cosmic source (Weekes et al., 1989). This was done by making use of sophisticated techniques to discriminate between proton and $\gamma$-ray initiated showers, by exploiting the difference between the light image of the proton initiated showers compared to $\gamma$-ray showers.

The physics department of the then Potchefstroom University for CHE (now North-West University) became involved in the studies of $\gamma$-rays with the MK I $\gamma$-ray telescope which became operational in the 1980’s and later the MK II became operational in 1989 (Raubenheimer, 1995).

1.1.1 The development of the Atmospheric Cherenkov Imaging Technique (ACIT)

As mentioned above, the first generation of telescopes did not provide the means to discriminate between light pulses from $\gamma$-rays or the more numerous cosmic ray induced showers, which implied that their flux sensitivity was very limited. When it became apparent that the signal strength of possible $\gamma$-ray sources had been over-estimated, work began on methods to improve the flux sensitivity (Weekes, 2007).

To accomplish this, several differences between Cherenkov light from $\gamma$-rays and hadron initiated showers were exploited. The most important difference is that for $\gamma$-ray initiated showers the image of Cherenkov light superimposed on the night sky background has a more uniform distribution as well as being smaller than that of cosmic ray initiated showers. It was this fact that lead to the first credible detection of a TeV $\gamma$-ray source (Weekes, 2007).

When using only a single light detector as a $\gamma$-ray detector the arrival direction of the $\gamma$-ray shower can not be determined, and the angular resolution is no better than that of the field of view of the telescope. Boley (1964) used an array of photomultipliers tubes at the focal surface of a parabolic mirror to study the longitudinal development of large air showers. However it was only through the use of an image intensifier system that the first image of Cherenkov light from air showers was recorded. The advantage of this system, according to Jelley & Porter (1963), was that it had the means of separating the $\gamma$-ray signal originating at astrophysical point sources from that of cosmic ray air showers as well as achieving a higher angular resolution. The photographs obtained could make it possible to determine the true direction of the shower as well as the point of intersection with the ground in relation to the position of the equipment.

That being said the technology was very limited, implying that the technique was limited to $\gamma$-rays with energies $>100$ TeV, and did not result in a practical $\gamma$-ray telescope. In the 1970s, however, Grindlay et al. (1975) used a simple stereoscopic system, consisting of multiple light detectors separated by distances of about 100 meters, to reduce the hadron background and to pinpoint the shower arrival direction. This method, although showing promising results, was not developed further due to limited resources at the time and it being difficult to implement.
The foundations of the modern Atmospheric Cherenkov Technique would be laid through the use of electronic cameras consisting of matrices of phototubes. These would be situated in the focal planes of large reflectors, to record images of Cherenkov light from air showers, so as to reduce the energy threshold to below 100 GeV. Monte Carlo simulations suggested that below this threshold the ratio of Cherenkov light from $\gamma$-ray showers to cosmic ray showers would increase drastically (Weekes, 2007).

### 1.1.2 Atmospheric Cherenkov Imaging Telescopes (ACIT)

By 1996 the ACIT was seen as being very successful and several groups planned a third generation of ACIT’s. A single telescope has a severe limitation (Weekes, 2007), in that at low trigger thresholds it was impossible to distinguish low energy $\gamma$-ray events from the much more numerous background of partial muon rings. It became clear that the next generation of ACIT would involve arrays of reflectors with apertures in excess of 10m, with better optics, more sophisticated cameras, and with data acquisition systems capable of handling high data rates. Such systems, however, would require a financial investment that was almost an order of magnitude greater than that for the previous generation of detectors.

To realize the full potential of the imaging atmospheric Cherenkov telescopes there has to be detection of air showers with two or more imaging telescopes. Stereoscopic systems give several advantages, discussed in great detail by Aharonian & Akerlof (1997). These advantages are as follows:

**Energy threshold:**

The energy threshold is determined by the mirror area and the quantum efficiency. The mirror area could be increased either by increasing the size of a single ACT or by using similar ACT’s in a distributed array and adding up the elements. The first option turns out to be very expensive. The cost of the telescope would increase faster than the cube of the diameter of the telescope while the threshold would decrease linearly. However, the distribution of telescopes is not of critical importance, meaning that an array of small telescopes in close proximity to each other and operating in coincidence is the same as if their signals are added and is approximately the same as that of a single large telescope of the same total mirror area.

**Angular resolution:**

Stereoscopic observations of air showers allows for the determination of the arrival direction of $\gamma$-ray primaries with an accuracy of $\leq 0.1^\circ$. Stereoscopic imaging therefore improves the angular resolution. This was established with the use of just two telescopes with a separation of about 100 m, which meant that the telescopes were within the pool of the Cherenkov light, which is a circle with diameter of about 200 m. The angular resolution can be improved by greater separation of the telescopes, but beyond a separation of 100 m between two telescopes the effective $\gamma$-ray collection area starts to decrease.
Background discrimination:

The stereoscopic approach is powerful for discriminating against hadronic showers, through the exploitation of the differences between the electromagnetic and hadronic cascades, which implies an improvement in the signal-to-noise ratio. This makes it possible to detect VHE $\gamma$-rays from point sources at the 0.01 Crab flux level. However hadronic events that develop like an electromagnetic cascade cannot be identified with this method. The array approach does however make it possible to remove single local muons while the improved angular resolution can also narrow the acceptable arrival directions.

Energy resolution:

A single ACT will yield no precise information about the impact parameter of the shower axis at ground level. The lack of this information is the limiting factor for the determination of the energy of the $\gamma$-ray, since the intensity of the Cherenkov light is a function of the distance from the shower axis. However using an ACT array will give the core location of the shower on an event-by-event basis to an accuracy of about 10 m. This means that the energy resolution can be reduced to 10% compared to that using a single imaging ACT (Weekes, 2007).

To build the third generation of ACT’s, large collaborations had to be formed. These collaborations were MAGIC (Baixeras, 2003), VERITAS (Weekes et al., 2002), CANGAROO (Enomoto et al., 2002), and H.E.S.S. (Hinton, 2004)

1.2 VHE $\gamma$-ray emission from the Blazar PKS 2155-304

A simple interpretation of a blazar is that it is an object that possesses beamed emission from a relativistic jet which is aligned roughly toward the line of sight to the observer. Beamed emission from this jet dominates the radio through infrared spectrum. A BL Lac object is a class of blazar with greater flux variability.

The spectral energy distribution of PKS 2155-304 is shown in Figure 1.1 and shows the presence of two peaks. The mechanism for TeV radiation from blazars is thought to be inverse Compton scattering from a relativistic jet (Schönfelder, 2001). The lower energy peaks in the spectra are found at frequencies corresponding to hard X-rays and are interpreted as resulting from synchrotron emission. The second peak in the spectra is said to result from inverse Compton emission and is interpreted as Doppler shifted to higher energies, so that instead of being in the MeV range it is found somewhere in the GeV to the TeV range. The peak of the emission also shift during a major flare. Assuming a leptonic model (Schönfelder, 2001), it can be said that the shift in the peaks is caused by an acceleration to larger maximum energies of the leptons responsible for the emission.

Of particular interest is the BL Lac object, PKS 2155-304, which is one of the UV-brightest BL Lacs and one of the X-ray-brightest objects. It is also one of the few BL Lac objects observed at
TeV energies (Chadwick et al., 1999). Its broadband spectrum shows a peak due to synchrotron emission peaking in the UV and soft X-rays like most X-ray selected BL Lacs do. The other peak is around the $\gamma$-ray region, and is attributed to inverse Compton scattering by the same high-energy electrons that are radiating synchrotron photons (Zhang et al., 1999). It has a very hard $\gamma$-ray spectrum in the 0.1-10 GeV region, with a power-law spectral index of $\Gamma \approx 0.71$ (Vestrand et al., 1995) and time-averaged integral flux of $4.2 \times 10^{-11}$ erg cm$^{-2}$ s$^{-1}$ above 300 GeV (Chadwick et al., 1999).

According to Ulrich et al. (1997), studies of variability provide an important understanding of the physical processes in the central region of an AGN. Crucial information is contained in the time scales, spectral changes, and correlations, as well as delays between variations in different continuum or line components. These variables provide information on the nature
and location of the AGN components and their interdependencies. The largest flux variability is observed from the subclass of AGN called blazars, and therefore variability studies of blazars are crucial to understanding AGN (Aharonian, 2007). PKS 2155-304 has a history of rapid strong broadband variability, and is a typical compact flat-spectrum radio source, being one of the rare X-ray emitting BL Lacs established to be a VHE $\gamma$-ray emitter. The VHE $\gamma$-ray emission of PKS 2155-304 is reportedly variable on a timescale of days (Chadwick, 1999).

On the nights of 27/28 July 2006 and again on 29/30 July 2006 the High Energy Spectroscopic System (H.E.S.S.) detected two outbursts from PKS2155-304 at the level of about 30 Crab (Figure 1.2). This activity then decreased, reaching the steady state level of about 0.1 Crab within two weeks. The Crab nebula has an integral $\gamma$-ray flux of $(2.26 \pm 0.08_{stat} \pm 0.54_{sys}) \times 10^{-11}\,\text{cm}^{-2}\text{s}^{-1}$ above 1 TeV (Aharonian, 2006).

Figure 1.2 shows an example of a binned H.E.S.S. data set. The data consists of three runs of 28 minutes each and 3 minutes of dead time between each run. The binning was done in one minute intervals.

![Figure 1.3](image)

Figure 1.3: The Fourier power spectrum as calculated by (Aharonian, 2007). The gray area represents a 90% confidence interval for the power-law spectrum. The horizontal line represents the average noise level.

As mentioned above, studies of variability provide an important understanding of the physical processes taking place in an AGN. For this reason, Aharonian (2007) did a Fourier analysis of the July 2006 flaring event of PKS 2155-304, the results of which are shown in Figure 1.3. Here it was found that the Fourier power could be fitted with a power-law of $P_\nu \propto \nu^{-2}$, and that the burst shows variability up to $\sim 600$ s or $1.667 \times 10^{-3}$ Hz(Aharonian, 2007).

**A reanalysis of the PKS 2155-304 flare**

The Fourier power calculated by Aharonian (2007) was done using a binning size of one minute. However the signal to noise ratio (SNR) will increase with increasing binning size. This could possibly explain why the average noise level in Figure 1.3 deviates from the white noise level of one. Furthermore, the telescope switches off for 3 minutes after every 28 minute
run. These gaps can be seen clearly in Figure 1.2. The periodically occurring dead time results effectively in the presence of a square wave. This square wave will definitely affect the Fourier power spectrum. No mention of this is made in Aharonian (2007).

The periodic dead time implies a frequency of $\approx 6 \times 10^{-4}$ Hz to be present in the signal, which is in the very low frequency range where most of the Fourier power is located in Figure 1.3. This immediately raises the question whether the power-law behavior seen in Figure 1.3 is due to the periodic occurrence of the three minute dead time intervals. The low frequency range is also where most of the power of a square wave is found. Obviously, this is a problem as it may complicate the detection of periodic signals with periodicity in the low frequency region of the frequency domain. The dead time then results in the presence of two signals, that of the source (PKS 2155-304), and a square wave resulting from the dead time of the telescope.

In this thesis a study will be done of the effects of dead time as well as the effects of binning size on both the power-law as well as the variability time on the Fourier power. Furthermore, a study will also be done as to finding a way to compensate the unwanted square wave signal.

### 1.3 Vela X-1

Vela X-1 is a wind-driven X-Ray binary system at a distance of 1.9 kpc from Earth (Sadakane et al., 1985) with an orbital period of 8.96 days and a pulse period of 283 seconds. The system consists of a neutron star with a magnetic field of at least $10^{12}$ G (Raubenheimer, 1990) and a mass $1.86 M_\odot$ (Barziv et al., 2001) as well as a $23 M_\odot$ (Kreykenbohm et al., 1999a) B0.5Ib supergiant HD 77581, separated by a distance $0.25$ AU.

According to Boynton et al. (1984), variations in the intrinsic pulse frequency of neutron star X-ray sources are believed to reflect changes in the rotation rate of the stellar crust produced by torques originating inside and outside the crust. The external torque is influenced by the flow pattern of the accreting plasma, whereas the internal torque is dependent on the state of the interior and its coupling to the crust. This implies that the study of the intrinsic frequency variations could provide information about the accretion flow as well as the state of the star itself. It was shown in Boynton et al. (1984) that the only adequate description for the intrinsic pulse frequency fluctuations was that of a white noise model (see Figure 1.4).

The closeness of the companion star in Vela X-1 results in the compact object being immersed in the strong stellar wind of HD 77581. The mass loss rate of HD 77581 is approximately $10^{-6} M_\odot \text{yr}^{-1}$ (van der Klis & Bonnet-Bidaud, 1984) with a terminal wind velocity of 1100 kms$^{-1}$ (Watanabe et al., 2006). The X-ray emission from binary sources are thought to be the result of matter being accreted by the companion star onto the neutron star, which will gain kinetic energy and will convert the gained energy into thermal energy when the matter hits the surface of the neutron star.

If it is assumed that the emitted X-ray emission is a result of black-body radiation from the
surface of the neutron star, then the typical temperature of the radiating matter would have to be around $10^7$ K. It was found, however, that the observed X-ray spectrum from Vela X-1 could not be explained by the spectra from thermal bremsstrahlung or black-body radiation. Rather the spectra could be fitted by a power law modified by a high energy cutoff (Kreykenbohm et al., 1999b).

Figure 1.4: The long term history of angular frequency of Vela X-1 according to BATSE. Vertical bars represent 1σ confidence intervals and are only shown where they are larger than the symbols (BATSE, 2012).

Protheroe et al. (1984) first claimed detection of pulsed TeV γ-ray emission. Later, North et al. (1987) found marginal evidence of pulsed TeV γ-ray emission at a period lower than that of the X-ray data, and determined Vela X-1 as a source of γ-rays at a confidence level of 97%. The γ-ray properties of Vela X-1 were further analysed by Raubenheimer et al. (1989), and Raubenheimer et al. (1994). However, the earlier γ-ray telescopes had several problems, in particular the ability to distinguish between source and background. The newer telescopes have several improvements, and can distinguish between a source and background much better than earlier telescopes. No further evidence of pulsed emission from Vela X-1 has been reported. In this study an attempt will be made to give a final verdict on whether Vela X-1 is a source of TeV γ-rays, using data gathered by the H.E.S.S. telescope.

A brief overview of γ-ray emission mechanisms and sources will be given in the next chapter. This will then be followed by a short overview of the H.E.S.S. experiment. The necessary signal processing background will be given in Chapter 4. A signal will then be simulated
using Monte Carlo methods and the techniques from Chapter 4 will then be applied to it. Results regarding the dead time and the effects of binning will be drawn from the simulated signals before applying these techniques to actual data of PKS 2155-304 and Vela X-1.
Chapter 2

\(\gamma\)-ray production mechanisms and sources

2.1 Introduction

Although the main theme of this thesis is about the analysis of time series of \(\gamma\)-ray measurements, the underlying production mechanisms and sources of \(\gamma\)-rays should be kept in mind. The purpose of this chapter is to give a brief summary of the physical mechanisms and sources through which \(\gamma\)-rays are produced.

In contrast to for example thermal emission from stars and the dense interstellar medium, black body radiation and thermal radio emission, \(\gamma\)-ray production is exclusively through non-thermal processes. In summary these processes are: inverse Compton scattering, \(\pi^0\)-decay, bremsstrahlung, and synchrotron radiation. Considering that we are dealing with photons of energy of \(10^9\) to \(10^{12}\) eV it should be clear that all of these processes involve highly relativistic particles. Thus, it can be argued that interpreting the results of the analysis of the time series done here, will eventually require some understanding of the production mechanisms, and sources for \(\gamma\)-rays as well as the acceleration of particles to very high energies. In this chapter the production mechanisms and sources of \(\gamma\)-rays will be briefly summarized.

2.2 Inverse Compton scattering

This occurs when low energy photons are scattered by relativistic electrons (Thomson scattering). The high energy electrons scatter low energy photons so that the electrons loose energy while the photons gain energy. This describes one of the most important \(\gamma\)-ray production processes.

The angle averaged total cross section for the Klein-Nishina regime (where the energy of the photon is much more than the rest mass energy of the electron) for the mildly relativistic case is given by Coppi & Blandford (1990)

\[
\sigma_{IC} = \frac{3\sigma_T}{8E_eE_p} \left\{ 1 - \frac{2}{E_eE_p} - \frac{2}{(E_eE_p)^2} \right\} \ln(1 + 2E_eE_p) + \frac{1}{2} + \frac{4}{E_eE_p} - \frac{1}{2(1 + 2E_eE_p)^2} \right\} \tag{2.1}
\]

where \(\sigma_T = \frac{8\pi r_e^2}{3}\) is the Thomson cross-section, \(r_e\) is the classical electron radius, while \(E_e\) and \(E_p\) are the electron and photon energy respectively. Equation 2.1 is accurate within 10% for all energies. For the nonrelativistic regime the expression approaches the Thomson cross section \(\sigma_{IC} \approx \sigma_T(1 - \frac{2E_eE_p}{b})\), while for the ultrarelativistic case \(\sigma_{IC} = \frac{3\sigma_T}{8E_eE_p} \ln(4E_eE_p)\).

For isotropically-distributed relativistic electrons with energy \(E\) being penetrated by a monoenergetic beam of low energy photons with frequency \(\omega_0\), the spectrum of radiation scattered at an angle \(\theta\) relative to the direction of the initial photon beam, written in terms of the Thomson cross-section is (Aharonian & Atoyan, 1981)

\[
\frac{d^2N(\theta, \omega)}{d\omega d\Omega} = \frac{3\sigma_T}{16\pi\omega_0 E^2} \left[ 1 + \frac{z^2}{2(1 - z)} - \frac{2z}{b(1 - z)} + \frac{2z^2}{b^2(1 - z)^2} \right] \tag{2.2}
\]

where \(b_\theta = 2(1 - \cos\theta)\omega_0 E\), \(z = \omega/E\), \(b = 4E_e\omega_0\), \(E_e\) is the electron energy, \(d\Omega\) is the solid angle and \(\omega\) varies within the limits of

\[
w_0 << \omega << \frac{b_\theta}{1 + b_\theta} E. \tag{2.3}
\]

The integration of Equation 2.2 over \(\theta\) for the case of isotropically distributed electrons and photons yields (Blumenthal & Gould, 1970)

\[
\frac{dN(\omega_\gamma)}{d\omega_\gamma} = \frac{3\sigma_T}{4\omega_0 E^2} \left[ 1 + \frac{z^2}{2(1 - z)} + \frac{z}{b(1 - z)} - \frac{2z^2}{b^2(1 - z)^2} + \frac{z^3}{2b(1 - z)^2} - \frac{2z}{b(1 - z) \ln b(z - 1)} \right]. \tag{2.4}
\]

The domain of the scattering can be determined by setting the parameter \(\beta = \omega_0 E_e\). The case \(\beta \ll 1\) corresponds to the Thompson limit. In this regime only a fraction of the primary electron energy is released to the upscattered photon. When \(\beta \gg 1\) the scattering is in the deep Klein-Nishina domain, where a substantial amount of the electron energy can be transferred to an upscattered photon in just one interaction.

For a power law distribution of electrons, where \(N_e = K_e E_e^{-p}\), the resulting \(\gamma\)-ray spectrum in the non-relativistic regime has a power-law form with a photon index of \(-(p + 1)/2\), first derived by Ginzburg & Syrovatskii (1964). The spectrum in the ultrarelativistic regime is found to steepen so that it has a photon index of \(-(p + 1)\) (Blumenthal & Gould, 1970).
For relativistic electrons in a monoenergetic field of photons with energy $w_0$ and number density $n_{ph}$ the energy loss rate is given by (Aharonian & Atoyan, 1981)

$$-\frac{dE_e}{dt} = \frac{2\pi r_0^2 m^2 c^5}{w_0 b} \left[ \left( \frac{6 + b}{2} + \frac{6}{b} \right) \ln(1 + b) - \ln^2(1 + b) - 2Li \left( \frac{1}{1 + b} \right) - \frac{(11/12)b^3 + 8b^2 + 13b + 6}{(1 + b)^2} \right]$$

(2.5)

where

$$Li(x) = \int_x^1 \frac{\ln(y)dy}{y - 1}.$$

In the Thomson and the Klein-Nishina domains, Equation 2.5 reduces to

$$-\frac{dE_e}{dt} = \frac{4}{3} \sigma_T c w_0 n_{ph} E_e^2$$

(2.6)

for $\beta \ll 1$ and

$$-\frac{dE_e}{dt} = \frac{3}{8} \sigma_T c n_{ph} \frac{E_e^2}{w_0} \left( \ln b - 11/6 \right)$$

(2.7)

for $\beta \gg 1$, as shown by Blumenthal & Gould (1970). The energy loss-rate in the Thomson limit has an $E_e^2$ dependence, while almost being energy independent in the Klein-Nishina limit. This implies that for the Thomson limit the steady-state electron spectrum becomes steeper. In the Klein-Nishina limit the electron spectrum becomes flatter compared to the Thomson limit, but in the extreme Klein-Nishina limit the energy loss does not have the same meaning as in the Thomson limit, in that the electron looses its energy in discreet amounts. A possible way in which the electron would loose its energy in the extreme Klein-Nishina limit is shown in Blumenthal & Gould (1970).

