INAUGURAL LECTURE

TOPIC:

Symmetries, Solutions and Conservation Laws of Differential Equations

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1 Introduction

1.1 History of differential equations

Many problems that arise naturally from physics and other sciences can be described by **differential equations**. The importance of differential equations may be realized from their ability to mathematically describe, or model, real-life situations. The differential equations come from the diverse disciplines of architecture, chemical kinetics, civil engineering, demography, ecology, economics, electrical engineering, mechanical engineering, meteorology, and physics

Differential equations form part of **calculus**. The invention of the calculus was one of the great intellectual achievements of the 1600s. The historian Carl Boyer called the calculus "the most effective instrument for scientific investigation that mathematics has ever produced" [1]. The invention of calculus is normally ascribed to two brilliant contemporaries, the Englishman Isaac Newton (1642-1727) and the German Gottfried Wilhelm Leibniz (1646-1716). Both men published their researches in the 1680s, Liebniz in 1684 in the recently founded journal Acta Eruditorum and Newton in 1687 in his great treatise, the Principia. However, a lot of contribution was also made by Cavalieri, Torricelli, Barrow, Descartes, Fermat and Wallis to the subject [2].

Coming back to Differential Equations, there is no doubt that the Swiss mathematician and physicist, Leonhard Euler (1707-1783), is most responsible for the methods of solution to differential equations, used today in first-year calculus courses. Also, even many of the problems appearing in current calculus textbooks can be traced back to the great treatises Euler wrote on the calculus −*Institutiones Calculi Differentialis* (Petersburg, 1755) and *Institutiones Calculi Integralis* (Petersburg, 1768 1770, 3 vols.). Euler also contributed to the use of integrating factors and the systematic methods of solving linear differential equations of higher order with constant coefficients. He also distinguished between the linear homogeneous and nonhomogeneous differential equations and between the particular and general solutions of differential equations.

During this time solving ordinary differential equations had become one of the most important problems in applied mathematics.

Daniel Bernoulli (1700-1782), for example, solved the second-order ordinary differential equation

$$y'' + Ky = f(x)$$

independently of Euler and about the same time in 1739-1740, Jean d'Alembert (1717-1783), as well as Euler, had general methods for solving completely the linear differential equations.

However, most differential equations cannot easily be reduced to simple quadratures, and would require ingenious substitutions or algorithms for their solution. One of the interesting differential equations of the eighteenth century was the so-called Riccati equation

$$y' = p(x)y^2 + q(x)y + r(x).$$

It was Euler who first noted that if a particular solution $v = \phi(x)$ is known, then the substitution $y = v + 1/z$ converts the Riccati equation into a linear differential equation in z, so that a general solution can be found. Today, in the literature, the linear differential equation with variable coefficients

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y^{(0)} = f(x)$$

bears Euler's name (where exponents included within parentheses indicate orders of differentiation). It can easily be transformed to a linear equation having constant coefficients by the substitution $x = e^t$. Euler also made contribution to partial differential equations by giving the wave equation

$$u_{tt} = a^2 u_{xx}$$

the solution

$$u = f(x + at) + g(x - at).$$

Euler produced four volumes of *Institutiones* that contained by far the most exhaustive treatment of the calculus up to that time. A theorem known as "Euler's theorem on homogeneous functions", namely, if $f(x, y)$ is a homogeneous function of order n, then $xf_x + yf_y = nf$, is taught in the calculus courses even today.

1.2 Basics concepts of differential equations

Definition 1. A *differential equation* (DE) is an equation involving unknown functions and their derivatives.
If the functions are real functions of one real variable, the derivatives occurring are ordinary derivatives, and the equation is called an *ordinary* DE. If the functions are real functions of more than one real variable, the derivatives occurring are partial derivatives, and the equation is called a *partial* DE.
Definition 2. An ordinary DE is said to be *linear* if and only if it can be written in the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

where f and the coefficients $a_0, a_1, \cdots, a_n$ are continuous functions of x.
All other DEs are called *nonlinear* equations. Thus a linear DE is linear in y and its derivatives.
Definition 3. The *order* of a DE is the order of the highest derivative appearing in the equation.
Definition 4. By a *solution* of the DE we mean a function $y = \phi(x)$ which satisfies the given equation.
Remark. Some DEs have infinitely many solutions while others have no solution.
Definition 5. By a *general solution* of an nth-order DE, we mean a solution containing n essential arbitrary constants.

Remark. The general solution of the first-order ODE

$$\frac{dy}{dx} = f(x)$$

is given by

$$y = \int f(x)dx + C,$$

where C is an arbitrary constant and we say that the ODE is *integrable by quadrature.*
Definition 6. A *particular solution* of a DE is any one solution.
It is usually obtained by assigning specific values to the constants in the general solution.
A particular solution can be represented as a curve in the $xy-$plane called an *integral curve.*
The general solution is the set of all integral curves, or the $family$ of all integral curves.
Definition 7. A solution of the DE that cannot be obtained from a general solution by

assigning particular values to the arbitrary constants is called a *singular solution.*
The solution of a second-order DE generally involves *two* arbitrary constants. By specifying two subsidiary conditions one can determine the values of these constants to obtain a *unique* solution. The conditions may be given at the *same* value of x, such as $y(a) = k_1$, $y'(a) = k_2$. Such conditions are called *initial conditions* (IC).
Definition 8. A DE together with IC's is called an *initial-value problem* (IVP).
If the conditions are given at two *different* values of x, such as $y(a) = k_1$, $y(b) = k_2$, they are called *boundary conditions* (BC).
Definition 9. A *boundary-value problem* (BVP) is a DE together with BCs.
These definitions can be extended to DEs of any order.
Remark. The IVP generally possess a unique solution. However, on the other hand a BVP does not always possess a solution and even if a solution exits, it may not be unique. BVPs are intrinsically more difficult than IVPs, although they have equal importance in applications. The theory of IVPs is much more complete than that of the BVPs. Only for linear DEs has the BVP been thoroughly investigated.

1.3 Examples of Ordinary Differential Equations

(1) The first-order ODE

$$\frac{dN}{dt} = \rho N \tag{1.1}$$

models various phenomena such as radioactive decay, population growth and future balances of investments earning interest at rates compounded continuously.

(2) Newton's second law of motion can be stated in terms of an object's acceleration, namely

$$F = ma = m\frac{d^2x}{dt^2}, \tag{1.2}$$

where F is the net force applied, m is the mass of the body, and a is the body's acceleration.

(3) The damped harmonic oscillators satisfy the second-order differential equation

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0, \tag{1.3}$$

where ω_0 is the undamped angular frequency of the oscillator and ζ is a constant called the damping ratio.

(4) For a particle of mass m under a potential $V(x)$, the one-dimensional, time-independent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x), \tag{1.4}$$

where $\psi(x)$ is the wave function, in general complex, and $\hbar$ is the Planck constant h divided by $2/\pi$.

(5) In the study of stellar structure the Lane-Emden equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^r = 0, \tag{1.5}$$

where r is a constant, models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics.

(6) The modelling of several physical phenomena such as pattern formation, population evolution, chemical reactions, and so on, gives rise to the systems of Lane-Emden equations

$$\frac{d^2u}{dt^2} + \frac{n}{t}\frac{du}{dt} + f(v) = 0, \tag{1.6}$$

$$\frac{d^2v}{dt^2} + \frac{n}{t}\frac{dv}{dt} + g(u) = 0, \tag{1.7}$$

where n is real constant and $f(v)$ and $g(u)$ are arbitrary functions of v and u, respectively.

1.4 Examples of Partial Differential Equations

(1) The Burgers equation

$$u_t + uu_x = \nu u_{xx}, \tag{1.8}$$

where u is the velocity and ν is the viscosity coefficient, occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow.

(2) The well-known Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.9}$$

governs the dynamics of solitary waves. Firstly it was derived to describe shallow water waves of long wavelength and small amplitude. It is a crucial equation in the theory of integrable systems because it has infinite number of conservation laws and gives multiple-soliton solutions.

(3) An essential extension of the KdV equation is the Kadomtsev-Petviashvili (KP) equation given by

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{1.10}$$

This equation models shallow long waves in the $x-$direction with some mild dispersion in the $y-$direction. The inverse scattering transform method can be used to prove the complete integrability of this equation. This equation gives multiple-soliton solutions.

(4) The dimensionless form of the nonlinear Schrödinger equation is given by

$$iq_t + aq_{xx} + b|q|^2q = 0, \tag{1.11}$$

where the dependent variable q represents the vibrational coordinate of the amide-I vibrations. The second term is a dispersion term representing the dipole-dipole coupling and arises from the effective mass of the exciton while the third term is nonlinear term representing coupling to hydrogen bonds.

(5) A two-dimensional integrable generalization of the Kaup-Kupershmidt (KK) equation is given by

$$\begin{aligned}&u_t + u_{xxxxx} + \frac{25}{2}u_xu_{xx} + 5uu_{xxx} + 5u^2u_x + 5u_{xxy} - 5\partial_x^{-1}u_{yy} + 5uu_y\\&+5u_x\partial_x^{-1}u_y = 0\end{aligned} \tag{1.12}$$

and arises in various problems in many areas of theoretical physics.

(6) The generalized coupled variable-coefficient modified Korteweg-de Vries (CVCmKdV) system

$$u_t - \alpha(t)\left(u_{xxx} + 6(u^2 - v^2)u_x - 12uvv_x\right) - 4\beta(t)u_x = 0, \tag{1.13a}$$
$$v_t - \alpha(t)\left(v_{xxx} + 6(u^2 - v^2)v_x + 12uvu_x\right) - 4\beta(t)v_x = 0 \tag{1.13b}$$

models a two-layer fluid and is applied to investigate the atmospheric and oceanic phenomena such as the atmospheric blockings, interactions between the atmosphere and ocean, oceanic circulations and hurricanes or typhoons.