### 2.3 $\pi^0$ decay

Neutral $\pi$ mesons decay into $\gamma$-rays according to the following reaction

$$\pi^0 \rightarrow 2\gamma$$

(2.8)

where the $\pi^0$ are created through p-p interactions. The $\gamma$-ray production source function for the $\gamma$-ray spectrum, according to Mori (1997), and first calculated by Stecker (1970), is

$$q(E_\gamma) = 4\pi n_H \int_{T_{\pi,\min}}^{\infty} dT_p j_p(T_p) \langle \xi(T_p) \rangle \times \int_{E_\gamma + m_\pi^2}^{\infty} dE_\pi \frac{2dN(T_p, T_\pi)}{\sqrt{E_\pi^2 - m_\pi^2}}$$

(2.9)

where $j_p$ is the cosmic ray proton flux, $d\sigma(T_p, T_\pi)/dT_\pi$ is the differential cross section for the production of a $\pi^0$ particle with kinetic energy $T_\pi$ in the galactic rest frame as the result of
collision of a cosmic ray proton of kinetic energy \( T_p \) with a H atom at rest. \( E_{\pi} \) is the total pion energy and \( m_{\pi} \) is its mass and \( n_H \) is the atomic hydrogen density. The quantity \( T_{\pi}^{\text{min}}(T_p) \) is the minimum proton kinetic energy that contributes to the production of a pion with energy \( T_{\pi} \). Here \( \xi \) is the \( \pi^0 \) multiplicity. \( \langle \xi \sigma(T_p) \rangle \) is the inclusive cross section for the reaction \( p + p \to \pi^0 + \text{anything} \), and \( \xi \) is the \( \pi^0 \) multiplicity.

For \( \pi^0 \) to be produced, the kinetic energy of the proton has to be greater than \( T_p \approx 280 \) MeV (Aharonian, 2004). The \( \pi^0 \) will immediately decay according to Equation 2.8, where the mean lifetime for \( \pi^0 \) decay is \( 8.4 \times 10^{-17} \) s, which is much shorter than the lifetime of charged \( \pi \)-mesons. At high energies all three types of pions are produced, but the spectral form of \( \pi \)-mesons is generally determined by a few leading particles rather than a large number of low energy secondaries.

The \( \gamma \)-ray spectrum of neutral pion decay has a distinct peak at about 70 MeV and then drops rapidly toward higher energies. The peak is a result of the kinematics of the \( \pi^0 \) decay. The reaction kinematics are shown in Stecker (1966).

The precise calculation of the spectrum require extensive integration over differential cross sections, as can be seen from Equation 2.9, which is experimentally obtained from particle accelerators. However the emissivity of \( \gamma \)-rays from an arbitrary broad energy distribution of protons can be derived analytically and with good accuracy for a broad \( \gamma \)-ray energy range, according to Aharonian (2004)

\[
\epsilon_\gamma(E_\gamma) = 2 \int_{E_\gamma + m_{\pi}^2/4E_\gamma}^{\infty} \frac{\epsilon_\pi(E_\pi)}{\sqrt{E_\pi^2 - m_{\pi}^2}} dE_\pi
\] (2.10)

It is shown by Aharonian & Atoyan (2000) that this approach for the calculation of the emissivity of secondary particles from inelastic proton-proton interactions achieves good accuracy when compared to Monte Carlo calculations by Mori (1997). The cosmic ray proton spectra, with a spectral index \( \Gamma \geq 2.4 \) reaches its maximum emissivity at \( E \approx 1 \) GeV, after which for lower energies the spectrum sharply declines.

The cooling time is almost independent of energy in the region above 1 GeV, where the nuclear losses will dominate over the ionization losses, and the initial acceleration spectrum of protons will remain unchanged. The characteristic cooling time of relativistic protons due to p-p interactions in the hydrogen medium with number density \( n_0 \) is almost independent of energy. Assuming the average cross section, \( \sigma_{pp} \), at very high energies is about 40 mb (Aharonian, 2004), and taking into account that on average the proton losses is about half of its energy during every interaction. With a coefficient of inelasticity, \( f \approx 0.5 \), the cooling time is given by

\[
t_{pp} = (n_0\sigma_{pp}fc)^{-1} \approx 5.3 \times 10^7 (n/1cm^{-3})^{-1} \text{yr}.
\]

The \( \gamma \)-ray spectrum repeats the spectrum of the protons from which they originate. This im-
plies that high energy γ-rays carry direct information about the acceleration spectrum of protons.

2.4 Bremsstrahlung

Bremsstrahlung is the radiation associated with the acceleration of charged particles like electrons in the electrostatic fields of ions and the atomic nuclei. The exact bremsstrahlung cross section can only be derived through the use of quantum electro-dynamics and can be found in Jauch & Rohrlich (1955) as well as Heitler (1954). A classical approach can be found in Longair (1994), where Blumenthal & Gould (1970) use both a quantum mechanical as well as a semi-classical approach to solve the problem. The high energy electron bremsstrahlung cross section on an unshielded static charge, Z\(e\), is given by Blumenthal & Gould (1970) as

\[
d\sigma = 4Z^2\alpha r_0^2 \frac{dw}{w} \frac{1}{E_{ei}} \left( \ln \frac{2E_{ei}E_{ef}}{mc^2\hbar w} - \frac{1}{2} \right) \left( E_{ei}^2 + E_{ef}^2 - 2/3E_{ei}E_{ef} \right)
\] (2.11)

Equation 2.11 is based on the Born approximation, which is valid for high energies, where the effects of the Coulomb field of the scatter on the incoming and outgoing electron are negligible. This equation gives the cross section for a photon of energy within \(\hbar dw\) in the scattering of an electron of initial energy \(E_{ei}\) and final energy \(E_{ef} = E_{ei} - \hbar dw\).

Another method for obtaining Equation 2.11 is through the use of the Weizsäcker – Williams method. This method treats bremsstrahlung as Compton scattering by the incoming electron off the virtual photons of the Coulomb field of the scattering center. This method is quite useful for cases of electron-electron collisions and pair production by charged particles, and can be found in Heitler (1954) as well as Blumenthal & Gould (1970).

The radiation length, which is the average distance over which an ultra relativistic particle loses a fraction \(1 - 1/e\) of its energy due to bremsstrahlung, is given as (Aharonian, 2004)

\[
X_0 = \frac{7}{9n\sigma_0}.
\] (2.12)

The parameter \(X_0\) is also the mean free path of γ-rays.

The average energy loss rate for electrons during bremsstrahlung is, according to Aharonian (2004),

\[
-\frac{dE_e}{dt} = \left( \frac{cm_{\mu}n}{X_0} \right) E_e,
\] (2.13)

from which the lifetime of electrons due bremsstrahlung is

\[
t_{br} = \frac{E_e}{-dE_e/dt} \approx 4 \times 10^7 (n/1cm^{-3})^{-1} \text{yr},
\] (2.14)
where \( n \) is the number density of the ambient gas.

It is interesting to note that the electron energy loss rate Equation 2.13 is proportional to the electron energy, and as a result the lifetime given by Equation 2.14 is energy independent. The implication of this, according to Aharonian (2004), is that for an initial power-law spectrum \( K_e E_e^{-\Gamma} \), the bremsstrahlung losses will not change the original electron spectrum.

If it is assumed that there is an infinite, uniform distribution of sources, each of which inject high energy electrons with a power-law spectrum of \( K_e E_e^{-\Gamma} \), Longair (1994) gives the steady state spectrum \( N(E_e) \) as

\[
N(E_e) = -\left( \frac{dE_e}{dt} \right)^{-1} \int K_e E_e^{-\Gamma} dE_e,
\]

(2.15)

where it was assumed that \( N(E_e) \to 0 \) as \( E_e \to \infty \). The above equation shows that for the case of a power-law spectrum of electrons the \( \gamma \)-ray spectrum would be the same as the electron spectrum, only when the energy loss is dominated by bremsstrahlung losses. This is a result of the \( 1/E^\gamma \) dependence of the differential cross section.

Equations 2.14 and 2.13 are derived for neutral gas i.e. in the presence of screening. The case of a fully ionized gas will change these equations somewhat as shown in Blumenthal & Gould (1970).

### 2.5 Synchrotron radiation

The treatment of synchrotron radiation can be found in Blumenthal & Gould (1970), Ginzburg & Syrovatskii (1964), and many others. Synchrotron radiation of relativistic and ultrarelativistic electrons is the process that most significantly contributes to high energy astrophysics, and is the radiation that is emitted by electrons that gyrate in a magnetic field. Synchrotron radiation is sometimes also referred to as magnetic bremsstrahlung since it is the magnetic field that distorts the trajectory of the charged particle. The energy loss rate can be found using classical electrodynamics (Griffiths, 1998) and is found to be

\[
-\frac{dE}{dt} = \frac{q^2 \bar{r}^2}{6\pi\varepsilon_0 c^3}
\]

(2.16)

where \( q \) is the charge of the particle and \( \bar{r} \) is the acceleration of the charge. The loss rate of a relativistic electron moving through a magnetic field \( H \) at an angle \( \theta \) with a squared velocity \( v^2 \) is given as

\[
-\frac{dE_e}{dt} = \frac{e^4 H^2 v^2 \gamma^2}{6\pi\varepsilon_0 c^3 m_e^2} \sin^2 \theta
\]

(2.17)
where \( m_e \) is the electron mass. In a magnetic field \( H \) an electron moves in a helix with an angular frequency

\[
w_H = \frac{eHmc^2}{mcE_e}.
\]

(2.18)

Consider the case where an electron’s velocity has a constant angle \( \theta \) with the direction of the magnetic field. The radiation is confined within a cone of angle \( \theta \) along the direction of motion relative to the magnetic field. An observer located on the surface of this cone at a large distance from the emitting particle would record successive pulses of radiation spaced at an interval of \( \tau = \frac{2\pi}{w_H} \), with a duration of

\[
\Delta t = \frac{mc}{eH_\perp} \left( \frac{mc^2}{E_e} \right)^2.
\]

(2.19)

where \( H_\perp = H \sin \theta \) is the magnetic field perpendicular to the velocity of the electron. If a decomposition of the pulse is done using Fourier analysis, as described in Longair (1994), it is found that the electron radiates at harmonics of the gyrofrequency \( w_{H_l} \) where \( l \) has integer values \( l = 1, 2, 3, \ldots \). For higher harmonics the radiation can be regarded as a continuum spectrum since \( \Delta t \approx \frac{1}{w_H} \).

The critical frequency of a relativistic electron is defined as

\[
w_c = \frac{3c}{2} \gamma^3 w_H \sin \theta.
\]

(2.20)

The spectral distribution of synchrotron radiation for relativistic electrons has broad maximum that is roughly centered at \( w_c \), and is given as (Aharonian, 2004) or for example in Rybicki & Lightman (1986) and Jackson (1975)

\[
P = \sqrt{2} \frac{e^3H}{\hbar mc^2} F(x)
\]

(2.21)

where \( x = \frac{w}{w_c} \) and \( F(x) = \int_x^{\infty} K_{5/3}(x)dx \) is the modified Bessel function of order \( 5/3 \).

The above results can be obtained through the use of classical electrodynamics. However, to find the limits to where classical electrodynamics is relevant it is useful to introduce the parameter

\[
\chi = \frac{E_e}{m_e c^2} \frac{H_\perp}{H_{cr}}
\]

(2.22)

where \( H_{cr} \) is the critical magnetic field, with \( H_{cr} = 4.41 \times 10^{13} \) G. Classical electrodynamics is useful for \( \chi \ll 1 \). In strong magnetic fields where \( \chi \gg 1 \) electromagnetic showers can occur and in super strong fields when dealing with high energy electrons (photon energy \( \geq 2mc^2 \)) the
development of electron-positron pairs becomes possible. The resulting quantized synchrotron radiation by pairs will convert to second generation of pairs, allowing an electromagnetic cascade to develop. For pair production to develop the synchrotron radiation has to approach the quantum threshold of $E H \approx 10^7$ TeV Gauss. For non-zero probabilities for synchrotron radiation and pair production, require strong magnetic fields and high energies and depends on the parameter $\chi$. Calculations for the probability (cross section) of synchrotron radiation and pair production as function of $\chi$ are discussed in Anguelov & Vankov (1999). The cross section mentioned here does not have the same meaning as the cross sections shown in previous subsections. In processes like bremsstrahlung or inverse Compton scattering, the interacting particles have comparative sizes. However, for synchrotron radiation it is a charged particle like an electron interacting with a magnetic field that is setup by some sort of interstellar body. Figure 2.1, taken from Anguelov & Vankov (1999), shows the total cross section as function of $\chi$.

Figure 2.1: Total cross section of synchrotron radiation and magnetic pair production (Anguelov & Vankov, 1999).

It can be seen from Figure 2.1 that, for values of $\chi \ll 1$, the cross sections for synchrotron radiation remain almost constant, while for magnetic pair production the cross section drops off dramatically, proportional to $\exp(-8/3\chi)$. It can also be seen from Figure 2.1 that the cross section for magnetic pair production achieves a maximum at a value of $\chi \approx 10$, after which, according to Erber (1966) the cross section decreases as $\chi^{-1/3}$. The cross section for synchrotron radiation shows similar behavior for large values of $\chi$, but its absolute value exceeds that of the value for pair production by a factor of 3 (Aharonian, 2004).

It was shown by Ginzburg & Syrovatskii (1964) that the synchrotron losses for nuclei or protons, for the ultra relativistic case, with atomic number $Z$ and mass $M$ are of the order $(Zm/M)^4$ less than those for electrons with the same energy, in the same perpendicular magnetic field. The characteristic time by which the energy is halved for nuclei or protons is given as
where $A$ is the mass number. Ginzburg & Syrovatskii (1964) show that for protons in a field of $H_\perp \approx 10^{-6}$ Oe that the corresponding $t_{sy} \approx 10^{12}$ years. This shows that the synchrotron radiation of protons and nuclei occurs in a time of order $t \approx 10^{10}$ years, which is the “cosmological time”. However, under certain conditions the synchrotron cooling time of protons is comparable or even shorter than other timescales that characterize the acceleration and confinement regions of ultrarelativistic protons.

In compact objects like pulsars, the highest energy protons are accelerated and lose energy via synchrotron or curvature losses, resulting in very hard spectra. The upper limit for protons seems to be below $10^{20}$ eV (Derishev et al., 2003). The important equations for the synchrotron radiation of protons, like the critical frequency are similar to those for electron synchrotron radiation.

### 2.6 Pulsars, Pulsar Winds and Plerions

A brief review of pulsars, pulsar wind nebulae, magnetosphere radiation, etc, is given below, as one of the aims of this work is to determine whether Vela X-1 is a source of $\gamma$-rays.

Pulsars are rapidly rotating, magnetized neutron stars and were the first astrophysical source of $\gamma$-rays discovered in high energy astrophysics, with the Vela pulsar as the brightest persistent source of GeV $\gamma$-rays (Aharonian, 2004).

The Crab and the Vela pulsars were some of the earliest discoveries of $\gamma$-ray emitters where the Crab shows pulsed as well as unpulsed emission. The unpulsed emission is interpreted as resulting from synchrotron radiation below a few GeV and inverse Compton radiation up to TeV energies (Schönfelder, 2001).

High energy $\gamma$-rays from rotation powered pulsars can be produced through several radiation mechanism in three physically distinct regions viz. the magnetosphere, the relativistic wind and the synchrotron nebula.

#### 2.6.1 $\gamma$-radiation from the Magnetosphere

The environment immediate to the pulsar is referred to as the magnetosphere. Pulsars may be treated as non-aligned rotating magnets, where the magnetic field strength at the surface is $\sim 10^{12}$ G (Sturrock, 1971). The result of this strong magnetic field is that the Lorentz force is far greater than that of gravity. Furthermore, the induced electric field at the surface is so powerful that it can exceed the work done on an electron in the surface by the surface material. Electrons therefore can be freed from the surface so there must be plasma surrounding the neutron star (Aharonian, 2004).
The magnetic field lines which extend beyond the light cylinder are open and thus a particle dragged off at the poles can escape beyond the light cylinder. The polar gap model does not predict any TeV radiation from the magnetosphere since it is expected that high energy $\gamma$-rays will be absorbed before they can escape the magnetosphere (Aharonian, 2004). However for millisecond pulsars with modest magnetic fields, the effect of $\gamma$-ray absorption is significantly reduced, thereby making it possible for high energy radiation to escape from the magnetosphere. According to Bulik et al. (2000), for these pulsars $\gamma$-ray emission is expected between 1 MeV and a few hundred GeV, where the main spectral component should be due to curvature radiation of primary particles. At higher energies, inverse Compton scattering could result in energies of up to 1 TeV.

The outer magnetosphere gap model (Cheng et al., 1986) predicts a TeV component of magnetospheric $\gamma$-radiation resulting from inverse Compton scattering from pulsars with large enough spin rates and strong magnetic fields, such as the Crab and Vela.

### 2.6.2 $\gamma$-rays from unshocked pulsar winds

Rotation powered pulsars eject plasma by means of relativistic winds that carry off most of the rotational energy. In the case of the Crab Nebula (the best known example) the pulsar ejects a relativistic wind which is terminated at a distance of 0.1 pc by a standing reverse shock. This shock accelerates electrons to energies of up to $10^{15}$ eV and randomizes their pitch angles. This results in the formation of an extended synchrotron source in the region downstream from the source (Aharonian, 2004).

Pulsar winds are characterized by the magnetization, or $\sigma$ parameter, which is the ratio of electromagnetic energy flux to kinetic energy flux of particles. For $\sigma \geq 1$ the wind is Poynting flux dominated, whereas for $\sigma \leq 1$ the wind is kinetic energy dominated (Aharonian, 2004).

The formation of a kinetic energy dominated wind is an unsolved problem in pulsar physics, since $\sigma \approx 10^3 - 10^4$ at the base of the plasma, meaning that the base of the plasma should be Poynting dominated. The electrons in the wind may have energies as large as $10^{13}$ eV. If, however, they have velocity vectors parallel to the magnetic field, these electrons will not radiate synchrotron radiation. However, inverse Compton radiation is unavoidable due the illumination of the wind by low-energy photons of a different origin (Aharonian, 2004).

### 2.6.3 Synchrotron nebula

As already mentioned, pulsars lose their rotational energy by driving ultrarelativistic particle winds. If a pulsar is surrounded by a supernova remnant, its wind is considered to terminate at a collisionsless shock front, where the pressure of the relativistic outflow balances the pressure within the nebula (Rees & Gunn, 1974).
A synchrotron nebula is formed when the ultrarelativistic kinetic energy dominated wind is confined in a slowly expanding shell of a supernova remnant. A major fraction of the rotational energy is thus released through nonthermal synchrotron radiation which can extend to the $\gamma$-ray energies.

- **Bremsstrahlung**

For a mean gas density in the Crab nebula of $n \approx 5 \text{ cm}^{-3}$, the bremsstrahlung flux of $\gamma$-rays cannot exceed 15% of the flux from inverse Compton $\gamma$-rays (Aharonian, 2004). For a uniform distribution of relativistic electrons throughout the nebula the effective gas density is defined by the mean density of the nebula, $n_{\text{eff}} \approx n$. If, however, electrons are at least partially trapped in the regions of high density, then $n_{\text{eff}} > n$. For the Crab Nebula, the bremsstrahlung flux could explain the measured GeV flux as well as the modified spectrum at very high energies (Atoyan & Aharonian, 1996).

- **$\pi^0$ decay $\gamma$-rays**

The interactions of the nucleonic component of accelerated particles with the ambient gas for the Crab nebula leads to the production of $\gamma$-rays through the decay of secondary $\pi^0$ particles. Given that the average gas density in the nebula is low, the contribution from this mechanism should only be significant for the case of partial confinement of relativistic particles in the filaments, so that $n_{\text{eff}} > n$. The detection of $\gamma$-rays up to 50 TeV from the Crab Nebula (Tanimori et al., 1998) could have significant implications concerning the content of the wind and propagation or interaction of accelerated particles in the filaments, since Bednarek & Protheroe (1997) among others, predicted that $\pi^0$ decay $\gamma$-rays may be seen at TeV energies and beyond.

### 2.7 Microquasars

According to Weekes (1992), there is no reason to expect X-ray binaries to be sources of high energy $\gamma$-rays, however these objects played an important role in early ground based $\gamma$-ray telescopes. The early data was however treated with much skepticism, more so after the failure of new generation ground based $\gamma$-ray telescopes to confirm the claims of detection of TeV $\gamma$-rays from X-ray binaries and cataclysmic variables.