(7) A coupled KdV system which is formulated in the KP sense is given by

$$\left(u_t + u_{xxx} + 3uu_x + 3ww_x\right)_x + u_{yy} = 0, \tag{1.14a}$$

$$\left(v_t + v_{xxx} + 3vv_x + 3ww_x\right)_x + v_{yy} = 0, \tag{1.14b}$$

$$\left(w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right)_x + w_{yy} = 0. \tag{1.14c}$$

2 Symmetries of Differential Equations

About 200 years after Leibniz and Newton had introduced the concept of the derivative and the integral of a function, solving ordinary differential equations had become one of the most important problems in applied mathematics. It was then that Sophus Lie (1842-1899), a Norwegian mathematician, got interested in this problem. He got inspiration from Galois (1811-1832) theory for solving algebraic equations that had become widely known around 1850 mainly due to the efforts of Liouville. Lie showed that the majority of the known methods of integration of ordinary differential equations, which until then had seemed artificial, could be derived in a unified manner using his theory of continuous transformation groups. This method is known today as Lie group analysis and is indeed the most powerful tool to find the general solution of differential equations. In the last six decades there have been considerable developments in Lie group methods for differential equations as can be seen by the number of research papers [3–12], books [13–16] and new symbolic softwares [17–23] devoted to the subject.

A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. Determining all the symmetries of a differential equation is a formidable task. However, Sophus Lie realized that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries.

Many differential equations of physical interest involve parameters, arbitrary elements or functions, which need to be determined. Usually, these arbitrary parameters are determined experimentally. However, the Lie symmetry approach through the method of group classification has proven to be a versatile tool in specifying the forms of these parameters systematically [3–8, 16]. The problem of group classification was first carried out by Lie [14] in 1881 for a linear second-order partial differential equation with two independent variables. Later, various people including Ovsiannikov applied Lie's methods to a wide range of physically important problems. The problem of group classification of a differential equation involving an arbitrary element, say H, consists of finding the Lie point symmetries of the differential equation with arbitrary H, and then determining all possible forms of H for which the principal Lie algebra can be extended.

3 Conservation laws

The notion of conservation laws plays an important role in the solution process and reduction of differential equations. Conservation laws are mathematical expressions of the physical laws, such as conservation of energy, mass, momentum and so on. In the literature, conservation laws have been extensively used in studying the existence, uniqueness and stability of solutions of nonlinear partial differential equations (see for example, [24–26]). Conservation laws have also been applied in the development and use of numerical methods (see for example, [27,28]). Recently, conserved vectors associated with Lie point symmetries have been used to find exact solutions (by exploiting a double reduction method) of some classical partial differential equations [29–31]. Thus, it is important to derive all the conservation laws for a given differential equation.

For variational problems the celebrated Noether (1882-1935) theorem [32] provides an elegant way to construct conservation laws. Infact, it gives an explicit formula for determining a conservation law once a Noether symmetry associated with a Lagrangian is known for an Euler-Lagrange equation. Thus, the knowledge of a Lagrangian is essential in this case. However, there are differential equations, such as scalar evolution differential equations, which do not have a Lagrangian. In such cases, several methods [15,33–43] have been developed by researchers about the construction of conserved quantities. Comparison of several different methods for computing conservation laws can be found in [37,43].

4 My research in differential equations

4.1 Research Topics

(1) Existence and uniqueness of solutions of certain hyperbolic partial differential equations
(with Everitt)

(2) Symmetries and First Integrals
(with Mahomed, Muatjetjeja, Ntsime)

(3) Symmetries and conservation laws
(with Mahomed, Kara, Johnpillai, Muatjetjeja, Baikov, Pai, AR Adem, KR Adem, Moleleki, Magalakwe, Mogorosi, Nkwanazana, Matjila)

(4) Approximate symmetries and conservation laws of differential equations
(with Gazizov, Ünal, Wafo, Diatta)

(5) Contact symmetries of scalar ordinary differential equations
(with Ibragimov, Mahomed)

(6) Application of symmetries to problems in Fluid Mechanics
(with Hayat, Asghar, Pakdemirli, Aksoy, Hanif, Matebese, AR Adem, Najama)

(7) Application of symmetries to problems in Mathematical Finance
(with Molati, Motsepa, Lekalalaka)

(8) Application of symmetries to problems in Astrophysics
(with Muatjetjeja, Mahomed, Molati, Ntsime)

(9) Fractional differential equations
(with Jafari, Baleanu, Chun, Yousefi, Firoozjaee, Momanic)

(10) Stochastic differential equations
(with Ünal, Iyigunler, Turkeri)