The discovery of microquasars (X-ray binaries with bipolar jets) changed this view by establishing that non-thermal processes do play a non-negligible role in these accretion driven objects, as well as allowing for a better understanding of accretion disks and jets due to their proximity. It has been established that the non-thermal power of synchrotron jets during strong radio flares could be compared to the thermal X-ray luminosity of the central compact object (Aharonian, 2004). If the electrons are highly accelerated then the spectrum of the synchrotron
radiation of the jets could extend to X-rays or soft $\gamma$-rays (Markoff et al., 2001), only for the case of weak disk luminosity. The high density photon fields around the compact object, provided by the companion star, as well as being produced by the jet itself, could possibly create conditions necessary for the effective production of inverse Compton $\gamma$-rays inside the jet.

The electrons that are accelerated by a shock, created by the jet propagating through the supersonic wind driven by the companion star (Atoyan et al., 2002) would result in quasi stationary, high energy inverse Compton $\gamma$-rays, where the optical (UV) target photons are being supplied by the hot optical star (Paredes et al., 2000). Generally the shocks should accelerate protons as well. For $p - p$ interactions to be effective the gas densities should be high. This could be the case where the old atmospheric target or target cross beam scenarios can result in high energy $\gamma$-rays from hadronic origin (Aharonian, 2004).

It should also be mentioned that besides the generation of $\gamma$-rays in small scale jets of microquasars it could be expected that persistent $\gamma$-radiation might be extended from the synchrotron lobes formed by electrons accelerated at the interface between the relativistic jet and the interstellar medium (Aharonian & Atoyan, 1998). Another possibility is protons interacting with dense molecular clouds (Heinz & Sunyaev, 2002). The fact that the termination shock of relativistic jets could accelerate particles up to very high energies implies that extended emission from inverse Compton radiation, caused by the cosmic microwave background and the interstellar medium, as well as synchrotron radiation resulting from the interaction with the ambient interstellar magnetic field, would take place (Aharonian & Atoyan, 1998).

### 2.8 Large scale jets of radio galaxies and quasars

Radio galaxies and quasars belong to the class of the so-called active galactic nuclei. Radio galaxies are sources of vast fluxes of high energy particles and magnetic fields, as well as being similar to Seyfert galaxies which also have strong broad emission lines in their optical spectra. Quasars are among one of the most extreme examples of active galactic nuclei that are known. These objects are characterized by a stellar appearance and very great distances. Quasars can also be classified as radio-quiet or radio-loud (Aharonian, 2004).

The cluster of galaxies contain many Active Galactic Nuclei (AGN) with powerful jets which energise the the intercluster medium through the termination shocks accompanied by particle acceleration and magnetic field amplification. Particle acceleration also occurs inside the jets. The most energetic particles will escape from the jets and make a non-negligible contribution to the energy of the host galaxy cluster. Large scale AGN jets and cluster galaxies are believed to be among the most important contributors to cosmic ray acceleration (Aharonian, 2004).

The acceleration of the highest energy particles in AGN jets and clusters of galaxies is still a theoretical conviction, however the presence of non-thermal particles of lower energy in these objects is an established fact (Aharonian, 2004). Furthermore, it is probable that the formation
and radiation of large scale extragalactic jets is a process that is strongly dominated by non-thermal processes.
Chapter 3

The H.E.S.S. experiment

3.1 Introduction

This chapter will give an introduction to some of the most important principles related to High Energy Stereoscopic System (H.E.S.S.) telescope. The H.E.S.S. project consists of four 12 m stereoscopic imaging atmospheric Cherenkov telescopes that are capable of observing $\gamma$-rays above 100 GeV. H.E.S.S. is located close to Windhoek at an altitude of 1800 m above sea level in Namibia.

Unlike most telescopes, for example radio or optical telescopes, data gathered using the H.E.S.S. telescope cannot be regarded as discretely sampled. For example, consider a telescope that measures electromagnetic intensity every $T$ seconds. In this example $T$ seconds is the sampling time, i.e. after every $T$ seconds the measured electromagnetic intensity is registered.

This is very different for the H.E.S.S. telescope. For this case, the telescope can be regarded as a photon counter. This is because it merely records time of arrival of a $\gamma$-ray event. The time of arrival of $\gamma$-rays is a random process. This is very different from the example mentioned earlier.

In this thesis we are interested in the analysis of time series measured by the H.E.S.S. telescope. A proper understanding of the format data requires knowledge of the instruments used to collect the data. A brief description of the H.E.S.S. telescope is presented below.

3.2 Air showers

3.2.1 Electromagnetic air showers

The components of an air shower, of which there can only be two types, are dependent on the nature of particle incident onto the atmosphere. If a hadron is the incident particle, the shower will consist of hadronic and electromagnetic sub-showers, while in the case of an electron or photon the resulting shower will be of an electromagnetic kind. For an electromagnetic cascade
there are three processes that dominate the longitudinal development viz. bremsstrahlung, pair production and the ionization of air molecules.

When a high-energy photon, with energy $E_0$, enters the earth’s atmosphere, an electron-positron pair is produced in the Coulomb field of an atomic nucleus. These secondary particles will radiate high-energy photons due to bremsstrahlung, which in turn will further produce an electron-positron pair. The electrons and positrons get deflected by nuclei and then emit photons through bremsstrahlung. There are thus two processes that create a cascade of particles and result in an electromagnetic shower. The cascade dies out when the ionization threshold energy, for air showers, of $E_c \approx 80$ MeV is reached (Schmidt, 2005).

The propagation of particles through the atmosphere is theoretically described by coupled transport and cascade equations that depend on the properties of the particles and their interactions as well as the structure of the atmosphere. This equation in matrix notation is given by (Gaisser, 1990), as

$$\frac{dN_i(E_i, X)}{dX} = -\left( \frac{1}{\lambda_i} + \frac{1}{d_i} \right) N_i(E_i, X) + \sum_j \int \frac{F_{ij}(E_i, E_j)}{E_i} N_j(E_j) \frac{dE_j}{\lambda_j}$$

Here $N_i(E, X)$ is the amount of particles of type $i$ at depth $X$, where $X$ is the slant depth measured from the top of the atmosphere downward along the direction of the incident nucleon, $\lambda_i$ is the mean free path and $d_i$ is the decay length. The depth, mean free path and decay length are all measured in $g/cm^2$. The function $F_{ij}(E_i, E_j)$ is known as the inclusive cross section for an incident nucleon of type $i$ with energy $E_i$ to collide with an air nucleon to produce an outgoing nucleon of type $j$ with energy $E_j$.

In the case of bremsstrahlung, $\lambda_i$ is called the radiation length and is usually measured in $g/cm^2$. In air this amounts to $37.2$ $g/cm^2$ (Berge, 2002). This is the mean distance by which a high-energy electron looses all but $1/e$ of its energy by means of bremsstrahlung.

The lateral shower development is dominated by multiple scattering of shower particles in the air. Pair production and bremsstrahlung also contribute to the lateral distribution of secondary particles. An electron undergoing bremsstrahlung will radiate photons in a cone in a forward direction so that the average opening angle of the cone is equal to the multiplicative inverse of the Lorentz factor. This implies that in the case of high energy electrons the directional divergence from the shower axis that originates from bremsstrahlung is very small and can be neglected.

The number of particles in the shower will increase exponentially until the primary energy is divided among $N = E_0/E_c$ particles, with each particle having an energy of $\approx E_c$. Thus, the maximum number of particles in the shower is proportional to the primary energy. The shower maximum for TeV energies is reached at an atmospheric depth of $\sim 200$ $g/cm^2$ which corresponds to a height of 8 km for vertical showers (Schmidt, 2005). The shower development
is shown in Figure 3.1. The shower can be described as a cone with a radius of 80 m at sea level and contains 90% of the energy of the particles (Schlenker, 2001).

![Diagram of electromagnetic air shower](image)

**Figure 3.1:** The development of an electromagnetic air shower (Schmidt, 2005).

### 3.2.2 Hadronic air showers

Hadronic cosmic rays (mainly protons), dominate the cosmic ray flux that enters the earth’s atmosphere. These cosmic rays enter at a rate approximately a thousand times greater than that of the γ-rays at TeV energies. When a cosmic ray nucleus hits the atmosphere, the first interaction is dominated by inelastic scattering with air particles, which is quite different from the electromagnetic shower development. The collision between a nucleus and a high-energy cosmic ray particle results in the fragmentation of the target nucleus into several lighter nuclei \textit{i.e.}, hadrons and mainly pions. Figure 3.2 shows the hadronic shower development. The pions have large transverse momenta which results in larger lateral development as compared to the electromagnetic showers.

While the nuclei will undergo further inelastic scattering, the charged pions will decay into either muons and neutrinos or into γ-rays, inducing electromagnetic sub-cascades. Thus the hadronic shower has a considerably larger lateral and longitudinal spread than that of the electromagnetic shower. This is illustrated in Figure 3.3.

### 3.2.3 Cherenkov radiation

A charged particle exceeding the speed of light in air polarizes the surrounding medium and induces constructive interference of electromagnetic waves. This creates a cone of light emis-
The development of a hadronic air shower taken from Schlenker (2001).

Figure 3.2: The development of a hadronic air shower taken from Schlenker (2001).

The emission with a characteristic angle between the direction of the radiation and the trajectory of the particle

$$\cos \theta_c = \frac{1}{\beta n},$$

where $n$ is the refractive index in air. Here $\beta = v/c_n$, with $v$ the particle velocity and $c_n$ the speed of light in air while $\theta_c$ refers to the emission angle.

The threshold energy for the emission of Cherenkov light depends on the mass of the particle $m_0$ and the refractive index of the medium, and is given by (Schmidt, 2005)

$$E_{\text{min}} = \frac{m_0 c^2}{\sqrt{1 - n^{-2}}}.$$

As particles and photons travel at nearly the same speed during the shower development, the Cherenkov light is concentrated in a flash that has a duration of a few nanoseconds, while the shower itself develops over several microseconds. It is the short duration of the flash that makes it detectable, since a longer flash would be dominated by background light from the night sky.

The Cherenkov emission from a single shower electron at a height $h$ results in a ring of light with radius $R_{\text{ring}}$ that depends on the observation altitude of $h_{\text{obs}}$. The average radius $R_{\text{ring}}$ is given by (Schmidt, 2005)
The resulting, almost symmetric ring has a radius of about 100 - 120 m.

During hadronic air showers, muons are produced in the decay of pions. These also emit Cherenkov radiation, but this radiation is concentrated in a much narrower cone than that of the electrons, usually reaching only a single telescope and is generally mapped into a muon ring as shown in Figure 3.4 (Schmidt, 2005).

### 3.3 The H.E.S.S. experiment

#### 3.3.1 The Atmospheric Cherenkov Imaging Technique (ACIT)

This technique uses an array of photomultipliers (PMTs) in the focal plane of a large optical reflector to record the Cherenkov image of an air shower. The use of ACIT makes it possible to detect point sources of cosmic ray air showers. Charged primary particles are rendered isotropic by the intervening interstellar magnetic fields, which implies that neutral quanta such as γ-ray photons and neutrons will not lose their directional properties.
3.3.2 Telescope design

- Telescope arrangement

The High Energy Stereoscopic System (H.E.S.S) consists of an array of four identical telescopes arranged in a square with 120 m sides to provide stereoscopic views of air showers (Schlenker, 2001). The greater the separation of the telescopes, the better the angular resolution, however increasing the separation beyond 100 m will begin to reduce the effective γ-ray collection area (Weekes, 2007). The spacing can thus be seen as a compromise between a large distance between the telescopes for good stereoscopic viewing of the shower, which increases the resolution of the shower reconstruction, and the requirement that at least two telescopes have to be well within the Cerenkov light pool, which has a radius of about 200 m.

- Telescope characteristics

Each telescope has a diameter of 13 m and has four arms that support the photomultiplier camera, giving it a focal length of 15 m. The reflector consists of 380 individual quartz-coated round mirror facets, 60 cm in diameter, and arranged with Davies-Cotton optics (Hinton, 2004).

The dish and the camera arms are mounted in alt-az fashion on a rotating base frame which is supported at the ends of the elevation axis by two towers. The base frame rotates around a central bearing on six wheels running on a 13.6 m diameter rail. In both the azimuth and elevation, the telescope is driven by friction drives acting on special 15 m diameter drive rails. This is explained in more detail in Bernlöhr et al. (2003).

- Camera

The cameras used by H.E.S.S. consist of 960 pixels of 0.16° angular size providing a total field of view of 5°. Every pixel consists of a photomultiplier tube (PMT) with a Winston
cone light collector to avoid photon losses. The Winston cone light collector also focus the light into the active part of the area of the PMT. The PMT’s are organized into drawers of 16 PMT’s each, with associated read-out electronics. All the triggering and read-out electronics are contained within the body of the camera. For more detail consult Schlenker (2001)

- **Absolute telescope pointing accuracy**

The pointing accuracy is determined by the telescope’s tracking system, the mirror alignment, as well as the correction of misalignments and deformations of the telescope structure. To allow remote alignment, each mirror is equipped with two alignment motors. The alignment procedure uses the image of a star on the closed lid of the PMT camera, which is viewed by CCD camera, mounted on the center hub of the dish. The optical point spread function of the telescope is measured in the same way, described in detail by Cornils et al. (2003).

When the absolute pointing accuracy was measured, using images of stars on the camera lid, uncorrected star images deviated from nominal positions with a RMS error of 28”. A pointing precision of 8” is achieved. By using a guide telescope attached to the dish for online corrections, the pointing error can be further reduced to 2.5” RMS. It is therefore possible for H.E.S.S. to locate bright \(\gamma\)-ray sources to within a few arc-seconds.

Figure 3.5 shows the optical point spread function for the H.E.S.S. telescopes. The 80% light containment radius is smaller than half the pixel diameter up to 2° off-axis. The point spread function is well contained within a single hexagonal pixel, and the on-axis RMS width of the two-dimensional distribution is 0.34 mrad, which is about 70”. The large mirror area, and the narrow point spread function of the H.E.S.S. telescopes, allow for a very high signal to noise ratio.

### 3.3.3 Analysis techniques

- **Stereoscopy**

Stereoscopic systems consist of two or more telescopes placed within the Cherenkov light shower. The idea is to view the same shower with different telescopes. This offers several advantages, since an increased mirror area implies an enhanced sensitivity to air showers; the light distribution can be examined at different points on the ground; the measurements are independent and they allow for a more accurate energy estimation of the primary particle.

By requiring more than one telescope at a time to be triggered, the night sky background can be rejected. Specifically, an atmospheric muon that triggers only one telescope will not trigger the system as a whole. The trigger threshold may also be lowered, thereby increasing the sensitivity to low-energy showers.
Furthermore, stereoscopic observation allows for a three dimensional reconstruction of the inclination and the impact point of a shower. Assuming a vertical $\gamma$-ray shower observed by a system of two telescopes, the impact point or shower core corresponds to the point of intersection of the straight lines of the observation level defined by the major axis of the two ellipses in each camera as shown by the left side image of Figure 3.6. For non-vertical showers the direction of the initial particle can be obtained by the superposition of both camera images into a single camera plane coordinate system. It is then possible to reconstruct the source direction in the camera field of view by intersecting the two major axes of the shower images, as shown in the right side of Figure 3.6. The process of reconstructing the shower image is explained by Schlenker (2001) in greater detail.

Finally, viewing the Cherenkov light image from different angles significantly reduces the uncertainty of the overall light amount, and thus the error, in the energy reconstruction.

- Trigger

The H.E.S.S. trigger uses a two-level system, described by Funk et al. (2005). The first level consists of a camera trigger which triggers each telescope independently. The trigger system divides the camera into 38 overlapping trigger sectors, where each trigger sector corresponds to a number of pixels. Whether or not the camera triggers depends on a certain number of pixels within a trigger sector of the camera being above a certain threshold.

The camera triggers are sent to a central trigger system. At the second level a decision is made by the central trigger system. If at least two telescopes have been triggered within a certain time frame a system trigger will be issued and all telescopes will be read out. The
The central trigger system is also responsible for issuing a time stamp to each sample. The time stamp is derived from the GPS clock of the central trigger system.

### 3.3.4 Signal extraction

- **Reconstruction of shower geometry**
  
The shower reconstruction yields the position of the source of $\gamma$-rays in the sky as explained above. The impact parameter, which is the distance of each telescope from the shower axis, can also be calculated. These values will be overdetermined when more than two telescopes are used. In such a case a more reliable estimate would be obtained by taking a weighted average of the values derived from all telescope pairings.

- **Hillas parameters**
  
The Hillas parameters were first discovered by Hillas (1985) and have become the standard for analyzing shower images. This method uses a set of second moments, which is effectively an ellipse, to analyze shower images. A detailed calculation and description of these moments is presented by Berge (2002). The end result is five parameters that can be used for the reconstruction of shower parameters. These parameters are illustrated in Figure 3.7.
These parameters are the RMS spread of light in directions parallel and perpendicular to the image axis, referred to as the length ($l$) and width ($w$) of the image. The distance ($d$) of the center of the camera to the center of the ellipse is called the nominal distance. The MISS, is the perpendicular distance of the center of the field from the image axis. Lastly the azimuthal-width ($\alpha$), the RMS width of the image relative to a new axis that joins the source to the center of the image.

As mentioned previously hadronic showers are found to be more isotropic. In relation to the Hillas parameters this implies that hadronic showers are longer, wider and are not systematically aligned to the source as well as having a larger MISS and azimuthal-width. It is the width and length that allows for a separation of hadronic and $\gamma$-ray events.
Chapter 4

Discrete signal processing

4.1 Introduction

This chapter forms the basis of this thesis. It will begin by giving the necessary mathematical and signal processing background after which analysis methods like the periodogram and the modified periodogram will be discussed. Finite impulse response filtering, explained later in this chapter, will be used to avoid aliasing as well as the filtering of unwanted signals.

These techniques will be used extensively in the analysis of both simulated periodic signals as well as data gathered with the H.E.S.S. telescope. The mathematical background, background on Fourier analysis and signal processing can be found in Kreyszig (1999), Chatfield (2003), Jenkins & Watts (1968), Morrison (1994), and Smith (1997), among others. The author is however heavily indebted to Oppenheim & Schafer (1989). The discussion presented below should not be considered as the author’s own work but as a background study on techniques to process simulated and actual signals. All graphs are the product of the author’s own codes and were used to evaluate the validity of the techniques and routines that were utilized.

Data gathered by the H.E.S.S. telescope cannot be regarded as discretely sampled. Instead the telescope should be regarded as a photon counter that merely records the time of a \( \gamma \)-ray event that was registered. Only after binning these events in time can the data be regarded as discretely sampled. The binning process is the discretization process while the binning size, that can be chosen by the user, should then be considered as the sample time.

After binning, the data or time series is then considered a discrete sampling of a continuous event. Continuous to discrete conversion is explained through the sampling theorem, a result arrived at later in this chapter. It also introduces important concepts like the Nyquist frequency and Nyquist rate.
4.2 Fourier Analysis

4.2.1 Fourier series

The Fourier series is given as (Kreyszig, 1999)

\[ f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \]  \hspace{1cm} (4.1)

where \( m = 1, 2, 3, \ldots \) and is used in an attempt to represent a given periodic function \( f(x) \). Equation 4.1 gives rise to the eigenfunctions \( 1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots \). It can be shown that these eigenfunctions are both orthogonal and orthonormal and form a complete set. Equation 4.1 is a trigonometric series whose coefficients are determined from \( f(x) \) through the Euler formulas

\[
\begin{align*}
    a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
    a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos nx \, dx \\
    b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin nx \, dx
\end{align*}
\]  \hspace{1cm} (4.2)

for \( n = 1, 2, 3, \ldots \). The coefficients given by Equation 4.3 are called the Fourier coefficients of \( f(x) \).

- Convergence and sum of Fourier series

To illustrate the convergence and the sum of the Fourier series we will use the square wave shown in Figure 4.1. For any periodic function that is piecewise continuous in the interval \(-L \leq x \leq L\) and has a left-hand and a right-hand derivative at each point of that interval we can calculate the Fourier coefficients using Equation 4.3 and represent the function using the corresponding Fourier series Equation 4.1.

Now consider the function \( f(x) \) of Equation 4.1 to be the square wave of Figure 4.1, which oscillates between the values of 1 and \(-1\). Figure 4.2 shows how Equation 4.1 approximates this function for various values of \( m \). From Figure 4.2 it can be seen that \( f(x) \) is better represented by larger values of \( m \). It can thus be said that the sum and the convergence of the Fourier series improves as more terms are used (larger values of \( m \)) to represent a function \( f(x) \).
The complex Fourier series
As the Fourier series Equation 4.1 is often written in complex form, this notation will often be used throughout this thesis. This is done using the Euler formula

\[ e^{imx} = \cos mx + i \sin mx, \]
\[ e^{-imx} = \cos mx - i \sin mx. \]

The result is that the Fourier series can then be written as

\[ f(x) = \sum_{m=-\infty}^{\infty} c_m e^{im\pi/L} \]

(4.3)

and the Fourier coefficients as

\[ c_m = \frac{1}{2L} \int_{-L}^{L} f(x)e^{im\pi x/L} dx \]

(4.4)

These are known as the complex Fourier coefficients. According to Morrison (1994) any physically realizable periodic wavefront can be expressed as a sum of complex exponentials.