(11) Soliton solutions
(with Biswas, AR Adem, KR Adem, Moleleki)

(12) Industrial Mathematics Problems
(with Ockendon, Stacey, Please, Mason, Anthonyrajah, Mureithi, Ngnotchouye, Hutchinson, van der Merwe, Yilmaz, Fanucchi, Foy, Oliphant)

(13) Exact solutions of partial differential equations using various techniques
(with AR Adem, KR Adem, Moleleki, Magalakwe, Mogorosi, Nkwanazana, Mhlanga, Mothibi, Matjila)

5 Lane-Emden-Flower type equations

5.1 Introduction

In the study of stellar structure the Lane-Emden equation [44, 45]

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^r = 0, \tag{5.1}$$

where r is a constant, models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. This equation was proposed by Lane [44] (see also [46]) and studied in detail by Emden [45]. Fowler [47,48] considered a generalization of Eq. (5.1), called Emden-Fowler equation [49], where the last term is replaced by $x^{\nu-1}y^r$.

The Lane-Emden equation (5.1) also models the equilibria of nonrotating fluids in which internal pressure balances self-gravity. When spherically symmetric solutions of Eq. (5.1) appeared in [50], they got the attention of astrophysicists. In the latter half of the twentieth century, some interesting applications of the isothermal solution (singular isothermal sphere) and its nonsingular modifications were used in the structures of collisionless systems such as globular clusters and early-type galaxies [51, 52].

The work of Emden [45] also got the attention of physicists outside the field of astrophysics who investigated the generalized polytropic forms of the Lane-Emden equation (5.1) for specific polytropic indices r. Some singular solutions for $r = 3$ were produced by Fowler [47,48] and the Emden-Fowler equation in the literature was established, while the works of Thomas [53] and Fermi [54] resulted in the Thomas-Fermi equation, used in atomic theory. Both of these equations, even today, are being investigated by physicists and mathematicians. Other applications of Eq. (5.1) can be found in the works of Meerson et al [55], Gnutzmann and Ritschel [56], and Bahcall [57,58].

Many methods, including numerical and perturbation, have been used to solve Eq. (5.1). The reader is referred to the works of Horedt [59, 60], Bender [61] and Lema [62, 63], Roxbough and Stocken [64], Adomian et al [65], Shawagfeh [66], Burt [67], Wazwaz [68] and Liao [69] for a sample. Exact solutions of Eq. (5.1) for $r = 0, 1$ and 5 have been obtained (see for example Chandrasekhar [50], Davis [70], Datta [71] and Wrubel [72]). Usually, for $r = 5$, only a one-parameter family of solutions is presented. A more general form of (5.1), in which the coefficient of y' is considered an arbitrary function of x, was investigated for first integrals by Leach [73].

Many problems in mathematical physics and astrophysics can be formulated by the generalized Lane-Emden equation

$$\frac{d^2y}{dx^2} + \frac{n}{x}\frac{dy}{dx} + f(y) = 0, \tag{5.2}$$

where n is a real constant and $f(y)$ is an arbitrary function of y. For $n = 2$ the approximate analytical solutions to the Eq. (5.2) were studied by Wazwaz [68] and Dehghan and Shakeri [74].

One form of $f(y)$ is given by

$$f(y) = (y^2 - C)^{3/2}. \tag{5.3}$$

Inserting (5.3) into Eq. (5.1) gives us the "white-dwarf" equation introduced by Chandrasekhar [50] in his study of the gravitational potential of degenerate white-dwarf stars. In fact, when $C = 0$ this equation reduces to Lane-Emden equation with index $r = 3$.

Another nonlinear form of $f(y)$ is the exponential function

$$f(y) = e^y. \tag{5.4}$$

Substituting (5.4) into Eq. (5.1) results in a model that describes isothermal gas spheres where the temperature remains constant.

Equation (5.1) with

$$f(y) = e^{-y}$$

gives a model that appears in the theory of thermionic currents when one seeks to determine the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium was thoroughly investigated by Richardson [75].

Furthermore, the Eq. (5.1) appears in eight additional cases for the function $f(y)$. The interested reader is referred to Davis [70] for more detail.

The equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + e^{\beta y} = 0, \tag{5.5}$$

where β is a constant, has also been studied by Emden [45]. In a recent work [76] an approximate implicit solution has been obtained for Eq. (5.5) with $\beta = 1$.

Furthermore, more general Emden-type equations were considered in the works [77–81]. See also the review paper by Wong [82], which contains more than 140 references on the topic.

The so-called generalized Lane-Emden equation of the first kind

$$x\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta x^{\nu} y^{n} = 0, \tag{5.6}$$

and generalized Lane-Emden equation of the second kind

$$x\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta x^{\nu} e^{ny} = 0, \tag{5.7}$$

where α, β, ν and n are constants, have been recently studied in [83,84]. In Goenner [84], the author uncovered symmetries of Eq. (5.6) to explain integrability of (5.6) for certain values of the parameters considered in Goenner and Havas [83]. Recently, the integrability of the generalized Lane-Emden equations of the first and second kinds has been discussed in Muatjetjeja and Khalique [85].