The discrete Fourier series of periodic sequences
It would be useful to first introduce a method for writing discretely sampled signals or sequences. A sequence of numbers \( y \) in which the nth number in the sequence is denoted as \( y[n] \) is formally written as
Figure 4.2: The representation of a square wave using the partial sums of a Fourier series.
where \( n \) is an integer. For example, consider the square wave of Figure 4.1 as a discretely sampled signal with a sample period \( T \). The numeric value of the \( n \)th number in the sequence is equal to the value of the continuous signal \( f(t) \), where \( x \) is now replaced by time \( t \), so that we are dealing with a discrete-time signal; i.e.

\[
y[n] = f(Tn) - \infty < n < \infty
\]

Although it is technically correct to draw graphs or figures of discrete signals as a sequence of data points rather than a continuous line, it is the latter format that will often be used in this thesis. It is thus important to note that \( y[n] \) is defined only for integer values of \( n \), and it is not correct to think of \( y \) as being zero for non-integer \( n \). Throughout this thesis the notation of Equation 4.5 will be used, as we will be dealing with a discrete-time data set.

Now consider a periodic sequence \( y[n] \) with a period of \( N \) so that \( y[n] = y[n + aN] \) for any integer value of \( a \). This sequence can be represented by a Fourier series, which corresponds to a sum of harmonically (frequency) related complex exponentials with frequencies that are integer multiples of the fundamental frequency \( 2\pi/N \) associated with the periodic sequence \( y[n] \). These periodic complex exponentials \( e_k[n] \) have the form

\[
e_k[n] = e^{j(2\pi/N)kn} = e_k[n + aN]
\]

where \( k \) is an integer, and the Fourier series representation has the form

\[
y[n] = \frac{1}{N} \sum_k Y[k] e^{j(2\pi/N)kn}
\]

The representation of a continuous-time signal by a Fourier series requires infinitely many harmonically related complex exponentials. However, the Fourier series of a discrete-time signal with period \( N \) requires only \( N \) harmonically related complex exponentials. This can be seen by noting that the harmonically related complex exponentials \( e_k[n] \) in Equation 4.6 are identical for values of \( k \) separated by \( N \), i.e., \( e_0[n] = e_N[n], e_1[n] = e_{N+1}[n] \) and in general

\[
e_k[n] = e_{k+bN}[n]
\]

where \( b \) is an integer. This implies that the set of complex exponentials \( e_0[n], e_1[n], ..., e_{N-1}[n] \) defines all the distinct complex exponentials with frequencies that are integer multiples of \( (2\pi/N) \). It can now be concluded that the Fourier series representation of a periodic function \( y[n] \) only has to contain \( N \) periodic complex exponentials and the Fourier series has the form
\[ y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] e^{j(2\pi/N)k} \quad (4.9) \]

The sequence of the Fourier series coefficients \( Y[n] \) will be obtained by using the sequence property of the sequence of complex exponentials. It is shown in Oppenheim & Schafer (1989) that Equation 4.9 can be written as

\[ y[n] = \frac{1}{N} \sum_{n=0}^{N-1} Y[k] e^{j(2\pi/N)(k-a)n} = \begin{cases} 1, & k - a = mN \\ 0, & \text{otherwise} \end{cases} \quad (4.10) \]

where \( m \) is an integer, which implies that \( Y[n] \) becomes

\[ \sum_{n=0}^{N-1} y[n] e^{-j(2\pi/N)an} = Y[n]. \quad (4.11) \]

This means that the Fourier coefficients of a periodic discrete-time series of period \( N \), such that \( Y[0] = Y[N], Y[1] = Y[N+1] \), can be found using Equation 4.11. The Fourier coefficients given by Equation 4.11 can be interpreted as a sequence of finite length for \( k = 0, 1, 2, ..., N - 1 \) and zero otherwise, or as a periodic sequence defined for all \( k \). Although both of these interpretations are acceptable, there is an advantage to interpreting the coefficients as a periodic sequence, namely that there exists a duality between the frequency domain (when the Fourier coefficients are plotted as function of frequency) and time domain (where the signal \( y[n] \) is plotted as function of time). Equations 4.11 and 4.9 are together referred to as an analysis and synthesis pair, and is known as the discrete Fourier series.

- **Properties of the discrete Fourier series**

The discrete Fourier series has several important properties that make it applicable to signal processing. Some of these properties will be discussed in this section.

**Shift of a sequence:**

Consider a periodic sequence \( y[n] \) that has the Fourier coefficients \( Y[n] \). Then the shifted version of \( y[n] \) would be \( y[n - m] \), so that

\[ y[n - m] = \sum_k Y[k] e^{-j(2\pi/N)km} \quad (4.12) \]

That is, the shift in the time domain must also be represented in the frequency domain. Since the sequence of the Fourier series coefficients is a periodic sequence, a similar result is expected for a shift in the Fourier coefficients by an integer \( a \), such that

\[ \sum_n y[n] e^{-j(2\pi/N)an} = Y[k - a] \quad (4.13) \]
Duality:
The strong similarity that exists between the Fourier analysis and syntheses equations in continuous time implies that there is a duality between the time and frequency domain. From Equations 4.9 and 4.11 it can be seen that the discrete Fourier series analysis and synthesis equation differs by a factor \( \frac{1}{N} \) and the sign of the exponent \( e^{j(2\pi/N)n} \). It should also be noted that a periodic sequence and its discrete Fourier series coefficients are similar functions in that they are both periodic sequences. When taking into account the factor \( \frac{1}{N} \) and the difference in sign in the exponent it then follows from Equation 4.13

\[
Ny[-n] = \sum_{k=0}^{N-1} Y[k] e^{j(2\pi/N)kn}
\]

or by interchanging \( n \) and \( k \)

\[
Ny[-k] = \sum_{n=0}^{N-1} Y[n] e^{j(2\pi/N)kn}.
\]

The above equations give the duality property.

- **Discrete-time systems**

  **Linear systems:**
  Linear systems are defined by the principle of supposition. If \( y_1[n] \) and \( y_2[n] \) are the responses of the system when \( x_1[n] \) and \( x_2[n] \) are the respective inputs, then the system is linear only if

\[
T(x_1[n] + x_2[n]) = Tx_1[n] + Tx_2[n] = y_1[n] + y_2[n]
\]

and

\[
Ta[n] = aT x[n] = ay[n]
\]

where \( T \) refers to the transform that maps the input sequence with values \( x[n] \) to the output sequence with values \( y[n] \), and \( a \) is an arbitrary constant.

  **Time-invariant systems:**
  A time-invariant system is one for which a time shift or delay of the input sequence corresponds to a shift in the output sequence. Suppose that a system transforms the input sequence with values \( x[n] \) into the output sequence with values \( y[n] \). The system is said to be time-invariant if for all \( n_0 \) the input sequence with values \( x[n - n_0] \) transforms to the output sequence \( y[n - n_0] \). There exists a particularly important class of systems that
are both linear and time-invariant, known as Linear time-invariant systems. These two properties in combination lead to very convenient representations for these systems.

**Stability:**
A system is said to be stable in the bounded-input bounded-output sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value $B_x$ such that

$$|x[n]| \leq B_x < \infty$$

for all $n$. The requirement for stability is that for every bounded output there exists a fixed positive finite value $B_y$ such that

$$|y[n]| \leq B_y < \infty$$

for all $n$. A formal proof is given by Zaanen (1989).

### 4.2.2 The Fourier transform

It was shown in Kreyszig (1999) and Zaanen (1989) that the Fourier series of a periodic function, $f(x)$ that has a periodicity of $2L$ can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-iwx} dv \right] e^{iwx} dw$$

(4.14)

where it was assumed that $L \to \infty$. By allowing $L \to \infty$ the implication is that the Fourier integral gives a way of representing non-periodic functions. The expression in brackets is a function of $w$ (frequency) and will be denoted by $F(w)$. This is called the Fourier transform of $f(x)$, and by taking $v = x$

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

(4.15)

Plotting $F(w)$ as a function of frequency, $w$, is referred to as the frequency domain. By using the above result Equation 4.14 can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{iwx} dw$$

(4.16)

and is called the inverse transform of $F(w)$. For the Fourier transform to exist, $f(x)$ has to be piecewise continuous on every finite interval, and $f(x)$ has to be absolutely integrable on the x-axis. Up to this section the functions had to be periodic, and as already mentioned waveforms that are periodic can always have a Fourier series representation. However, it often happens that the waveform that has to be analyzed is not periodic. The Fourier integral thus makes it possible to analyze single pulses as well. It will be seen
later that periodic waveforms can often be thought of as single, infinitely long pulses that occur only once.

- The representation of sequences by Fourier transforms

For the Fourier transform of a discrete signal consider

$$y[n] = e^{jwn} \left( \sum_{k=-\infty}^{\infty} h[k]e^{-jwk} \right) \quad (4.17)$$

The function $y[n]$ can be written as

$$y[n] = H(e^{jw})e^{jwn} \quad (4.18)$$

where

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jwk} \quad (4.19)$$

The sequence $h[n]$ is known as the impulse response.

The eigenfunction of the system is $e^{jwn}$, and the associated eigenvalue is $H(e^{jw})$. It can thus be said, from Equation 4.18, that $H(e^{jw})$ describes the change in complex amplitude of a complex exponential as a function of frequency $w$. The eigenvalue $H(e^{jw})$ is referred to as the frequency response of the system. This then demonstrates the eigenfunction property of complex exponentials for discrete systems or signals.

Many signals or functions can be described by the Fourier integral, as given by Equation 4.14. For discrete signals or systems the Fourier integral can be written as

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jwn}dw \quad (4.20)$$

where $X(e^{jw})$ is given by Equation 4.19. Equation 4.20 and Equation 4.19 together form a Fourier representation for the sequence. Equation 4.20, which is the inverse Fourier transform, is a synthesis formula, as it represents $x[n]$ as a superposition of infinitesimally small complex sinusoids of the form

$$\frac{1}{2\pi} X(e^{jw})e^{jwn}dw$$

with $0 \leq w \leq 2\pi$ and with $X(e^{jw})$ determining the relative amount of each complex sinusoidal component, as described earlier in this section. In writing Equation 4.20, $w$ has been chosen to be between $\pi$ and $-\pi$, however any interval with length $2\pi$ could be chosen. The Fourier transform given by Equation 4.19 is an expression for computing $X(e^{jw})$ from the sequence of $x[n]$, or for analyzing the sequence $x[n]$ to determine how
much of each frequency component is necessary to synthesize $x[n]$ with Equation 4.20. Generally the Fourier transform is a complex-valued function of $w$, and is sometimes written in rectangular form as

$$X(e^{jw}) = X_R(e^{jw}) + jX_\Im(e^{jw})$$

or in polar form as

$$X(e^{jw}) = |X(e^{jw})| e^{j\angle X(e^{jw})}$$

Where $|X(e^{jw})|$ and $\angle X(e^{jw})$ are called the magnitude and phase of the Fourier transform. The Fourier transform is also referred to as the Fourier spectrum or simply the spectrum.

From Equation 4.19 it can be seen that the frequency response of a linear-time invariant system is simply the Fourier transform of the impulse response. This means that the impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} \, dw$$  \hspace{1cm} (4.21)

The frequency response is a periodic function as discussed above, and the Fourier transform is periodic with a period of $2\pi$. Equation 4.19 has the form of a Fourier series for the continuous-variable periodic function $X(e^{jw})$. Equation 4.20, which expresses the sequence values $x[n]$ in terms of the periodic function $X(e^{jw})$, is of the form of the integral that would be used for obtaining the coefficients in the Fourier series. Equations 4.19 and 4.20 show how a sequence $x[n]$ could be represented using the Fourier series representation of continuous-variable periodic functions and the Fourier transform representation of discrete-time signals. This is a result of the properties of the Fourier series that can be applied, with the correct interpretation of variables, to the Fourier transform representation of a sequence.

It is shown in Oppenheim & Schafer (1989) that Equation 4.20 is the inverse of Equation 4.19. It was further shown in Oppenheim & Schafer (1989) that a sequence can be represented as

$$p[n] = \sum_{k=-\infty}^{\infty} p[k] \delta[n - k]$$

where

$$\delta[n - k] = \frac{\sin \pi(n - m)}{\pi(n - m)}$$

$$= \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$
and is known as the unit sample sequence. The unit sample sequence plays the same role for discrete signals as the unit impulse function (Dirac delta function) for continuous systems.

It is shown in Oppenheim & Schafer (1989) that Equation 4.20 and Equation 4.19 can represent any sequence that has finite-length and is absolutely summable. In the context of linear time-invariant systems, any finite impulse response systems (where the impulse response only has a finite number of nonzero samples) will be stable and therefore will have a finite, continuous frequency response.

Absolute frequency is a sufficient condition for the existence of Fourier transform representation, and it also guarantees uniform convergence. However, some sequences are not absolutely summable, but are square summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Sequences like this can be represented by a Fourier transform if the condition of uniform convergence of the infinite sum defining $X(e^{jw})$ is relaxed. In other words we are concerned with the convergence of the sum of $|X(e^{jw})|^2$, this then illustrates the convergence of a sequence that has infinite length.

A system with a frequency response of

$$H(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c < |w| \leq \pi \end{cases}$$

will have an impulse response of

$$h[n] = \frac{\sin w_c n}{\pi n}$$

for $-\infty < n < \infty$. It should be noted that $h[n]$ is non-zero for $n < 0$, and that $h[n]$ is not absolutely summable. The sequence values approach zero as $n \to \infty$, but only as $1/n$. This is because $H(e^{jw})$ is discontinuous at $w = w_c$. Since $h[n]$ is not absolutely summable

$$\sum_{n=-\infty}^{\infty} \frac{\sin w_c n}{\pi n}$$

does not converge uniformly for all values of $w$. To illustrate some important features of this, consider $H(e^{jw})$ as the sum of a finite number of terms

$$H_m(e^{jw}) = \sum_{n=-m}^{m} \frac{\sin w_c n}{\pi n} e^{-jwn}$$

If the function $H(e^{jw})$ is evaluated as a function of $w$ for several values of $m$, as shown in Figure 4.3, the oscillatory behavior at $w = w_c$, known as the Gibbs affect, is clearly visible.
Figure 4.3: Convergence of the Fourier transform. The graphs are drawn for \( H(e^{jw}) \) as function of \( w \). The oscillatory behavior at \( w = w_c \) is called the Gibbs phenomenon.
It can be seen that as \( m \) increases, the oscillatory behavior increases, however the size of the ripples decrease. It can further be shown that as \( m \to \infty \) the maximum amplitude of the oscillations does not go to zero but the oscillations converge in location toward the point \( w = w_c \). Thus the infinite sum does not converge uniformly to the discontinuous function \( H(e^{jw}) \) of Equation 4.22. However \( h[n] \) is square summable, which means that \( H_m(e^{jw}) \) converges in the mean square sense to \( H(e^{jw}) \) so that

\[
\lim_{m \to \infty} \int_{-\pi}^{\pi} |H(e^{jw}) - H_m(e^{jw})|^2 dw = 0
\]

- **Fourier transform theorems**

**Linearity of the Fourier transform:**

If

\[
x_1[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jw})e^{jw} \, dw
\]

and

\[
x_2[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(e^{jw})e^{jw} \, dw
\]

then

\[
ax_1[n] + bx_2[n] = a \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jw})e^{jw} \, dw + b \frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(e^{jw})e^{jw} \, dw
\]

**Time shifting and frequency shifting:**

If

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jw} \, dw
\]

then, for the time-shifted sequence,

\[
x[n - n_d] = e^{-jn_d} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jw} \, dw
\]

while for the frequency-shifted Fourier transform

\[
e^{jw_0}x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{j(w-w_0)} \, dw
\]

**Parseval’s theorem:**

If
\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jwn} \, dw \]

then

\[ E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{jw})|^2 \, dw, \]

where the function \(|X(e^{jw})|\) is called the energy density spectrum, since it denotes how the energy is distributed in frequency. The energy density spectrum is defined only for finite-energy signals.

**The convolution theorem:**

If

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jwn} \, dw \]

and

\[ h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw})e^{jwn} \, dw \]

and if

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \]

then

\[ Y(e^{jw}) = X(e^{jw})H(e^{jw}). \]

Thus the convolution of sequences implies multiplication in the corresponding Fourier transforms.

**The modulation, or windowing, theorem:**

If

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})e^{jwn} \, dw \]

and

\[ w[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{jw})e^{jwn} \, dw \]

and if
\[ y[n] = x[n]w[n] \]

then

\[ Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})W(e^{j(\omega-\theta)})d\omega \]

The above equation is a periodic convolution, a convolution of two periodic functions with the limits of integration extending over only one period. The duality inherent in most Fourier transform theorems becomes evident when the convolution and modulation theorems are compared. However, in contrast to the continuous-time case where the duality is complete, in the discrete-time case fundamental differences arise because the Fourier transform is a sum while the inverse Fourier transform is an integral with a periodic integrand. For the continuous-time case it can be stated that convolution in the time domain is represented by multiplication in the frequency domain and vice versa. For discrete-time this statement has to be modified. Specifically, discrete-time convolution of sequences (the convolution sum) is equivalent to multiplication of corresponding periodic Fourier transforms, while multiplication of sequences is equivalent to periodic convolution of corresponding Fourier transforms.

### 4.2.3 Sampling of continuous-time signals

Discrete-time signals arise mostly as a result of the representation of continuous-time signals. This is partly due to the fact that the processing of continuous-time signals is often done by discrete-time signals of sequences obtained by sampling. Under reasonable constraints a continuous-time signal can be adequately represented by samples.

### 4.2.4 Periodic sampling

The typical method for obtaining a discrete-time is through periodic sampling, where a sequence of samples \( x[n] \) is obtained from a continuous-time signal \( x_c \) according to the relation

\[ x[n] = x_c(nT) \]

where \(-\infty < n < \infty\) and \( T \) is the sampling period while its reciprocal, \( f_s = 1/T \), is the sampling frequency. A system that converts from a continuous-time to a discrete-time system is referred to as a continuous-to-discrete-time (C/D) converter and is illustrated in Figure 4.4.

It is convenient to represent the sampling process in two stages, which consist of an impulse train modulator followed by a conversion of the impulse train to a sequence. This means that the continuous-time signal \( x_c(t) \) is represented by the periodic impulse train \( s(t) \) such that \( x_s(t) = x_c(t)s(t) \).
**Frequency-domain representation of sampling**

To derive the input and output of an ideal C/D converter, the conversion of \( x_c(t) \) to \( x_s(t) \) will first be considered. The modulating signal \( s(t) \) is a periodic impulse train

\[
s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

where \( \delta(t) \) is the unit impulse function or Dirac delta function. This means that

\[
x_s(t) = x_c(t) s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT).
\]

(4.23)

According to Oppenheim & Schafer (1989) the Fourier transform of Equation 4.23 can be written as

\[
X_S(jw) = \frac{1}{2\pi} X_c(jw) * S(jw)
\]

(4.24)

or

\[
X_S = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(jw - kjwt)
\]

(4.25)
where

\[ S(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(w - kw_s) \]

and \( w_s = \frac{2\pi}{T} \) is the sampling frequency in radians/2.

The above equation provides the relationship between the Fourier transform of the input and the output of the impulse train modulator. From this equation it can be seen that the Fourier transform of \( x_s(t) \) consists of periodically repeated copies of the Fourier transform of \( x_c(t) \). The copies of \( X_c(jw) \) are shifted by integer multiples of the sampling frequency and then superimposed to produce the periodic Fourier transform of the impulse train of samples.

Consider a band limited signal i.e. a signal limited to certain frequencies such as Figure 4.5a, so that the highest frequency with a nonzero value for \( X_c(jw) \) is at \( w_N \). Then, if \( X_c(jw) \) would be convolved with \( S(jw) \), it would be evident that when \( w_s - w_N > w_N \) or \( w_s > 2w_N \) the replicas of \( X_c(jw) \) do not overlap, as in Figure 4.5b. Therefore, when they are added together in Equation 4.25, there would remain, within a scaling factor of \( 1/T \), a replica of \( X_c(jw) \) at each integer multiple of \( w_N \). This then implies that \( x_c(t) \) can be recovered from \( x_s(t) \) with an ideal lowpass filter. This can be achieved with an impulse train modulator followed by a linear time-invariant system with frequency response \( H_r(jw) \).