Here, firstly, a generalized Lane-Emden-Fowler type equation

$$x\frac{d^2y}{dx^2} + n\frac{dy}{dx} + x^\nu f(y) = 0, \tag{5.8}$$

where n and ν are real constants and $f(y)$ is an arbitrary function of y will be studied. We perform the Noether symmetry analysis of this problem. It should be noted that Eq. (5.8) for the power function $F(y) = y^r$ is related to the Emden-Fowler equation $y'' + p(X)y^r = 0$ by means of the transformation on the independent variable $X = x^{1-n}$, $n \neq 1$ and $X = \ln x$, $n = 1$.

Secondly, we consider a generalized coupled Lane-Emden system, which occurs in the modelling of several physical phenomena such as pattern formation, population evolution and chemical reactions. We perform Noether symmetry classification of this system and compute the Noether operators corresponding to the standard Lagrangian. In addition the first integrals for the Lane-Emden system will be constructed with respect to Noether operators.

5.2 Noether classification and integration of (5.8) for different fs

In this section we perform a Nother point symmetry classification of Eq. (5.8) with respect to the standard Lagrangian. We then obtain first integrals of the various cases, which admit Noether point symmetries and reduce the corresponding equations to quadratures.

It can easily be verified that a Lagrangian of Eq. (5.8) is given by

$$L = \frac{1}{2}x^n y'^2 - x^{n+\nu-1}\int f(y)dy. \tag{5.9}$$

The determining equation (see [86]) for the Noether point symmetries corresponding to L in (5.9) is

$$X^{[1]}(L) + LD(\xi) = D(B), \tag{5.10}$$

where X given by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{5.11}$$

is the generator of Noether symmetry and $B(x,y)$ is the gauge term and D is the total differentiation operator defined by [87]

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots . \tag{5.12}$$

After some calculations the solution of Eq. (5.10) results in

$$\xi = a(x),$$

$$\eta = \frac{1}{2}[a' - nx^{-1}a]y + b(x), \tag{5.13}$$

$$B = \frac{1}{4}x^n\left[a'' - n\left(\frac{a}{x}\right)'\right]y^2 + b'x^n y + c(x), \tag{5.14}$$

$$\begin{aligned}[-(n+\nu-1)x^{n+\nu-2}a - a'x^{n+\nu-1}]\int f(y)dy + [-\frac{1}{2}x^{n+\nu-1}a'y \\ +\frac{1}{2}nx^{n+\nu-2}ay - x^{n+\nu-1}b]f(y) = \frac{1}{4}a'''x^n y^2 + \frac{1}{2}nx^{n-2}a'y^2 \\ -\frac{1}{2}nx^{n-3}ay^2 - \frac{1}{4}n^2x^{n-1}\left(\frac{a}{x}\right)'y^2 + b''x^n y + b'nx^{n-1}y + c'(x).\end{aligned} \tag{5.15}$$

The analysis of Eq. (5.15) leads to the following eight cases:

Case 1. $n \neq \dfrac{1-\nu}{2}$, $f(y)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6.

We find that $\xi = 0$, $\eta = 0$, $B =$ constant and we conclude that there is no Noether point symmetry.

Noether point symmetries exist in the following cases.

Case 2. $n = \dfrac{1-\nu}{2}$, $f(y)$ arbitrary.

We obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B =$ constant. Therefore we have a single Noether symmetry generator $X = x^{\frac{1-\nu}{2}}\partial/\partial x$. For this case the integration is trivial even without a Noether symmetry. The Noetherian first integral [86] is

$$I = \frac{1}{2}x^{1-\nu}y'^2 + \int f(y)dy$$

from which, setting $I = C$, one gets quadrature.

Case 3. $f(y)$ is linear in y.

We have five Noether point symmetries associated with the standard Lagrangian for the corresponding differential equation (5.8) and $sl(3,\Re)$ symmetry algebra. This case is well-known, see, e.g., [88].

Case 4. $f = \alpha y^2 + \beta y + \gamma$, $\alpha \neq 0$

There are four subcases. They are as follows:

4.1. If $n = 2\nu + 3$, $\beta = 0$ and $\gamma = 0$, we obtain $\xi = x$, $\eta = -(\nu+1)y$ and $B =$ constant. This is contained in Case 5.1 below.

4.2. If $n = 2\nu + 3$, $\nu \neq -1$, $\beta^2 = 4\alpha\gamma$, we get $\xi = x$, $\eta = -(\nu+1)(y + \beta/2\alpha)$ and $B = \dfrac{\beta\gamma}{6\alpha}x^{3\nu+3}$. We have

$$X = x\frac{\partial}{\partial x} - (\nu+1)(y+\beta/2\alpha)\frac{\partial}{\partial y}.$$

In this case the Noetherian first integral [86] is

$$I = -\frac{1}{2}x^{2\nu+4}y'^2 - \frac{1}{3}\alpha x^{3\nu+3}y^3 - \frac{1}{2}\beta x^{3\nu+3}y^2 - \gamma x^{3\nu+3}y - (\nu+1)x^{2\nu+3}yy' \\ -(\nu+1)\frac{\beta}{2\alpha}x^{2\nu+3}y' - \frac{\beta\gamma}{6\alpha}x^{3\nu+3}.$$

Thus the reduced equation is

$$\frac{1}{2}x^{2\nu+4}y'^2 + \frac{1}{3}\alpha x^{3\nu+3}y^3 + \frac{1}{2}\beta x^{3\nu+3}y^2 + \gamma x^{3\nu+3}y + (\nu+1)x^{2\nu+3}yy' \\ +(\nu+1)\frac{\beta}{2\alpha}x^{2\nu+3}y' + \frac{\beta\gamma}{6\alpha}x^{3\nu+3} = C, \tag{5.16}$$

where C is an arbitrary constant. We now solve Eq. (5.16). For this purpose we use an invariant of X (see [89]) as the new dependent variable. This invariant is obtained by solving the Lagrange's system associated with X, viz.,

$$\frac{dx}{x} = \frac{dy}{-(\nu+1)(y+\beta/2\alpha)},$$

and is

$$u = x^{\nu+1} y + \frac{\beta}{2\alpha} x^{\nu+1}.$$

In terms of u Eq. (5.16) becomes

$$C = \frac{1}{2}(\nu+1)^2 u^2 - \frac{1}{2} x^2 u'^2 - \frac{1}{3}\alpha u^3,$$

which is a first-order variables separable ordinary differential equation. Separating the variables we obtain

$$\frac{du}{\pm\sqrt{(\nu+1)^2 u^2 - (2/3)\alpha u^3 - 2C}} = \frac{dx}{x}.$$

Hence we have quadrature or double reduction of our Eq. (5.8) for the given f.

4.3. If $n = (\nu+4)/3$, $n \neq (1-\nu)/2, -1$, $\beta = 0$ and $\gamma = 0$, we find $\xi = x^{(2-\nu)/3}$, $\eta = -\dfrac{\nu+1}{3} x^{-(\nu+1)/3} y$ and $B = \dfrac{(\nu+1)^2}{18} y^2 + k$, k a constant. This is subsumed in Case 5.2 below.

4.4. If $n = (1-\nu)/2$, $n \neq (\nu+4)/3$, β and γ are arbitrary, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$. This reduces to Case 2.

Case 5. $f = \alpha y^r$, $\alpha \neq 0, r \neq 0, 1$.
Here we have two subcases.

5.1. If $n = \dfrac{r+2\nu+1}{r-1}$, we obtain $\xi = x$, $\eta = \dfrac{\nu+1}{1-r} y$ and $B =$ constant. The solution of Eq. (5.8) for the above n and f is given by

$$y = u x^{\frac{\nu+1}{1-r}}, \tag{5.17}$$

where u satisfies

$$\int \frac{du}{\pm\sqrt{(\nu+1)^2 (1-r)^{-2} u^2 - 2\alpha (1+r)^{-1} u^{1+r} - 2C_1}} = \ln x C_2, \tag{5.18}$$

in which, C_1 and C_2 are arbitrary constants of integration.

We note that when $r = 5$ and $\nu = 1$, we get $n = 2$. This gives us the Lane-Emden equation $y'' + (2/x)y' + y^5 = 0$. Its general solution is given by Eq. (5.18) and we recover the solution given in [90]. Only a one-parameter family of solutions is known in the other literature, namely, $y = [3a/(x^2 + 3a^2)]^{1/2}$, a = constant (see, e.g., [70] or [91]). Here we have determined a two-parameter family of solutions. Another almost unknown exact solution of $y'' + (2/x)y' + y^5 = 0$, which is worth mentioning here, is given by

$$xy^2 = \left[1 + 3\cot^2\left(\frac{1}{2}\ln\frac{x}{c}\right)\right]^{-1}, \tag{5.19}$$

where c is an arbitrary constant.

5.2. If $n = \dfrac{r+\nu+2}{r+1}$, with $r \neq -1$, we have $\xi = x^{\frac{r-\nu}{r+1}}$, $\eta = -\left(\dfrac{\nu+1}{r+1}\right)x^{-\frac{\nu+1}{r+1}}\,y$ and $B = \dfrac{(\nu+1)^2}{2(r+1)^2}\,y^2 + k$, where k is a constant.