Since

\[ X_r(jw) = H_r(jw)X_s(jw) \]

it follows that

\[ w_N < w_c < (w_s - w_N) \]

and hence

\[ X_r(jw) = X_c(jw) \]

If \( w_s - w_N > w_N \) or \( w_s > 2w_N \) do not hold, that is, when \( w_s \leq 2w_N \) as in Figure 4.5c, the copies of \( X_c(jw) \) will overlap so that when they are added together \( X_c(jw) \) is no longer recoverable by lowpass filtering. It this case, the reconstructed output is related to the original continuous-time input through a distortion referred to as aliasing. This distortion forms the basis for what is known as the Nyquist sampling theorem.

**Nyquist sampling theorem**

Let \( x_c(t) \) be a bandlimited signal with \( X_c(jw) = 0 \) for \( |w| > w_N \). Then \( x_c(t) \) is uniquely determined by its samples \( x[n] = x_c(nT), n = 0 \pm 1, \pm 2, \ldots \) if
Figure 4.5: The effect in the frequency domain of sampling in the time domain.
The frequency \( w_N \) is known as the Nyquist frequency and the frequency \( 2w_N \) that must be exceeded by the sampling frequency is called the Nyquist rate.

Until now only the impulse train modulator has been considered. However, the objective is to express \( X(e^{j\Omega}) \), which is the discrete Fourier transform of the sequence \( x[n] \), in terms of \( X_s(jw) \) and \( X_c(jw) \). Oppenheim & Schafer (1989) shows that this can be done by applying the Fourier transform, and is found to be

\[
X(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega - j\frac{2\pi k}{T})
\]

\( X(e^{j\Omega}) \) is simply a frequency-scaled version of \( X_s(jw) \), meaning that \( X_s(jw) = X(e^{j\Omega})_{\Omega=wT=X(e^{j\Omega})} \), with the frequency scaling specified by \( \Omega = wT \). Alternatively, this scaling can be thought of as a normalization of the frequency axis so that the frequency \( w = w_s \) in \( X_s(jw) \) is normalized to \( \Omega = 2\pi \) for \( X(e^{j\Omega}) \). The fact that there is a frequency scaling or normalization in the transformation from \( X_s(jw) \) to \( X(e^{j\Omega}) \) is directly associated with the fact that there is a time normalization in the transformation from \( x_s(t) \) to \( x[n] \). Specifically, \( x_s(t) \) retains a spacing between samples equal to the sample period \( T \). However, the “spacing” of sequence values \( x[n] \) is always unity, meaning the time axis is normalized by a factor of \( T \). Correspondingly, in the frequency domain, the frequency axis is normalized by a factor of \( f_s = 1/T \).

• The reconstruction of a signal from its samples

This section will be concerned with the exact reconstruction of a bandlimited continuous-time signal from its samples. According to the reconstruction theorem this should be possible if the samples are taken frequently enough, using information from the sampling period. Impulse train modulation will be used as a means of understanding the process of reconstructing the continuous-time bandlimited signal from its samples.

If the conditions of the sampling theorem are met and the modulated impulse train is filtered by an appropriate lowpass filter, then the Fourier transform of the filter output will be identical to the Fourier transform of the continuous-time signal \( x_c(t) \), and thus the output filter will be \( x_c(t) \). It is shown in Oppenheim & Schafer (1989) that for a sequence of samples, \( x[n] \)

\[
x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T},
\]

where \( t = nT \) and \( T \) is the sampling period. This is the ideal reconstruction filter and has a gain \( T \) and a frequency cutoff between \( w_N \) and \( w_s - w_N \). The most commonly used value
for $w_c$ is $w_s/T = \pi/T$, an appropriate choice that avoids aliasing. It then follows that the n-th sample is associated with the impulse at $t = nT$, where $T$ is the sampling period. If this impulse train is the input to an ideal lowpass continuous-time filter with frequency response $H_r(jw)$ and impulse response $h_r(t)$, then the output of the filter will be the ideal reconstruction filter with gain $T$ and a cutoff frequency $w_c$ between $w_N$ and $w_s - w_N$. A convenient and commonly used choice for the cutoff frequency is $w_c = w_s/2 = \pi/T$. This is the appropriate choice for any relationship between $w_s$ and $w_N$ that avoids aliasing. It then follows that

$$x_r(mT) = x_c(mT)$$

for all integer values of $m$. This means that the signal reconstructed with Equation 4.26 has the same values at the sampling times as the original continuous-time signal, independent of the sampling period $T$.

### 4.2.5 The discrete Fourier transform

The discrete Fourier transform is a sequence rather than a function of a continuous variable. This is unlike to what has been considered until now in this section, where the samples are equally spaced in frequency of the Fourier transform of the signal.

According to Smith (1997), the discrete Fourier transform is used for signals that extend from negative to positive infinity. However, for practical reasons, signals cannot go on to infinity. The signal can be truncated so that it can be imagined that the signal has an infinite number of points to the left and the right of the actual data.

The discrete Fourier transform analysis and synthesis equations are given as

**Analysis equation**:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jkn2\pi/N} \quad (4.27)$$

and

**Synthesis equation**:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{jnk2\pi/N}. \quad (4.28)$$

The above equation is written without eliminating the inherent periodicity. Defining the above equations in this way means that we are only interested in values of $x[n]$ on the interval $0 \leq x[n] \leq N - 1$ because $x[n]$ is really zero outside that interval. The only values of interest for $X[k]$ are on the interval $0 \leq k \leq N - 1$, because those are the only values needed for the synthesis.
4.3 Design of Finite Impulse Response (FIR) filters by windowing

There are basically three well-known classes of design techniques for FIR filters, all of which are explained in Rabiner & Gold (1975) as well as Oppenheim & Schafer (1989), however as mentioned the window method will be used here, and will therefore be the focus of the discussion below.

Filters are an important class of linear time-invariant systems. The term “frequency-selective filter” suggests a system that passes certain frequency components and totally rejects all others. In a broader context any system that modifies certain frequencies relative to others is also called a filter.

In practice, the desired filter is often implemented by means of digital computing. These filters are used to filter a signal that is derived from a continuous-time signal by means of periodic sampling, followed by analog-to-digital conversion, which can be viewed as an approximation of an ideal C/D (continuous to discrete) converter.

![Diagram of discrete-time filtering of continuous-time signals](image)

Figure 4.6: Basic system for discrete-time filtering of continuous-time signals.

When a discrete-time filter is to be used for discrete-time processing of continuous-time signals in the configuration of Figure 4.6, the specifications for both the discrete-time filter and the effective continuous-time filter are typically given on the frequency domain. This is especially common in frequency selective filters such as lowpass, bandpass, and highpass filters. If a discrete-time system is used as in Figure 4.6, and if the input is band limited, and the sampling frequency is high enough to avoid aliasing. Then the overall system behaves as a linear time-invariant continuous-time system with frequency response.
\[
H_{\text{eff}}(jw) = \begin{cases} 
H(e^{jwT}), & |w| < \pi/T \\
0, & |w| > \pi/T 
\end{cases}
\]

In cases like this it is relatively easy to convert from specifications on the effective continuous-time to specifications on the discrete-time filter by means of the relation \( \Omega = wt \). This means that \( H(e^{j\omega}) \) is specified over one period by the equation

\[
H(e^{j\omega}) = H_{\text{eff}} \left( j \frac{\omega}{T} \right) \tag{4.28}
\]

for \(|\omega| < \pi\). This will be illustrated in the following example from Oppenheim & Schafer (1989).

Consider using a discrete-time filter to lowpass filter a continuous-time signal with a system such as that described by Figure 4.6. The system of Figure 4.6 must have the following properties when the sample rate is \( 10^4 \text{ samples/s} \), so that \( T = 10^{-4} \text{ s} \). Firstly, the gain \( |H_{\text{eff}}(jw)| \) should be within \( \pm 0.01 \) of unity in the frequency band \( 0 \leq w \leq (2\pi)(2000) \), and secondly, the gain in the frequency band \( 2\pi(3000) \leq w \) should be no greater than \( 0.001 \). These specifications on \( |H_{\text{eff}}(jw)| \) can be depicted as in Figure 4.7a. The parameters for the example of Figure 4.7a would be

\[
\begin{align*}
\delta_1 &= 0.01(20 \log_{10}(1 + \delta_1)) = 0.086 \text{ db} \\
\delta_2 &= 0.001(20 \log_{10} \delta_2) = -60 \text{ db} \\
w_p &= 2\pi(2000) \\
w_s &= 2\pi(3000)
\end{align*}
\]

Since the sampling rate is \( 10^4 \text{ samples/s} \), the gain of the ideal system is identically zero above \( w = 2\pi(5000) \) due to the ideal discrete-to-continuous converter. The tolerance for the filter is shown in Figure 4.7b. This is the same as in Figure 4.7a except that it is plotted as function of normalized frequency and it need only be plotted in the range \( 0 \leq \Omega \leq \pi \). This is because the remainder can be inferred from symmetry properties (assuming that \( h[n] \) is real) and the periodicity of \( H(e^{j\Omega}) \). From Equation 4.28 it is clear that the passband, within which the magnitude of the frequency response must approximate unity with an error of \( \pm \delta_2 \), is

\[
(1 - \delta_1) \leq |H(e^{j\Omega})| \leq (1 + \delta_1)
\]

for \( |\Omega| \leq \Omega_p \), where for this example \( \delta_1 = 0.01 \) and \( \Omega_p = 2\pi(2000)10^{-4} = 0.4\pi \text{ radians} \). The other approximation band is the stopband in which the magnitude response must approximate zero with an error less than \( \delta_2 \), meaning that

\[
|H(e^{j\Omega})| \leq \delta_2
\]
for $\Omega_s \leq |\Omega| \leq \pi$. For this example, $\delta_2 = 0.001$ and $\Omega_s = 2\pi (3000) 10^{-4} = 0.6\pi$. To approximate the ideal lowpass filter in such a way with a reliable system, a transition band of nonzero width $(\Omega_s - \Omega_p)$ in which the magnitude response changes in a smooth fashion from passband to stopband is needed. The dashed curve in Figure 4.7b is the magnitude response of a system that meets the prescribed specification.

As the title of this section points out, the focus will be on FIR filters, which have applications almost entirely restricted to discrete-time systems. As a consequence of this, the design techniques for FIR filters are based on directly approximating the desired frequency response of the discrete-time system.

The simplest method of FIR filter design is called the window method. This method generally begins with an ideal desired frequency response that can be represented as

$$H_d(e^{jw}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-jwn}, \quad (4.29)$$

where $h_d[n]$ is the corresponding impulse response. Many idealized systems are defined by piecewise-constant or piecewise-functional frequency responses with discontinuities at the boundaries between bands. The result of this is that they have impulse responses that are noncasual and infinitely long. The most straightforward approach to obtain a casual FIR approximation to such systems is to truncate the ideal response. Equation 4.29 can be thought of as a Fourier series representation of the periodic frequency response $H_d(e^{j\omega})$, with the sequence $h_d[n]$ interpreted as Fourier coefficients. This means that the approximation of an ideal filter by truncation of the ideal impulse response is the same as the convergence of a Fourier series, where the Gibbs phenomenon is an important concept from this theory. In order to control the convergence of the Fourier series, a weighting function is used to modify the Fourier coefficients. This time limited weighting function is called a window (Rabiner, 1971). The following discussion will show how this nonuniform convergence phenomenon shows itself in the design of FIR filters.

The simplest way to obtain a casual FIR filter from $h_d[n]$ is to define a new system with impulse response $h[n]$ given by

$$h[n] = \begin{cases} h_d[n], & 0 \leq n \leq M \\ 0, & \text{otherwise.} \end{cases} \quad (4.30)$$

Generally $h[n]$ can be represented as the product of the desired impulse response and finite-duration “window” $w[n]$

$$h[n] = h_d[n]w[n]$$

where, for a simple truncation like that of Equation 4.30, the window is called the rectangular window.
4.3. DESIGN OF FINITE IMPULSE RESPONSE (FIR) FILTERS BY WINDOWING

(a) The specifications for effective frequency response of the overall system in Figure 4.6 for the case of a lowpass filter.

(b) The corresponding specifications for the discrete-time system in Figure 4.6.

Figure 4.7: The frequency response of a lowpass filter for a system like that of Figure 4.6.
\[ w[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]

It follows from the modulation, or windowing theorem, that

\[ H(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(w-\theta)}) d\theta, \quad (4.31) \]

Implying that \( H(e^{jw}) \) is the periodic convolution (the convolution of two periodic functions with the limits of integration extending over only one period) of the desired ideal frequency response with the Fourier transform of the window. This means that the frequency response \( H(e^{jw}) \) will be a “smeared” version of the desired response \( H_d(e^{jw}) \).

Figure 4.8a shows the typical functions \( H_d(e^{j\theta}) \) and \( W(e^{j(w-\theta)}) \), as required by Equation 4.31.

If \( w[n] = 1 \) for all \( n \), then \( W(e^{jw}) \) is a periodic impulse train with period \( 2\pi \), and therefore \( H(e^{jw}) = H_d(e^{jw}) \). According to this interpretation, if \( w[n] \) is chosen so that \( W(e^{jw}) \) is concentrated in a narrow band of frequencies around \( w = 0 \), then \( H(e^{jw}) \), will look like \( H_d(e^{jw}) \), except where \( H_d(e^{jw}) \) changes very quickly. It can be seen that the choice of a window is governed by the choice to have \( w[n] \) as short in duration as possible to minimize computation in the implementation of the filter, while having \( W(e^{jw}) \) approximate an impulse. This means that \( W(e^{jw}) \) has to be highly concentrated in frequency so that the convolution reproduces the desired frequency response.

Figure 4.8a depicts the frequency response, \( W(e^{j(w-\theta)}) \), of a rectangular window. As \( M \) increases, the width of the mainlobe decreases. The mainlobe is usually defined to be the region between the zero crossing on either side of the origin. For the rectangular window, the mainlobe width is \( w_m = 4\pi/(M+1) \). For the rectangular window the sidelobes are significantly large. The peak amplitudes of the mainlobe and the sidelobes grow in a manner such that the area under each lobe is a constant while the the width of each lobe decrease as \( M \) increases. As a consequence of this, the integral of \( \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(w-\theta)}) d\theta \) will oscillate as each sidelobe of \( W(e^{j(w-\theta)}) \) moves past the discontinuity of \( H_d(e^{j\theta}) \) with increasing \( w \).

The result of this is depicted in Figure 4.8b. Since the area under each under each lobe remains constant with increasing \( M \), the oscillations occur more rapidly but do not decrease in amplitude as \( M \) increases.

### 4.3.1 Properties of a some commonly used windows

The definitions of some commonly used windows:

**Rectangular:**

\[ w[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]
4.3. DESIGN OF FINITE IMPULSE RESPONSE (FIR) FILTERS BY WINDOWING

(a) The implied convolution process achieved by the truncation of the ideal impulse response.

(b) Typical approximation that results from windowing the ideal impulse response.

Figure 4.8: The convolution process.
Hanning:

\[ w[k] = \begin{cases} 
0.5 - 0.5 \cos(2\pi k/M), & 0 \leq k \leq M \\
0, & \text{otherwise}
\end{cases} \]

Hamming:

\[ w[k] = \begin{cases} 
0.54 - 0.46 \cos(2\pi k/M), & 0 \leq k \leq M \\
0, & \text{otherwise}
\end{cases} \]

where \( k \) has only integer values. The windows mentioned above are plotted in Figure 4.9.

All the abovementioned windows have the property that their Fourier transforms are concentrated around \( w = 0 \), as well as being easily computable. This is shown and discussed in more detail by Kaiser (1963). The function \( 20 \log_{10} |W(e^{j\omega})| \) is plotted as function of radian frequency for all of these windows in Figure 4.10. It is clear from these graphs that the rectangular window has the narrowest mainlobe (which is one of the desired characteristics of a window). Therefore, for a given length, it should yield the sharpest transitions of \( H(e^{j\omega}) \) at a discontinuity of \( H_d(e^{j\omega}) \). Another desired characteristic of a window is that the energy in the sidelobes should decrease rapidly as \( w \) tends to \( \pi \).
4.3. DESIGN OF FINITE IMPULSE RESPONSE (FIR) FILTERS BY WINDOWING

(a) Rectangular window.

(b) Hanning window.

(c) Hamming window.

Figure 4.10: The frequency response of some commonly used windows.
• The incorporation of generalized linear phase

When designing many types of FIR filters it is desirable to obtain casual systems with generalized linear phase response. Note that all the windows mentioned until now have the property

$$w[n] = \begin{cases} w[M - n], & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \quad (4.32)$$

meaning that they are symmetric about the point $M/2$, this was done with the generalized linear phase requirements in mind. As a result, their Fourier transforms are of the form

$$W(e^{jw}) = W_e(e^{jw})e^{-jwM/2},$$

where $W_e(e^{jw})$ is a real, even function of $w$. The convention of Equation 4.32 is the basis of casual filters in general. If the desired impulse response is also symmetric about $M/2$, i.e. $h_d[M - n] = h_d[n]$, then the windowed impulse response will be symmetric and the resulting frequency response will have a generalized linear phase, that is

$$H(e^{jw}) = A(e^{jw})e^{-jwM/2}$$

where $A_e(e^{jw})$ is a real and even function of $w$. Similarly, if the desired impulse is antisymmetric about $M/2$, i.e. $h_d[M - n] = -h_d[n]$, then the windowed impulse response will also be antisymmetric about $M/2$. Then resulting frequency response will have a generalized linear phase with a $90^\circ$ constant phase shift, so that

$$H(e^{jw}) = jA_0(e^{jw})e^{-jwM/2}$$

where $A_0(e^{jw})$ is real, and is an odd function of $w$. Suppose $h_d[M - n] = h_d[n]$, then

$$H_d(e^{jw}) = H_e(e^{jw})e^{-jwM/2}, \quad (4.33)$$

where $H_e(e^{jw})$ is real and even.

It shown in Oppenheim & Schafer (1989) that Equation 4.31 can be written as

$$H(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta})e^{-j\theta M/2}W_e(e^{-j(w-\theta)M/2})e^{-j(w-\theta)M/2}d\theta.$$ 

A simple manipulation of the phase factors leads to

$$H(e^{jw}) = A_e(e^{jw})e^{-jwM/2}$$

where
Thus, the resulting system has a generalized linear phase and, moreover, the real part $A(e^{j\omega})$ is the result of periodic convolution of the real functions $H(e^{j\omega})$ and $W(e^{j\omega})$.

The detailed behavior of the convolution of Equation 4.34 determines the magnitude response of the filter that results from windowing.

### 4.3.2 The Kaiser window

The tradeoff between mainlobe width and sidelobe area can be quantified by finding a window function that has most of its energy concentrated at low frequencies in the frequency domain and is of limited duration in the time domain in the frequency domain. Such a window was found by Kaiser (1974), and is defined as

$$w[n] = \begin{cases} 
I_0(\beta(1-(n-\alpha)/\alpha)^{1/2})/I_0(\beta), & 0 \leq n \leq M \\
0, & \text{otherwise}
\end{cases}$$

where $I_0$ is a modified Bessel function of the first kind, and $\alpha = M/2$. In contrast to the other windows, the Kaiser window has two parameters, namely the length $(M+1)$ and the shape parameter $\beta$. The tradeoff between the mainlobe width and sidelobe area can be achieved by varying $(M+1)$ and $\beta$. Figure 4.11 shows the Kaiser window for different values of $(M+1)$ and $\beta$. In Figure 4.11c it can be seen that increasing $M$ while keeping $\beta$ constant causes the mainlobe to decrease in width but does not effect the amplitude of the sidelobes. Through extensive numerical experimentation, Kaiser (1974) found a pair of formulas that would permit the filter designer to predict in advance the values of $M$ and $\beta$ needed to meet a given frequency-selective filter specification. Kaiser (1974) found that over a usefully wide range of conditions, the peak approximation error is determined by the choice of $\beta$. Given that $\delta$ is fixed, the passband cutoff frequency $w_p$ of the lowpass filter is defined to be the highest frequency such that $|H(e^{j\omega})| \geq 1 - \delta$. The stopband frequency is defined to be the lowest frequency such that $|H(e^{j\omega})| \leq \delta$. Therefore the transition region has width

$$\Delta = w_s - w_p$$

for the lowpass filter approximation. By specifying

$$A = -20 \log_{10} \delta$$

Kaiser (1974) determined that the value of $\beta$ needed to achieve a specified value of $A$, which is given by
\[ \beta = \begin{cases} 
0.1102(A - 8.7), & A > 50 \\
0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \leq A \leq 50 \\
0, & A < 21 
\end{cases} \] (4.36)

Furthermore, to obtain the prescribed values of \( A \) and \( \Delta w \), \( M \) must satisfy

\[ M = \frac{A - 8}{2.285\Delta w}. \] (4.37)

Equation 4.37 predicts \( M \) to be within \( \pm 2 \) over a wide range of values for \( \Delta w \) and \( A \). Using the above specifications, the impulse response of the filter can be computed as

\[ h[n] = \begin{cases} 
\sin \omega_c(n-\alpha) \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{\pi(n-\alpha)} I_0(\beta), & 0 \leq n \leq M \\
0, & \text{otherwise} 
\end{cases} \] (4.38)

where \( \omega_c = (\omega_p + \omega_s)/2 \) and is known as the critical frequency. Note that if \( \beta \) is zero the Kaiser window is equal to the rectangular window. However, unlike the rectangular window or the other windows mentioned, the Kaiser window gives the user control over the allowed error in the frequency response as well as allowing the user to calculate the length of the needed window without trial and error.