In this case the solution of the corresponding Eq. (5.8) is

$$y = ux^{-\frac{\nu+1}{r+1}}, \tag{5.20}$$

where u is given by

$$\int \frac{du}{\pm\sqrt{C_1 - 2\alpha(r+1)^{-1}u^{r+1}}} = \left(\frac{r+1}{\nu+1}\right)x^{\frac{\nu+1}{r+1}} + C_2, \tag{5.21}$$

in which, C_1 and C_2 are arbitrary constants.

5.3. If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and B = constant. This reduces to Case 2.

Case 6. $f = \alpha\exp(\beta y) + \gamma y + \delta$, $\alpha \neq 0, \beta \neq 0$.
Here again we have two subcases.

6.1. If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

6.2. If $n = 1$, $\nu \neq -1$, $\gamma = 0$ and $\delta = 0$, we deduce that $\xi = x$, $\eta = -(\nu+1)/\beta$ and $B = k$, k a constant.

The solution of the corresponding Eq. (5.8) for this case to be

$$y = \frac{\nu+1}{\beta}\ln\left(\frac{u}{x}\right), \tag{5.22}$$

where u is defined by

$$\int \frac{du}{\pm u\sqrt{1-2\alpha\beta(\nu+1)^{-2}u^{\nu+1}+2C_1\beta^2(\nu+1)^{-2}}} = \ln xC_2, \tag{5.23}$$

in which, C_1 and C_2 are integration constants.

Case 7. $f = \alpha \ln y + \gamma y + \delta$, $\alpha \neq 0$.
If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

Case 8. $f = \alpha y \ln y + \gamma y + \delta$, $\alpha \neq 0$.
If $n = \dfrac{1-\nu}{2}$, we obtain $\xi = x^{\frac{1-\nu}{2}}$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

5.3 Systems of Lane-Emden-Fowler equations

The modelling of several physical phenomena such as pattern formation, population evolution, chemical reactions, and so on (see, for example [92]), gives rise to the systems of Lane-Emden equations, and have attracted much attention in recent years. Several authors have proved existence and uniqueness results for the Lane-Emden systems [93,94] and other related systems (see, for example [95–97] and references therein). Here we consider the following generalized coupled Lane-Emden system [98]

$$\frac{d^2u}{dt^2} + \frac{n}{t}\frac{du}{dt} + f(v) = 0, \tag{5.24}$$

$$\frac{d^2v}{dt^2} + \frac{n}{t}\frac{dv}{dt} + g(u) = 0, \tag{5.25}$$

where n is real constant and $f(v)$ and $g(u)$ are arbitrary functions of v and u, respectively. Note that system (5.24)-(5.25) is a natural extension of the well-known Lane-Emden equation. We will classify the Noether operators and construct first integrals for this coupled Lane-Emden system.

It can readily be verified that a Lagrangian of the system (5.24)-(5.25) is

$$L = t^n \dot{u}\dot{v} - t^n \int f(v)dv - t^n \int g(u)du. \tag{5.26}$$

The determining equation (see [98]) for the Noether point symmetries corresponding to L in (5.26) is

$$X^{[1]}(L) + LD(\tau) = D(B), \tag{5.27}$$

where X is given by

$$X = \tau(t,u,v)\frac{\partial}{\partial t} + \xi(t,u,v)\frac{\partial}{\partial u} + \eta(t,u,v)\frac{\partial}{\partial v}, \tag{5.28}$$

with first extension [99]

$$X^{[1]} = X + (\dot{\xi} - \dot{u}\dot{\tau})\frac{\partial}{\partial \dot{u}} + (\dot{\eta} - \dot{v}\dot{\tau})\frac{\partial}{\partial \dot{v}}, \tag{5.29}$$

where $\dot{\tau}, \dot{\xi}$ and $\dot{\eta}$ denote total time derivatives of τ, ξ and η respectively. Proceeding as in Section 5.2, (see details of computations in [98]) we obtain the following seven cases:

Case 1. $n \neq 0$, $f(u)$ and $g(v)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6.

We find that $\tau = 0$, $\xi = 0$, $\eta = 0$, $B =$ constant and we conclude that there is no Noether point symmetry.

Noether point symmetries exist in the following cases.

Case 2. $n = 0$, $f(u)$ and $g(v)$ arbitrary.

We obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B =$ constant. Therefore we have a single Noether symmetry generator

$$X_1 = \frac{\partial}{\partial t} \tag{5.30}$$

with the Noetherian integral given by

$$I = \dot{u}\dot{v} + \int f(u)du + \int g(v)dv.$$

Case 3. $f(v)$ and $g(u)$ constants. We have eight Noether point symmetries associated with the standard Lagrangian for the corresponding system (5.24)-(5.25) and this case is well-known.

Case 4. $f = \alpha v + \beta, \quad g = \gamma u + \lambda,$ where α, β, γ and λ are constants, with $\alpha \neq 0$ and $\gamma \neq 0$.