### 4.4 Periodogram analysis

Some processes generate signals that are too complex for a reasonable deterministic model. These processes are called random processes. Typically, when the input to a linear time-invariant system is modeled as a random process, many of the essential characteristics of the input and output are adequately represented by average properties such as the mean value, the variance (average power), the autocorrelation function or the power density spectrum, which is also often called the power spectrum. A typical estimate for the mean value of a stationary random process from a finite-length segment of data is the sample mean.

The sample mean and the sample variance are unbiased and asymptotically unbiased estimators, respectively. This means that the expected value of \( \hat{m}_x \) is the true mean \( m_x \) and the expected value of \( \hat{\sigma}_x^2 \) approaches the true variance \( \sigma_x^2 \), as \( L \) approaches infinity. Furthermore, these are both consistent estimators, meaning that they improve with increasing \( L \), since their variances approach zero as \( L \) approaches infinity. The remainder of this chapter will be devoted to studying the estimation of the power spectrum of a random signal using the discrete Fourier transform. This will be done by using the periodogram, which is based on a direct Fourier transformation of finite-length segments of the signal.
4.4. PERIODGRAM ANALYSIS

Figure 4.11: The Kaiser window with $\beta$ or $M$ varied.

(a) The Kaiser window for different values of $\beta$.

(b) The frequency domain of the Kaiser window for different values of $\beta$.

(c) The frequency domain of the Kaiser window for different values of $M$. 

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(a) The Kaiser window for different values of $\beta$.

(b) The frequency domain of the Kaiser window for different values of $\beta$.

(c) The frequency domain of the Kaiser window for different values of $M$. 

Figure 4.11: The Kaiser window with $\beta$ or $M$ varied.
4.4.1 The periodogram

Consider estimating the power density spectrum $P_{ss}(\Omega)$ of a continuous-time signal $s_c(t)$, where $s_c(t)$ is assumed to be a stationary random signal. This can be done by using a system such as that shown in Figure 4.12, where an anti-aliasing filter is incorporated to minimize or eliminate the effects of aliasing when the continuous-time signal is converted to a sequence. The signal has to be multiplied by a window $w[n]$, as shown in Figure 4.12. This is done because of the finite-length requirement of the discrete Fourier transform.

![Figure 4.12: Processing steps in the discrete-time Fourier analysis of a continuous-time signal.](image)

The anti-aliasing filter creates a new stationary random signal with bandlimited power spectrum so that the signal can be sampled without aliasing. Then $x[n]$ is a stationary discrete-time random signal with a power density spectrum $P_{xx}(w)$ is proportional to $P_{ss}(\Omega)$ over the bandwidth of the anti-aliasing filter, such that

$$P_{xx}(w) = \frac{1}{T} P_{ss} \left( \frac{w}{T} \right)$$

for $|w| < \pi$, where it has been assumed that the cutoff frequency of the anti-aliasing filter is $\pi/T$, and that $T$ is the sampling period. A consequence of this is that a reasonable estimate of $P_{xx}(w)$ will yield a reasonable estimate for $P_{ss}(\Omega)$. As already mentioned, the window $w[n]$ in Figure 4.12 selects a finite-length segment (L samples) of $x[n]$, which will be denoted as $v[n]$. The Fourier transform of $v[n]$, then, is

$$V(e^{jw}) = \sum_{n=0}^{L-1} w[n] x[n] e^{-jwn} \quad (4.39)$$
As an estimate of the power spectrum, consider the quantity

$$I(w) = \frac{1}{LU} |V(e^{jw})|^2$$  \hspace{1cm} (4.40)

where \(U\) is a normalization constant needed to remove bias from the spectral estimate. When the rectangular window is used, then the power spectrum is called a periodogram. However if the window is not rectangular then the power spectrum is referred to as a modified periodogram. When the power spectrum is plotted as function of frequency rather than of period it sometimes referred to as a spectrogram (Chatfield, 2003). The periodogram does have some of the basic properties of the power spectrum \(viz.\) it is non-negative, and for real signals it is a real and even function of frequency. It can also be shown that

$$I(w) = \frac{1}{LU} \sum_{m=-L+1}^{L-1} c_{vv}[m] e^{-jwm}$$ \hspace{1cm} (4.41)

where

$$c_{vv}[m] = \sum_{n=0}^{L-1} x[n]w[n]x[n+m]w[n+m]$$ \hspace{1cm} (4.42)

It should be noted that the sequence \(c_{vv}[m]\) is the aperiodic correlation sequence for the finite-length sequence \(v[n] = x[n]w[n]\). Consequently, the periodogram is the Fourier transform of the aperiodic correlation of the windowed data sequence.

The explicit computation of the periodogram can be carried out, but only at discrete frequencies. It can be seen from Equations 4.39 and 4.40 that if the Fourier transforms of \(v[n] = x[n]w[n]\) are replaced by their discrete Fourier transform, samples of the discrete Fourier transform frequencies \(w_k = 2\pi/N, k = 0, 1, ..., N-1\) will be obtained. Samples of the periodogram are given by

$$I(w_k) = \frac{1}{LU} |V[k]|^2,$$

where \(V[k]\) is the N-point discrete Fourier transform of \(w[n]x[n]\).

If a random signal has a nonzero mean, its power spectrum will have an impulse response at zero frequency. If the mean is relatively large, this component will dominate the spectrum estimate, causing low-amplitude, low-frequency components to be obscured by leakage. In practice, this mean is often calculated and subtracted from the random signal before the computation of the random signal power estimate. Although the sample mean is only an approximate estimate of the zero frequency component, subtracting it from the random signal often leads to better estimates at neighbouring frequencies.
4.4.2 Properties of the periodogram

The nature of the periodogram estimate of the power spectrum can be determined by recognizing that for each value of $w$, $I(w)$ is a random variable. By computing the mean and variance of $I(w)$, it can be determined whether the estimate is biased or consistent. It is shown in Oppenheim & Schafer (1989) that, assuming the signal to be stationary,

$$E \{ I(w) \} = \frac{1}{2\pi LU} \int_{-\pi}^{\pi} P_{xx}(\theta)C_{ww}(e^{j(w-\theta)})d\theta,$$

(4.43)

where $C_{ww}(e^{jw})$ is the Fourier transform of the aperiodic autocorrelation of the window $c_{ww}[m] = \sum_{n=0}^{L-1} w[n]w[n+m]$ or

$$C_{ww}(e^{jw}) = |W(e^{jw})|^2.$$

Equation 4.43 shows that $E \{ I(w) \}$ is not equal to $P_{xx}$, which implies that the (modified) periodogram is a biased estimate of the power spectrum. This bias arises as a result of the convolution of the true power spectrum with the Fourier transform of the aperiodic autocorrelation of the data window. The main contribution to Equation 4.43 will occur in frequency bands where one or both integrand factors are large. If the window length is increased, it is expected that $W(e^{jw})$ should become more concentrated around $w = 0$, and thus $C_{ww}(e^{jw})$ should look increasingly like a periodic impulse train. Thus for large $N$, $I(w)$ will approximate the true spectrum, which shows that the estimate $I(w)$ is asymptotically unbiased. If the scale factor $1/(LU)$ is correctly chosen, then $E \{ I(w) \}$ should approach $P_{xx}(w)$ as $W(e^{jw})$ approaches a periodic impulse train. The scale can be adjusted by choosing the normalization constant $U$ so that

$$\frac{1}{2\pi LU} \int_{-\pi}^{\pi} |w(e^{jw})|^2dw = \frac{1}{LU} \sum_{n=0}^{L-1} (w[n])^2$$

$$= 1$$

or

$$U = \frac{1}{L} \sum_{n=0}^{L-1} (w[n])^2,$$

where $U$ is the energy in the window. For the rectangular window $U = 1$, but other data windows would require a value of $0 < U < 1$ if $w[n]$ is normalized to a value of 1. For ordinary deterministic signals the discrete Fourier transform will approach zero for large
values of the signal length $n$. A stationary random signal maintains the same range of values for large $n$ and small values of $n$. This implies that, as the signal length increases, $|V[k]|^2$ will keep increasing, on average, so that the $1/LU$ factor ensures convergences as $LU \to \infty$ (Schwartz & Shaw, 1975). Alternatively, the normalization can be absorbed into the amplitude of $w[n]$. To summarise, then, if properly normalized, the (modified) periodogram is asymptotically unbiased, meaning that the approaches zero as the window length increases.

To examine whether the periodogram is a consistent estimate or becomes a consistent estimate as the window length increases, it is necessary to consider the behavior of the variance of the periodogram. It is shown in Jenkins & Watts (1968) that over a wide range of conditions, as the window length increases,

$$\text{var}[I(w)] \approx P_{xx}^2.$$ 

This means that the variance of the periodogram estimate is approximately the same size as the power spectrum that is being estimated. It can therefore be said that since the variance does not asymptotically approach zero with increasing window length, the periodogram is not a consistent estimate.

Since the periodogram is basically a discrete Fourier transform, it will be useful to explain the analysis of signals using the discrete Fourier transform.

### 4.4.3 Fourier analysis of signals using the discrete Fourier transform

The basic steps for implementing the discrete Fourier transform to a continuous-time signal are shown in Figure 4.12. The anti-aliasing filter is incorporated to eliminate or minimize the effect of aliasing when the continuous-time signal is converted to a sequence. As already mentioned, the need for the window is a consequence of the finite-length requirement of the discrete Fourier transform. In many cases $s_c(t)$, and consequently $x[n]$, are very long or even infinitely long, therefore the window $w[n]$ is applied to $x[n]$ prior to computation of the discrete Fourier transform. The conversion of $x_c(t)$ to a sequence of samples $x[n]$ is represented in the frequency domain by periodic replication and frequency normalization, so that

$$X(e^{jw}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c \left( j \frac{w}{T} + j \frac{2\pi r}{T} \right). \quad (4.44)$$

In a practical implementation, the anti-aliasing filter cannot have infinite attenuation in the stopband, hence some nonzero overlap of the terms in Equation 4.44, or aliasing, can be expected. This, however, can be made negligibly small by a high quality continuous-time filter.
As mentioned earlier, the sequence \( x[n] \) is typically multiplied by a finite-duration window, \( w[n] \), since the input of the discrete Fourier transform must be of finite duration. The effect on the frequency domain is a periodic convolution, given by

\[
V(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(w-\theta)})d\theta.
\]

The convolution of \( W(e^{jw}) \) with \( X(e^{jw}) \) will have the effect of smoothing sharp peaks and discontinuities in \( X(e^{jw}) \). The final operation of Figure 4.12 is the discrete Fourier transform (DFT). The DFT of the windowed sequence is

\[
V[k] = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)kn},
\]

for \( k = 0, 1, 2, ..., N - 1 \), where it is assumed that the window length \( L \) is less or equal to the DFT length \( N \). \( V[k] \), the DFT of the finite-length sequence \( v[n] \), corresponds to equally spaced samples of the Fourier transform of \( v[n] \), such that

\[
V[k] = V(e^{jw})|_{w=2\pi k/N}.
\]

Since the pacing between DFT frequencies is \( 2\pi/N \), and the relationship between the normalized discrete-time frequency variable is \( \omega = \Omega T \), the DFT frequencies correspond to the continuous-time frequencies \( \Omega_k \), given by

\[
\Omega_k = \frac{2\pi k}{NT}.
\]

- **The effect of the window**

Oppenheim & Schafer (1989) show that the window smears or broadens the impulses in the theoretical Fourier representation, and thus the exact frequency is less sharply defined. Windowing also reduces the ability to resolve sinusoidal signals that are closely spaced in frequency. The spectral sampling inherent to the DFT has the potential to give a misleading or inaccurate picture of the true spectrum of the sinusoidal signal.

A wider (small \( L \)) smoothing window will yield a greater reduction in variance. However, if \( L \) is too large the average of \( V(e^{jw}) \) will not have the same value everywhere. When \( N \) is large the periodogram is unbiased as mentioned previously. The window causes a bias which is a result of the truncation of the signal. However it should be mentioned that even though the periodogram has no bias for large \( N \), it still has a high variance. This can be improved by the use of a window, or in other words, by smoothing as described in Schwartz & Shaw (1975).
For small values of $L$, different components of the periodogram are no longer indistinguishable, due to the window. This effect is called leakage, which is when one component of frequency leaks into the vicinity of another component.

Reduced resolution and leakage are two primary effects on the spectrum that occur as a result of applying a window to the signal. The resolution is primarily influenced by the width of the mainlobe of $W(e^{jw})$, while the degree of leakage depends on the relative amplitude of the mainlobe and the sidelobes of $W(e^{jw})$. In Section 4.3 it was shown that the width of the mainlobe and the relative sidelobe amplitudes depend primarily on the length $L$ of the window. It was also shown that the rectangular window has the narrowest mainlobe for a given length, but has the largest sidelobes of all the commonly used windows. The Kaiser window, given by Equation 4.35 and discussed in Section 4.3.2, has two parameters $\beta$ and $L$ which can be used in a tradeoff between between mainlobe width and relative sidelobe amplitude. The mainlobe width $\Delta_{ml}$ is defined as the symmetric distance between the central zero crossing.

The relative sidelobe level $A_{sl}$ is defined as the ratio in dB of the amplitude of the mainlobe ($P_1$) to the amplitude of the largest sidelobe ($P_2$), so that

$$A_{sl} = 10 \log \frac{P_1}{P_2} dB.$$  

Figure 4.11 shows the Fourier transform of Kaiser windows for different lengths and different values of $\beta$. When designing a Kaiser window for spectrum analysis, the designer wants to specify a desired value of $A_{sl}$ and determine the required value of $\beta$. It can be seen in Figure 4.11c that the relative sidelobe amplitude is essentially independent of the window length and thus depends only on $\beta$. The least squared approximation for $A_{sl}$ was determined to be

$$\beta = \begin{cases} 
0, & A_{sl} < 13.26 \\
0.76609(A_{sl} - 13.26)^{0.4} + 0.09834(A_{sl} - 13.26), & 13.26 < A_{sl} < 60 \\
0.12438(A_{sl} + 6.3), & 60 < A_{sl} < 120 
\end{cases} \quad (4.45)$$

Using the values of $\beta$ calculated from Equation 4.45 gives windows with actual values of $A_{sl}$ that differ by less than 0.36% from the desired value across the range $13.26 < A_{sl} < 120$ (Oppenheim & Schafer, 1989).

Figure 4.11c shows that the mainlobe width is inversely proportional to the length of the window. The tradeoff between mainlobe width, relative sidelobe amplitude and window length is given by

$$L \approx \frac{24\pi(A_{sl} + 12)}{155\Delta_{ml}} + 1 \quad (4.46)$$

Equations 4.35, 4.45 and 4.46 are the necessary equations for determining the Kaiser window with the desired values of the mainlobe width and relative sidelobe amplitude.
Chapter 5

Monte Carlo simulations

5.1 Introduction

As indicated in Chapter 1, the aim of this work is to re-examine the flare events of PKS 2155-304 for variable emission and to investigate whether Vela X-1 is indeed a source of periodic VHE $\gamma$-ray emission. Before applying the Fourier techniques explained in Chapter 4 to the real data of PKS 2155-304, it is necessary to do a theoretical study on synthetic data that have more or less the same characteristics. In particular, it is necessary to construct probability distributions of the relevant test statistics under the null hypothesis.

To determine the effects of white noise and dead time on the classical and the Lomb-Scargle periodograms, a signal will be simulated using Monte Carlo simulations. A known periodicity can be inserted into the signal, and after applying various spectral analysis techniques, the effectiveness of these techniques in calculating the power spectrum or determining the periodicity can be assessed. The effects of white noise and dead time on these techniques can also be determined.

Statistical significance will also be studied as described by Scargle (1982) for both the Lomb-Scargle periodogram as well as the classical periodogram. It is important to note here that we are dealing with photon counting events instead of sampling from a light curve. These events will have to be binned before the techniques mentioned above can be applied. It is important to note that none of the periodograms have been designed for the analysis of photon counting events. Binning has the effect of averaging a periodic signal’s instantaneous rate over the bin size (Ransom et al., 2002) as well as removing the exact time of the event. It can be expected that these factors will affect the distribution of the test statistic under the null hypothesis, which is, that the time series being analyzed consists only of white noise. Monte Carlo simulations are therefore necessary to determine the distribution of the test statistic under the null hypothesis.

The Rayleigh test (de Jager, 1987), on the other hand, can make use of these counting events without having to bin them. This test will also be studied and compared to the others.

Monte Carlo methods are explained, for example, in Gentle (2010). Monte Carlo simulations
will be used to simulate the random arrival times of γ-rays for the H.E.S.S. telescope. The simulated signal will consist of a random noise component as well as a periodic component.

5.2 Methodology

The random noise events measured by the telescope in a finite time interval are assumed to be a Poisson process. Thus, the probability of observing \( n \) events in a time interval \( t \) is

\[
P_N(n) = Pr[N = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}
\]

where \( n = 0, 1, 2, \ldots \) is the number of events, and \( \lambda \) is the mean rate of arrivals. Let \( T_k \), a random variable, be the time of the kth event. We then define \( F(t) \) as \( F(t) = Pr[T_k \leq t] \). Thus

\[
F(t) = Pr[T_k \leq t] = 1 - Pr[T_k > t]. \tag{5.1}
\]

In the Poisson process of the random arrival of photons, \( T_k > t \) corresponds to \( N = k - 1 \) events having taken place in the time interval from 0 to \( t \). The probability of \( k - 1 \) events occurring in time \( t \) is given by

\[
Pr[N = k - 1] = \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}.
\]

Therefore

\[
F(t) = 1 - \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}. \tag{5.2}
\]

This is the general expression for \( F(t) \). However, considering that we are only interested in the arrival time of the first event, it follows that \( k = 1 \), which implies that

\[
F(t) = Pr[T_1 \leq t] = 1 - e^{-\lambda t} \tag{5.3}
\]

The Poisson process is a memoryless process, so that Equation 5.3 also applies to the time interval between two consecutive events. The arrival times between two consecutive events can therefore be simulated by solving for \( t \) from

\[
1 - e^{-\lambda t} = R,
\]
where \( R \) is a random number which is uniformly distributed on the interval \([0, 1]\). The final solution for the arrival times is then \( t = -\ln \left(\frac{1-R}{\lambda}\right) \). Since this is the solution between consecutive events the time of the Nth event is \( t_1 + t_2 + \ldots + t_N \).

For the periodic signal a probability density function

\[
f(t) = \frac{[1 + \cos(wt)]}{K} \tag{5.4}
\]

is assumed, where \( w = 2\pi/P \). Here \( P \) is the period and \( K \) is a normalization constant. \( K \) can be found by solving the equation \( \int_0^P f(t)dt = 1 \) and has the value of \( P \). The cumulative distribution function is then found to be

\[
F(t) = \frac{[t + \sin(wt)]}{P} \tag{5.5}
\]

and the equation is then solved numerically, as before, using

\[
t + \sin(wt) = RP \tag{5.6}
\]

where again \( R \) is a random number which is uniformly distributed on the interval \([0, 1]\).

In the case of H.E.S.S., the total time for a run is 28 minutes. In general, the events recorded in such a 28 minute run will consist of a periodic signal as well as white noise. If there is a total of \( M \) events then the number consisting of white noise or the periodic signal component will be determined by the signal to noise ratio (SNR). If, for example, the signal consists of 40% periodic signal and 60% white noise then the number of periodic signal events would be \( S = 0.4M \) and the average count rate (per second) would be given as

\[
\lambda_S = \frac{S}{28 \times 60}.
\]

The same procedure would then be followed for the white noise.

Additionally, for the periodic component,

\[
t_S = y \times P + t
\]

where \( t \) is the solution of Equation 5.6, \( P \) is the periodicity and

\[
y = \left[\left(\frac{t'}{P}\right)\right]
\]

with \( t' \) a random variable generated in the same manner as the white noise component, and \([\cdot]\) indicates truncation.
The arrival times of the noise and the periodic signal are then added together in the same series. This series then represents the simulated signal of length 28 minutes (1 run) as it would be measured with the H.E.S.S. telescope.

5.3 Significance test

The test for the significance of signal detection for the classical periodogram and Lomb-Scargle periodogram was described by Scargle (1982). For the case of evenly spaced sampling the Lomb-Scargle periodogram reduces to the classical periodogram (Scargle, 1982). However, it is also shown by Scargle (1982) that there are still some differences between the results of these periodograms. The Lomb-Scargle periodogram is often used by astrophysicists and will be included in this study. The Lomb-Scargle periodogram is given as

\[
P_X = \frac{1}{2\sigma^2} \left\{ \left( \frac{\sum_j X_j \cos w(t_j - \tau)}{\sum_j \cos^2 w(t_j - \tau)} \right)^2 + \left( \frac{\sum_j X_j \sin w(t_j - \tau)}{\sum_j \sin^2 w(t_j - \tau)} \right)^2 \right\} \tag{5.7}
\]

where \(X_j\) denotes the counts, \(t_j\) the time, \(\sigma\) the standard deviation of \(X_j\) and \(\tau\) is defined as

\[
\tan(2w\tau) = \frac{\sum_j \sin(2wt_j)}{\sum_j \cos(2wt_j)}.
\]

Assuming the power of the noise to be exponentially distributed at a given frequency, the probability density of the power is given by

\[
p(z)dz = Pr(z < Z < z + dz) = \exp(-z)dz \tag{5.8}
\]

and the cumulative distribution function is

\[
F(z) = 1 - \exp(-z) \tag{5.9}
\]

where \(Z = P_X\).

For the maximum peak value \(Z_{\text{max}} = \max_n(P_n)\), where the maximum is taken over a set of \(N\) independent frequencies, then over the cumulative distribution function, taking into account the multiplicative property for independent random variables, which is given by

\[
Pr(Z_{\text{max}} < z) = 1 - F(z) = 1 - [1 - \exp(-z)]^N. \tag{5.10}
\]

This is the theoretically expected result and will be referred to as the theoretical cumulative distribution function. According to Scargle (1982) Equation 5.10 applies for both the classical
and the Lomb-Scargle periodogram. This result can be thought of as a null hypothesis. The null hypothesis would then be that only noise is present. As the power or peak $P_X$ becomes larger it becomes exponentially less likely that the high peak value could be produced by pure noise alone. If a threshold value $z$ is determined, where $z$ is the maximum power that could be produced by noise (could be determined using Monte Carlo simulations), then signal detection can only be claimed when $Z_{\text{max}} > z$. Then the probability that at least one of the powers of at least one frequency would exceed the threshold power $z$ is given by Equation 5.10.

According to Scargle (1982), Equation 5.10 contains a penalty for inspecting a large number of frequencies and selecting the largest value for $P_X$. It was shown by Scargle (1982) that the maximum of a pure noise spectrum over a set of independent frequencies is given as

$$< Z > = \sum_{k=1}^{N} \frac{1}{k}$$

and was interpreted to mean that: “if many frequencies are inspected for a spectral peak power expect to find a large peak power even if no signal is present”. This could be problematic since the number of frequencies to be inspected is dependent on the size of the photon binning, so that a small photon binning size would possibly result in a false detection of a signal power peak. However, it was concluded by Frescura et al. (2007) that the total number of independent frequencies in a time series is limited, so that the large peaks produced by white noise cannot increase without limit, and so over-sampling the periodogram does not dramatically increase the number of large peaks expected.

![Figure 5.1: Classical periodogram of white noise.](image)

Figure 5.1 shows the periodogram for white noise, generated as described above, when no
other signals are present. Here it can be seen that the power varies around one for all frequencies, as expected.

5.4 Periodogram analysis

5.4.1 General aspects related to H.E.S.S. observations

The classical periodogram was discussed in Section 4.4.1 and the Lomb-Scargle periodogram was described in the previous section. The Monte Carlo simulations simulate the photon counting process of the H.E.S.S. telescope, which consists of 28 minutes of observations and 3 minutes of dead time. After the photon arrival times are binned these simulations produce a data set like that shown, for example, in Figure 5.2. The average of the counts has been subtracted to avoid the presence of a delta function at 0 Hz.

As a result of the dead time, the data set consists not only of a periodic signal and noise, but effectively also of a square wave with a 28 minute periodicity (Figure 5.2). Using the linearity property of Fourier analysis, the signal can be written as

\[ v[n] = g[n] + f[n] \]

where \( g[n] \) is the square wave and \( f[n] \) is the signal from the astrophysical source or white noise or, more generally, a superposition of both. Then the Fourier transform of \( v[n] \) is

\[ V(e^{jw}) = G(e^{jw}) + F(e^{jw}) \]
Since \( F(e^{jw}) \) is the spectrum of interest we have

\[
F(e^{jw}) = V(e^{jw}) - G(e^{jw})
\]  \hspace{1cm} (5.11)

so that

\[
|F(e^{jw})|^2 = |V(e^{jw}) - G(e^{jw})|^2
\]

\[
= |V(e^{jw})|^2 - 2V(e^{jw})G(e^{jw}) + |G(e^{jw})|^2
\]  \hspace{1cm} (5.12)

Equation 5.11 then shows the procedure for subtracting the square wave while Equation 5.12 shows the resulting power spectrum. \(|F(e^{jw})|^2\) consists of a superposition of \(|G(e^{jw})|^2\), \(|V(e^{jw})|^2\), and an interference term. The presence of the interference term implies that it will never be possible to completely get rid of the effect of the square wave caused by the periodic dead time.

For calculating the Lomb-Scargle periodogram we note that

\[
V(e^{jw}) = \sum_j X_j \cos(w(t_j - \tau)) - i \sum_j X_j \sin(w(t_j - \tau)),
\]

and

\[
G(e^{jw}) = \sum_j C_j \cos(w(t_j - \tau)) - i \sum_j C_j \sin(w(t_j - \tau)),
\]

where \(C_j\) represents the counts of the square wave, and is calculated by taking an average for each 28 minute run, and taking the dead time as the zero values of the square wave. This then gives the values of \(C_j\), the time domain values for the square wave. Then we have

\[
V(e^{jw}) - G(e^{jw}) = \sum_j (X_j - C_j) \cos(w(t_j - \tau)) - i \sum_j (X_j - C_j) \sin(w(t_j - \tau)).
\]

Squaring the above equation leads to

\[
[V(e^{jw}) - G(e^{jw})]^2 = \left[\sum_j (X_j - C_j) \cos(w(t_j - \tau))\right]^2 + \left[\sum_j (X_j - C_j) \sin(w(t_j - \tau))\right]^2
\]

\[
- 2i \sum_n \sum_j (X_n - C_n) \cos(w(t_n - \tau))(X_j - C_j) \sin(w(t_j - \tau)).
\]

It was noted however in Chapter 4 that \(\cos x, \sin x, \cos 2x, \sin 2x, \ldots\) form an orthogonal basis which means that
\[ [V(e^{jw}) - G(e^{jw})]^2 = \left[ \sum_j (X_j - C_j) \cos w(t_j - \tau) \right]^2 + \left[ \sum_j (X_j - C_j) \sin w(t_j - \tau) \right]^2. \]

The periodograms are then found by applying the correct normalization or window to the above equation. The Lomb-Scargle periodogram when correcting for the square wave then becomes

\[
P_X = \frac{1}{2\sigma^2} \left\{ \frac{\sum_j (X_j - C_j) \cos w(t_j - \tau))^2}{\sum_j \cos^2 w(t_j - \tau)} + \frac{\sum_j (X_j - C_j) \sin w(t_j - \tau))^2}{\sum_j \sin^2 w(t_j - \tau)} \right\}. \]

This then demonstrates how to correct for the presence of dead time in the binned time series.

Figure 5.3: Classical periodogram of a square wave.

### 5.4.2 The effect of the square wave

- **Periodograms**

Figure 5.3 shows the classical periodogram of a square wave. The square wave has a periodicity of 28 minutes with a dead time of 3 minutes. The time series has a duration of 90 minutes, which is effectively the length of 3 runs with the H.E.S.S. telescope. It can be seen that most of the power is contained in the low frequency bands. The dashed line shows that a power-law with a gradient of \(-1.7 \pm 0.1\) can be fitted to the spectrum of the square wave.

The periodograms for white noise, without subtracting the square wave caused by the dead time, are shown in Figure 5.4, which also illustrates the effect of the square wave for
Figure 5.4: Classical periodogram of white noise with 5 second photon binning. The dashed line shows a power-law fit for the lower frequency range.
different dead time lengths in the low frequency region. It is clear from this figure that for lower frequencies there is a significant deviation from the expected white noise periodogram (Figure 5.1) where \( f = \frac{\nu}{2\pi} \). The power spectra of Figure 5.4 can be divided up into two regions, the low and high frequency range. The lower frequency region is where the square wave dominates (Figure 5.3). In this region a power-law can easily be fitted (the dashed line in Figure 5.4). This can also be interpreted as power-law noise, which would be incorrect. At the higher frequency region the white noise spectrum dominates and the spectrum converges to the expected white noise spectrum.

The value of the gradient of the dashed line in Figure 5.4a is \(-1.1 \pm 0.2\), for the case of 5 s binning. For 60 s binning the gradient is \(-0.7 \pm 0.5\). It seems that increasing the bin size flattens the power-law part of the spectrum. The gradient of the dashed line in Figure 5.4b is \(-0.51 \pm 0.08\), and the gradient of the dashed line in Figure 5.4c is \(-0.46 \pm 0.06\).

The square wave therefore causes the power to increase at lower frequencies, which might lead to difficulties when detecting signals at lower frequencies, as the signal will be “hidden” by the square wave spectrum, and the significance described previously (Equation 5.10 where only white noise is present) will show high significance for all lower frequencies. Since the signal is a linear system, the spectrum of the square wave can be subtracted from the spectra of the data as shown in Equation 5.11. The resulting periodogram should be significantly flattened at low frequencies. Figure 5.5 shows the result of this procedure, and it can be seen after comparing with Figure 5.1 that the result is closer to the expected white noise power spectrum.

- **Cumulative distribution function (CDF)**

Before applying the Fourier techniques mentioned in Section 4.4 and the Lomb-Scargle periodogram to the H.E.S.S. data it is necessary to know what the statistical behavior is
Figure 5.6: CDF for 1 and 3 runs with only white noise present, with 5 s binning. The dashed line shows the expected theoretical CDF as calculated by Scargle (1982). The solid line shows the CDF calculated using the Lomb-Scargle periodogram and the dot-dash line shows the CDF from the classical periodogram.
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Figure 5.7: CDF for 1 and 3 runs with 10 second binning and only white noise present. The dashed line shows the theoretical CDF as calculated by Scargle (1982). The solid line shows the CDF for the classical periodogram.

of the maximum power in the periodogram of a time series consisting of white noise. In particular this is to know what the probability is that the maximum power $Z_{max}$ is greater than a specific value, say $z$. As explained earlier, the null hypothesis is that the time series consists of white noise only. In statistical terminology, it is necessary to know the distribution of the test statistic, $Z_{max}$ in the power spectrum in this case, under the null hypothesis.

As the method of discretization used to generate the time series is quite different to sampling from a light curve, it was necessary to investigate and construct the distribution of $Z_{max}$ using Monte Carlo simulations. This will be done for the classical, as well as the
Figure 5.8: CDF for 1 run with 20s binning calculated using the classical periodogram.

Lomb-Scargle, periodogram. Furthermore, it is possible to show only a limited number of possible cases.

Figure 5.6 shows the Cumulative Distribution Functions (CDF) for the case of white noise. All graphs are an average of $10^4$ Monte Carlo generated time series. The horizontal dotted lines show significance levels for 1%, 5% and 10% probability that white noise caused the observed power. If the square wave has been dealt with perfectly, there would be no difference between the CDF of the experimental Lomb-Scargle periodogram and the theoretical CDF given by Scargle (1982). The CDF of the experimental periodograms was calculated and will be used later to determine the significance of observed powers.

Figure 5.6a shows the CDF for the case of one run, meaning that the dead time has no effect. The empirical curve (solid line) obtained from the Lomb-Scargle periodogram, is in close but not complete agreement with the expected CDF given Equation by 5.10 (dashed line), which might be a consequence of the binning of the photon arrival times. The dot dashed lines show the CDF for the classical periodogram and it can be seen that the observed powers are significantly higher than those of the Lomb-Scargle periodogram. As mentioned previously, there are differences in the behavior of the Lomb-Scargle periodogram (solid curve) and the classical periodogram (dot dash curve). For this reason the CDF’s resulting from the Lomb-Scargle as well as the classical periodogram were calculated.

Figure 5.6b shows the case of three runs where dead time is included. It can be seen that now the Lomb-Scargle periodogram CDF deviates significantly from the theoretical expectation for white noise without dead time. In this case the high powers become more likely. This is consistent with what has been said about the square wave effect, viz. that it results in high powers being present (Figures 5.3 and 5.4). It is also seen that the classical
5.4. PERIODGRAM ANALYSIS

(a) Square wave and sine signal of 200 s periodicity.

(b) The resulting spectrum with the square wave spectrum subtracted (solid line), and the spectrum for a sine wave with 200 s periodicity (dashed line).

(c) A filtered version of Figure 5.9b.

Figure 5.9: Classical periodogram of a sine wave with 5 second photon binning.
periodogram is strongly affected by the dead time when compared with Figure 5.6a.

Figure 5.6c shows again three runs but in this case the effect of the square wave has been corrected for, as in the case of Figure 5.5b. The empirical CDF still deviates from the theoretical CDF, however this is a great improvement on Figure 5.6b. For this case, low powers become less likely compared to the theoretical CDF, whereas higher powers become slightly more likely.

It was shown by Leahy et al. (1983) that binning has an effect on the Fourier transform, \textit{viz.} that it becomes frequency dependent by means of a sinc function. This effect binning has on the spectrum has not been accounted for in this work and could possibly explain the deviation from the theoretical CDF observed in Figure 5.6a. Leahy et al. (1983) showed the effect of binning for a specific case of a sine wave. The effect of binning on more complex pulse profiles remains unstudied.

The deviation observed in Figure 5.6a (with a binning of 5 s) between the experimental and theoretical CDF could possibly be explained by the effect of binning. Figure 5.7a shows the cumulative distribution for a signal consisting of white noise with a 10 s binning calculated using the classical periodogram. It can be seen that the experimental and theoretically expected curve are practically identical. This implies that the choice of bin size can deal with the effect that the binning would otherwise have on the spectrum. However, Figure 5.7b shows this same cumulative distribution, with dead time included but corrected for according to Equation 5.11. This again shows a deviation from the theoretically expected result. This will be explained in the next section.

Figure 5.8 shows the CDF for a binning of 20 s calculated using the classical periodogram. When the bin size is increased it decreases the number of data points in the Fourier transform. This implies less data points from which to calculate the CDF. This lack of data could explain the fluctuations observed. This behavior makes the use large bin sizes questionable.

- **The dead time effect on a periodic signal**

Figure 5.9 illustrates the effect of the square wave on a periodogram for a simple 200 second \((5 \times 10^{-3} \text{ Hz})\) periodic sine wave (no white noise). The periodogram of this sine wave superimposed on a square wave is shown in Figure 5.9a. Even though the periodicity can be seen in Figure 5.9a, this still clearly demonstrates how the periodicity of the sine wave can be “hidden” by the spectrum of the square wave when the significance test is applied blindly. While the local maximum due to the sine wave can be seen, its power is less than that expected, due to the effect of the periodic dead time. In cases where the SNR is lower, implying less power in the periodic signal, it will be even harder to detect since it wouldn’t be so prominent in the periodogram. Figure 5.9b shows the same periodogram, where the scale has been adjusted to include the expected periodogram (dashed curve), but with the spectrum of the square wave subtracted. It is expected that the resulting periodogram would consist only of the spectrum of the sine wave with all its power in
the frequency of $5 \times 10^{-3}$ Hz (dashed curve). Inspection of Figure 5.9b of the dashed and solid lines shows that this is clearly not the case. This simply illustrates that it is not possible to completely correct for the square wave (dead time) as was noted before.

Figure 5.9c shows the periodogram of Figure 5.9b after the application of a FIR bandpass filter with a Kaiser window. The application of the filter has produced only a slight improvement, meaning that frequencies other than $5 \times 10^{-3}$ Hz may still have higher than expected power. The higher than expected power is a direct result of Equation 5.12. It was mentioned earlier that this equation doesn’t just consist of the periodic signal, but a superposition of the square wave spectrum and the resulting spectrum as well an interference term. This the could also explain the observed deviation in Figure 5.7b.

It is clear that these deviations have severe consequences for the CDF since it can be influenced by a square wave and possibly binning. This implies that one cannot simply use the result as given by Scargle (1982), but will have to do a numerical experiment first to determine the CDF. It can also be said that Equation 5.10 cannot simply be assumed to be the null hypothesis, but that an experiment will have to determine the distribution of $Z_{max}$ under the null hypothesis. This experimental CDF should then be used to determine the significance of the observed powers.

5.4.3 The classical periodogram of a periodic noisy signal

As has been shown above, detecting the presence of a low frequency periodicity will be difficult using the periodogram without correcting for the effect of the square wave. If the power of a low frequency periodicity is in the same band where the square wave has a high power, then the periodicity will be “hidden”. Figure 5.10 shows the periodogram of a periodic signal with white noise where the effect of the square wave has been subtracted. Figure 5.10a shows a signal with a 100 second periodicity, while Figure 5.10b shows a signal with a periodicity of 283 seconds. The periodicity of 283 seconds was chosen because it is the suspected periodicity of Vela X-1.

Since the behaviour of the noise in Figure 5.10 is consistent with the expected behaviour of white noise, viz. the power around one as in Figure 5.1, it shows that subtracting the square wave is rather effective and the detected signal in Figure 5.10 is clearly visible. This then makes it possible to detect low frequency periodicity like that in Figure 5.10b.

5.4.4 The modified periodogram of a periodic noisy signal

The FIR windowing method, explained in Section 4.3, was used. The applied window was a Kaiser window with a length of 579 discrete points and with $\beta = 3$. Figure 5.11 shows the resulting modified periodogram for both 100 and 283 second periodicities.
(a) Periodogram of a 100 second periodic signal with white noise.

(b) Periodogram of a 283 second periodic signal with white noise.

Figure 5.10: Classical periodogram of noisy signal with 5 second photon binning with dead time corrections and SNR of 67% significances were calculated in Figure 5.6c.
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(a) Modified periodogram of a 100 second periodic signal with white noise.

(b) Modified periodogram of a 283 second periodic signal with white noise.

Figure 5.11: Modified periodogram of noisy signal with 5 second photon binning with dead time corrections and SNR of 67%.

When Figure 5.11 is compared to that of the classical periodogram it can be seen that a high degree of smoothing has been achieved. A smaller window would have resulted in a smoother spectrum as explained earlier. The window seems to have had the expected effect as explained in Chapter 4.4.3, in that a smoother periodogram has been achieved.

Tables 5.1 and 5.2 show the results for 100 Monte Carlo simulations for periodicities of 283 s and 100 s. The effect of the square wave on the spectra has been removed. The photon binning was 5 s, while the length of the modified periodogram was smaller (but not by much) than that
of the periodogram, and $\beta = 3$. The error on the frequency refers to the error on the average frequency of the 100 simulations.

In Table 5.1 it can be seen that the detected periodicity varies and is not as accurate as the detection made in Table 5.2 for the case of a 100 s periodicity. This is a result of the reduced resolution at low frequencies. However, the modified periodogram is more accurate than the classical periodogram. The Rayleigh test, as mentioned earlier, does not have to be binned. It can be seen that judging from the detected periodicity and error on that detection, that the modified periodogram and the Rayleigh test seem to be more accurate then the classical periodogram.

Table 5.1: Monte Carlo simulations for white noise and 283 s periodicity.

<table>
<thead>
<tr>
<th></th>
<th>Average frequency(\text{Hz})</th>
<th>Error(\text{Hz})</th>
<th>Detected periodicity(\text{s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical periodogram</td>
<td>4.03E-3</td>
<td>2.58E-4</td>
<td>248</td>
</tr>
<tr>
<td>Modified periodogram</td>
<td>3.47E-3</td>
<td>1.95E-5</td>
<td>288</td>
</tr>
<tr>
<td>Rayleigh test</td>
<td>3.70E-3</td>
<td>5.09E-8</td>
<td>270</td>
</tr>
</tbody>
</table>

Table 5.2: Monte Carlo simulations for white noise and 100 s periodicity.

<table>
<thead>
<tr>
<th></th>
<th>Average frequency(\text{Hz})</th>
<th>Error(\text{Hz})</th>
<th>Periodicity(\text{s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical periodogram</td>
<td>9.70E-3</td>
<td>2.22E-4</td>
<td>103</td>
</tr>
<tr>
<td>Modified periodogram</td>
<td>1.02E-2</td>
<td>4.13E-6</td>
<td>98.0</td>
</tr>
<tr>
<td>Rayleigh test</td>
<td>1.02E-2</td>
<td>4.70E-6</td>
<td>98.0</td>
</tr>
</tbody>
</table>

5.4.5 The effect of dead time on the Rayleigh test

The Rayleigh power is given as (de Jager, 1987)

$$R = \frac{1}{n} \left[ \left( \sum_{i=1}^{n} \cos \theta_i \right)^2 + \left( \sum_{i=1}^{n} \sin \theta_i \right)^2 \right], \quad (5.13)$$

where $\theta_i = t_i / P(\mod 2\pi)$ with $P$ the expected period and $t_i$ a set of arrival times. The $p$-value is obtained from

$$p - \text{level} = e^{-nR} \quad (5.14)$$

this then gives the significance as

$$\text{significance} = -\log 10(p - \text{level}). \quad (5.15)$$

If the periodicity is unknown, the Rayleigh test will have to be done for a large number of frequencies. Such a test was done in Figure 5.12, for white noise only.
Figure 5.12a shows the spectrum for white noise consisting of one run only, meaning that dead time is not present. This shows the expected white noise spectrum. However, Figure 5.12b (consisting of three runs, and therefore dead time) shows a lot of power concentrated in low frequencies. This is similar to what was found in the periodogram when the square wave spectrum was not subtracted.

It is expected then that for the case of Figure 5.12b a low frequency periodicity would then be “hidden”. The significance would then also be overestimated. These effects are the same as for the periodogram without correcting for the square wave spectrum, which is a result of the dead time.

The procedure for flattening the spectra obtained from the Rayleigh test would be more complicated. Monte Carlo simulations, simulating a square wave, would have to be done first. The resulting spectra of the square wave will then have to be subtracted from the power spectrum.

### 5.5 Summary and conclusions

It was shown that the telescope dead time results in a time series that consists of a square wave and some other signal.

The procedure for dealing with this square wave was outlined in Equation 5.11. This resulted in the spectrum of Equation 5.12. However, this result implied that the spectrum was still influenced by the spectrum of the square wave, even after its subtraction.

If the square wave spectrum is not subtracted, it results in high power at low frequencies compared to signals of only one run. The effect on the CDF is shown in Figure 5.6b, viz. that very high powers become more probable with respect to the null hypothesis of Equation 5.10.

The square wave power can easily be fitted by a power-law, as shown in Figure 5.3. If white noise is added to the time series, Figure 5.2 would at low frequencies still have high power. It should then be possible to fit a power-law at these low frequencies since the square wave spectrum would dominate. However at higher frequencies the white noise spectrum would dominate and converge to a power of $\approx 1$. Figure 5.4 shows the periodogram for the case of a square wave and white noise.

Figure 5.6a shows the possible effect of binning on the CDF. Since the CDF in Figure 5.6a was done only for 1 run there is no effect that can be attributed to dead time. Leahy et al. (1983) showed that binning makes the power spectrum frequency dependent. However, the authors tested for a specific case of a sine signal, however Figure 5.6a was done only for white noise. Figure 5.7a shows the CDF for 1 run and 10 s binning. It can be seen that the empirical and theoretically expected curve are practically the same. This implies that the effect of binning can be corrected for with an appropriate choice of bin size.

However, Figure 5.7b, which illustrates the CDF for white noise with dead time corrected for,
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(a) White noise done for one run.

(b) White noise done for three runs.

Figure 5.12: Rayleigh test for white noise.
shows significant deviation from that expected by theory. This deviation can be explained by again considering Equation 5.12, where the result is still dependent on the square wave and can lead to overestimation in the power at low frequencies.

This is better illustrated in Figure 5.9b. Even after subtracting the square wave, the resulting power is still overestimated. This power spectrum, by Equation 5.12, is still influenced by the square wave.

In conclusion, there are two influences that effect the spectrum. These are the binning and the dead time. The result of this is that the CDF will have be calculated from simulations, and signal detection can only be claimed through inspection of this calculated CDF.

Table 5.1 and Table 5.2 suggest that the modified periodogram, due to its smaller error in frequency detection, is better suited for periodicity detection of binned data then the classical periodogram. The Rayleigh test also shows small error margins. However, it was mentioned that correcting for the affect of dead time when using Rayleigh test would be more complicated and beyond the scope of this work.
Chapter 6

Data analysis

6.1 Introduction

Data was gathered using the High Energy Spectroscopic System (H.E.S.S.), described in Chapter 3. In total, 120 minutes of data were obtained for PKS 2155-304 from the H.E.S.S. group. An attempt will be made to determine whether Vela X-1 is a source of $\gamma$-rays or not, and if so, if it displays pulsed emission with 283s periodicity. The optimum integration region will also be determined.

The Fourier power spectrum of the 2006 flaring event of PKS 2155-304 will be determined with corrections made for the effect of telescope dead time, explained in the previous chapter, and compared it to the results of Aharonian (2007) already discussed in Chapter 1.

6.1.1 The $\gamma$-ray distribution of point sources

As different $\gamma$-ray sources have different spectra, they will as result have to undergo different cuts, where cuts are used to remove noise from the image obtained from the H.E.S.S. telescope. A detailed description of cuts can be found in Benbow (2005). Cuts make use of the Hillas parameters (Chapter 3) to discriminate between $\gamma$-rays and the more numerous hadrons (de Naurois, 2006). This could affect the integration region that has to be used for the hypothesis testing or source detection. Here we will try to determine the optimum integration region for sources with different spectra or a different photon index.

The point spread function for the H.E.S.S. telescope is given as

$$\frac{dN}{d\theta^2} = K(Ae^{-\frac{\theta^2}{2\sigma^2}} + Be^{-\frac{\theta^2}{2\sigma^2}}), \quad (6.1)$$

where $\sigma$ represents the standard deviation and $\theta$ is the integration region in degrees. The coefficients for the above equation can be found using the H.E.S.S. software. Integrating Equation 6.1 will yield the $\gamma$-rays from the source. With this known, the significance ($S$) can be calculated as (Li & Ma, 1983)
where \( N_{\text{on}} \) denotes the on-source counts, \( N_{\text{off}} \) the off-source counts and \( \alpha \) is the ratio of on-source time to off-source time. According to Li & Ma (1983), \( S \) is the ratio of excess counts above background to their standard deviation and an observational result with significance \( S \) can be called an \( S \) standard deviation result. Equation 6.1 was integrated in several steps of \( \theta^2 \). After every step, \( S \) was calculated and the values of \( \theta \) were plotted as a function of \( S \) in Figure 6.1.

The integration region also depends on the incident gamma-ray spectrum, particularly the photon index since this variable determines the cuts applied to the data. For power-law spectrum we have the following

\[
\text{photon index } 3 : \theta = 0.01 \log S^5 - 0.074 \log S^4 + 0.169 \log S^3 - 0.088 \log S^2 + 0.058 \log S + 0.059
\]

\[
\text{photon index } 2 : \theta = 0.015 \log S^5 - 0.106 \log S^4 + 0.24 \log S^3 - 0.142 \log S^2 + 0.063 \log S + 0.052
\]

\[
\text{photon index } 1.5 : \theta = 0.028 \log S^5 - 0.22 \log S^4 + 0.609 \log S^3 - 0.64 \log S^2 + 0.318 \log S + 0.003
\]

(6.2)

Vela X-1 was assumed to be a weak source with a photon index of 2, and after applying standard cuts the integration region was calculated for a significance of 5 and was determined to be 0.0859°. PKS 2155-304 is considered to be a strong source so the integration region does not have to be calculated, and all the data gathered can be used for the analysis.

### 6.2 Analysis of PKS 2155-304

Figure 6.2 shows the periodogram for PKS 2155-304 for 5 s photon binning. Figure 6.2a shows the periodogram without the square wave correction, whereas Figure 6.2b shows the periodogram with the square wave correction. In the low frequency domain, up to about \( 10^{-2} \) Hz, there are differences in both the shapes of the spectra as well as the power. This is consistent with what was found in the previous chapter regarding the effects of dead time on the Fourier transform. In the low frequency region of Figure 6.2a, a power-law can be fitted so that the power is proportional to \( f^{-1.3 \pm 0.2} \). For the low frequency range of Figure 6.2b a power-law can be fitted so that the power is directly proportional to \( f^{-0.9 \pm 0.2} \). Comparing the power-law fits of Figure 6.2a and Figure 6.2b it is clear that the steepening is caused by the square wave that has most of its power in the low frequency region, as discussed earlier. There is significant
power above the noise level (90% significance) up to about 600 s. This is in agreement with the result obtained by Aharonian (2007).

Figure 6.2c shows the Lomb-Scargle periodogram, and, when compared to Figure 6.2b, it can be seen that the shape of the spectrum is similar, however there is more power in the classical periodogram than in the Lomb-Scargle periodogram. This is consistent with the differences between the classical periodogram and the Lomb-Scargle periodogram as was found in Chapter 5. Using the significance levels read off the CDF calculated in Figure 5.6c, and comparing it to Figures 6.2b and 6.2c (dotted lines) it seems (with 99% certainty) that a signal other than white noise is present.

Figure 6.3 illustrates the periodogram with dead time corrections for PKS 2155-304 for 60 s binning. In the low frequency range, a power-law was fitted and it was found that the power is directly proportional to $f^{-1.6 \pm 0.4}$. Without correcting for the square wave the power is directly proportional to $f^{-1.6 \pm 0.5}$ which is close what was found by Aharonian (2007). The larger bin size seems to lessen the effect of the dead time as found in Chapter 5. This is different than the case of the 5 s bin size, where the square wave had a more significant effect, and a different power-law. This again illustrates the importance of appropriate bin size as well as correcting for the effect of dead time. For this analysis it was assumed that the signal from PKS 2155-304 consists of some unknown signal mixed with white noise. If the noise however consists of a power-law, the CDF used here would have to be adapted. Larger bin sizes were questioned earlier, where it was found that the reduced number of data points caused fluctuations in the CDF (Figure 5.8).
6.2. ANALYSIS OF PKS 2155-304

Figure 6.2: Periodograms of PKS 2155-304 with 5s photon binning.

(a) Classical periodogram without square wave correction

(b) Classical periodogram with square wave correction.

(c) Lomb-Scargle periodogram with square wave correction.
The modified periodogram for PKS 2155-304, using a FIR filter and the Kaiser window is shown in Figure 6.4. This periodogram again shows that some smoothing has been achieved, however the degree of smoothing is dependent on the length of the window. If the window is too small it will obscure the periodogram, resulting in a loss of information.

Figure 6.5 shows the result for the Rayleigh test, which is very similar to the periodogram without the corrections for dead time. All spectra reduce to white noise at higher frequencies as can clearly be seen in Figure 6.5. This is also consistent with the Monte Carlo simulations as discussed in Chapter 5.

This work does not, however, exclude the possibility of power-law noise. It was seen that in the case of 5 s binning there still remained power-law behavior in the low frequency part of the power spectrum even after the effect of dead time has partially been corrected for. As shown above, it seems as if the 60 s binning has reduced the effect of the power-law. This dependence on binning, and the square wave questions the correctness of the interpretation by Aharonian (2007) i.e the presence of $f^{-2}$ noise.
Figure 6.4: Modified periodogram of PKS 2155-304 with photon binning of 5 s.

Figure 6.5: Results of the Rayleigh test for PKS 2155-304.
6.3 Analysis of Vela X-1

6.3.1 Source detection

It is important to note that data for Vela X-1 was negatively influenced by weather. It was assumed that if Vela X-1 was a source of VHE $\gamma$-rays, then it would have a photon index of 2. The optimal expected integration region can then be calculated from Equation 6.2 and was found to be $0.0859^\circ$. Coordinates of events in H.E.S.S. data sets are given as right ascension (RA) and declination (DEC). Using spherical trigonometry, the region of an event $\theta_i$ was determined using

$$
\theta_i = \sqrt{((RA_i - 135.53)\cos(DEC_i))^2 + (DEC_i - 1 \times (-40.55))^2}.
$$

If for a particular event $\theta_i < 0.0859^\circ$ the event was assumed to be from the source ($N_b$). However, if $0.0859^\circ < \theta_i < 1^\circ$ then the events were said to be from the background and will be referred to as $N_A$.

The area of the source is $A_b = \pi(0.0859^\circ)^2$ while the area of the background is $A_A = \pi(1^\circ - 0.0859^\circ)^2$.

The data gathered was binned in time and $N^i\gamma$ calculated, where

$$
N^i\gamma = N^i_b - N^i_A \frac{A_b}{A_A}
$$

where $A_b/A_A$ is needed for normalization. The variance of $N_\gamma$ was calculated as

$$
VAR(N^i\gamma) = VAR(N^i_b - N^i_A \frac{A_b}{A_A}) = VAR(N^i_b) + VAR(N^i_A \frac{A_b}{A_A}) = N^i_b + \left(\frac{A_b}{A_A}\right)^2 N^i_A
$$

and

$$
\sigma_\gamma = \sqrt{VAR(N^i_\gamma)}
$$

thus

$$
s_i = \frac{\langle N^i_\gamma \rangle - \langle N_\gamma \rangle}{\sigma_\gamma(N^i_\gamma)} \approx N(0, 1)
$$
assuming the data to be normally distributed. The value of $\langle N^i_\gamma \rangle$ is $5.9 \times 10^{-4}$ and $\sigma_\gamma$ is $5.3 \times 10^{-3}$. This gives a value for $s_i$ of 0.11.

With the above calculations completed, the hypothesis tests to determine a source of $\gamma$-rays can be done. The null hypothesis is that we do not have a source of $\gamma$-rays, i.e $H_0 : \langle N_\gamma \rangle = 0$ and the alternate hypothesis is $H_A : \langle N_\gamma \rangle > 0$. When $H_0$ is found to be true than it can be said that the object, in this case Vela X-1, is not a source of $\gamma$-rays.

For Vela X-1 the null hypothesis postulated was that $\langle N_\gamma \rangle = 0$. With an $s_i$ value as low as 0.11 the null hypothesis seems correct for the available data i.e. Vela X-1 is not a source of $\gamma$-rays.
Chapter 7

Summary and Conclusions

7.1 Monte Carlo simulations

As mentioned previously, the H.E.S.S. telescope gathers data in 28 min intervals followed by 3 minutes of dead time. The dead time results effectively in a square wave being superimposed on the signal. The Fourier transform of the signal then consists of the Fourier transform of the source as well as the Fourier transform of the square wave. It was shown that the resulting spectrum has a large amount of the power in the low frequency range and that in this low frequency range the spectrum can be fitted with a power-law. This power-law could easily and incorrectly be interpreted as power-law noise. The square wave spectrum also affected the CDF, resulting in high powers becoming more probable with respect to Equation 5.10. It also had the effect of “hiding” periodicities that would otherwise be detected using the CDF.

The proposed method of dealing with this was to subtract the Fourier transform of the square wave from that of the signal. It was also shown that because of an interference term, it was not possible to completely remove the influence the square wave. However, this method did yield some positive results. It was shown by Monte Carlo simulations that it flattened the spectrum closer to that of the expected white noise spectrum, which in turn made it easier to detect periodicities in signals.

Since the H.E.S.S. data is given as photon counting events, the data has to be binned. Monte Carlo simulations showed that the binning affects the power spectrum. It was shown that the CDF of a binned data set deviates from that of Equation 5.10. These deviations in the CDF imply that Equation 5.10 cannot simply be used as the distribution of the test statistic under the Null hypothesis. The Null hypothesis CDF must be found experimentally through the use of Monte Carlo simulations.

It was explained that the time series is discretized through binning. The bin size should then be regarded as the sampling time. The sampling theorem, Equation 4.26, explains how a continuous signal can be determined by its samples. Then the deviations found between the experimental CDF and the CDF of Equation 5.10, taking into account the sampling theorem, are unexpected (for the case of one run). Put differently, the CDF should not depend on the binning
size. This could be explained by considering the difference between binning and say, sampling from a light curve. Increasing the bin size would increase the events or counts per bin. The sum over all the bins must equal the total amount of events or counts detected. This then implies that the binning procedure is an inconsistent measure. For example, increasing the sampling time when sampling from a light curve will not increase the amplitude per measurement. Furthermore, sampling from a light curve implies taking a measurement at a specific sampling time. However, binning implies integrating over the sampling time. The affect of binning remains a topic for further study.

It was suggested that the modified periodogram making use of a Kaiser window, was better suited to the analysis of data. This is mainly due to its smaller error in periodic detection (Tables 5.1 and 5.2). This FIR filter design was used as a low pass filter, where the signal is filtered to the Nyquist frequency. The Nyquist frequency is dependent on the binning size. Exactly why the modified periodogram gives a better result requires further study.

An FIR filter was also applied to simulated data in an attempt to filter out the effect of the square wave. This was, however, not entirely successful. FIR filters can be used to enhance periodic signals in data. However, the periodicity is often unknown, making the application of FIR filters difficult, and possibly biased. It would be biased in the sense that since the periodicity is unknown, the application of the filter could result in the periodicity being filtered out. For this work, the FIR filter was used only as an anti-aliasing filter when analyzing the the H.E.S.S. data using the classical or modified periodogram.

The Rayleigh test was found to be influenced by the square wave in the same manner as the other periodograms. However, for the case of the periodograms, the square wave could be corrected for quite easily. This is not the case for the Rayleigh test. For the Rayleigh test a Monte Carlo simulated square wave would have to be simulated. This is significantly more complicated then simply creating a square light curve, and hence beyond the scope of this study.

7.2 PKS 2155-304

The periodogram of PKS 2155-304 for the case of 60 s binning, without correcting for the square wave, yields a power spectrum that can be fitted with a power-law of $f^{-1.6 \pm 0.5}$. This is in close agreement with that found by Aharonian (2007). When the square wave corrections are included a power-law of $f^{-1.6 \pm 0.4}$ is found. This implies that the use of a larger bin size has reduced the effect of the dead time.

When the bin size was reduced to 5 s, and the square wave corrected for, the low frequency range power spectrum could be fitted by a power-law of $f^{-0.9 \pm 0.2}$, and $f^{-1.3 \pm 0.2}$ without correcting for the effect of dead time. This power-law is about half of what was reported by Aharonian (2007) and the case of 60 s bin size. It was shown with Monte Carlo simulations
that there are two important factors that have to be considered for the analysis, viz. the effect of dead time (the square wave), and binning. With these effects considered, it is very possible that the result of Aharonian (2007) is overestimated. Furthermore, most of the power (90% significance) in Figure 6.2b seems to be contained below about $1.6 \times 10^{-3}$ Hz (600 s). This result is consistent with that found by Aharonian (2007).

The differences in the power-law exponent can only be attributed to the differences in the binning size. It was shown with Monte Carlo simulations that binning size influences the CDF. The decision to adopt a binning size of 5 s was based on the results from the CDF of Figure 5.6c, such that the experimental CDF corresponds closely to that of the CDF given by Scargle (1982).

The case of 10 s binning is shown in Figure 5.7b. It can be seen that high powers become more probable compared to the CDF given by Scargle (1982). The case for 20 s binning is shown in Figure 5.8. Here the relatively large bin size has decreased the number of data points and caused fluctuations in the CDF. This implies that the use of a bin size of 60 s as used in Aharonian (2007) would severely decrease the amount of data points needed for a smooth CDF.

The modified periodogram of PKS 2155-304, for 5 s binning and square wave corrections, is shown in Figure 6.4. As mentioned earlier, the modified periodogram seems more accurate when it comes to periodicity detection as well as smoothing the periodogram. However, PKS 2155-304 shows no clear periodic behavior, instead it shows a power-law spectrum (at least in the low frequency range). It should also be noted that the signal from PKS 2155-304 is possibly not a stationary signal. This implies that even a relatively wide window will result in a large loss of information or large periodogram bias, compared to a simple stationary periodic signal, where the window could be relatively wide while not losing a large amount of information like the periodicity of the signal.

### 7.3 Vela X-1

In the previous chapter it was found that Vela X-1 is not a source of $\gamma$-rays. To explain why Vela X-1 is not a source $\gamma$-rays it might be useful to compare it to the High Mass X-ray binaries (HMXB) LS 5039 and LS +61°303. There are two plausible scenarios to explain the emission of these systems: where the compact object is a black hole, which would result in a microquasar scenario where particle acceleration and $\gamma$-ray production is possible throughout the entire jet (Khangulyan et al., 2008; Bednarek, 2006), and an alternate scenario, given by Dubus (2006) and Sierpowska-Bartosik & Torres (2008), where the compact object is a young pulsar emersed in the stellar wind of a companion star.

For the pulsar scenario, the compact object is a millisecond pulsar, which gives rise to a rotation-powered pulsar wind within by the stellar wind from the companion star. The emission arises
from this shocked pulsar wind (Dubus, 2006), and the shocked material then flows away to large distances in a comet-shaped tail. As the pulsar’s rotation gradually decreases (leads to increased period) these systems would become accretion-powered HMXB with probably no significant $\gamma$-ray emission.

Vela X-1 with a periodicity of 283s clearly does not have the energy to produce rotation-powered pulsar wind and would therefore not be expected to be a source TeV $\gamma$-rays.
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