There are three subcases, namely

4.1. For all values of $n \neq 0, 2$, we obtain $\tau = 0$, $\xi = a(t)$, $\eta = l(t)$ and $B = t^n \dot{l} u + t^n \dot{a} v - \lambda \int t^n a \, dt - \beta \int t^n l dt + C_1$, C_1 a constant. Therefore we obtain Noether point symmetry

$$X_1 = a(t)\frac{\partial}{\partial u} + l(t)\frac{\partial}{\partial v}, \tag{5.31}$$

where $a(t)$ and $l(t)$ satisfy the second-order coupled Lane-Emden system

$$\ddot{l} + \frac{n}{t}\dot{l} + \gamma a = 0, \; \ddot{a} + \frac{n}{t}\dot{a} + \alpha l = 0. \tag{5.32}$$

The first integral in this case is given by

$$I_1 \;\; = \;\; t^n \dot{l} u + t^n \dot{a} v - \lambda \int t^n a \, dt - \beta \int t^n l dt - a t^n \dot{v} - l t^n \dot{u}.$$

4.2. $n = 2$. In this subcase the Noether symmetries are X_1 given by the operator (5.31) and

$$X_2 = \frac{\partial}{\partial t} - u t^{-1}\frac{\partial}{\partial u} - v t^{-1}\frac{\partial}{\partial v}. \tag{5.33}$$

The value of B for the operator X_2 is given by $B = uv$.

The associated first integral for X_2 is given by

$$I_2 \;\; = \;\; uv + \frac{\alpha}{2}t^2 v^2 + \frac{\gamma}{2}t^2 u^2 + u t \dot{v} + v t \dot{u} + t^2 \dot{u}\dot{v}.$$

In this subcase, we note that the first integral corresponding to X_1 is subsumed in Case 4.1 above with β, $\lambda = 0$.

4.3. $n = 0$. Here the Noether operators are X_1 given by the operator (5.31) and

$$X_2 = \frac{\partial}{\partial t}, \text{ with } B = C_2, C_2 \text{ a constant.} \tag{5.34}$$

This reduces to Case 2.

We note also that the first integral associated with X_1 is contained in Case 4.1 above where $a(t)$ and $l(t)$ satisfy the coupled system

$$\ddot{l} + \gamma a = 0,\ \ddot{a} + \alpha l = 0. \tag{5.35}$$

Case 5. $f = \alpha v^r$, $g = \beta u^m$, $m \neq -1$ and $r \neq -1$ where α, β are constants, with $\alpha \neq 0$ and $\beta \neq 0$.

There are three subcases,viz.,

5.1. If $n = \dfrac{2m+2r+mr+3}{rm-1}$, $rm \neq 1, m \neq -1, m \neq 1$ and $r \neq -1$, we obtain $\tau = t$, $\xi = -\dfrac{(1+n)}{m+1}u$, $\eta = -\dfrac{(1+n)}{r+1}v$ and $B =$ constant.

Thus we obtain a single Noether point symmetry

$$X = t\frac{\partial}{\partial t} - \frac{(1+n)}{m+1}u\frac{\partial}{\partial u} - \frac{(1+n)}{r+1}v\frac{\partial}{\partial v} \tag{5.36}$$

with the associated first integral

$$I = \beta t^{n+1}\frac{u^{m+1}}{m+1} + \alpha t^{n+1}\frac{v^{r+1}}{r+1} + \frac{(n+1)}{m+1}t^n u\dot{v} + \frac{(n+1)}{r+1}t^n v\dot{u} + t^{n+1}\dot{u}\dot{v}.$$

We now consider the case when $m = -1$ and $r = -1$, in Case 5. Here we have two subcases

Case 5.2. $n = 0$, $(m = -1,\ r = -1)$.

This case provides us with two Noether symmetries namely,

$$X_1 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \text{ and } X_2 = \frac{\partial}{\partial t} \text{ with } B = 0 \text{ for both cases.} \tag{5.37}$$

We obtain the Noetherian first integrals corresponding to X_1 and X_2 as

$$I_1 = \dot{u}v - u\dot{v},\ I_2 = \dot{u}\dot{v} + \ln u + \ln v,$$

respectively.

Case 5.3. $n = -1$ $(m = -1,\ r = -1)$.

Here we obtain two Noether symmetry operators, viz.,

$$\begin{aligned} X_1 &= u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \text{ with } B = 0 \text{ and } X_2 = t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} \\ &\quad \text{with } B = -2\ln t \end{aligned} \tag{5.38}$$

and first integrals associated with X_1 and X_2 are given by

$$I_1 = \dot{u}vt^{-1} - u\dot{v}t^{-1}, I_2 = -2\ln t + \ln u + \ln v - 2u\dot{v}t^{-1} + \dot{u}\dot{v},$$

respectively.

Case 6. $f = \alpha\exp(\beta v) + \lambda,\ \ g = \delta\exp(\gamma u) + \sigma,\ \ \alpha, \beta, \lambda, \gamma, \delta,$ and σ are constants, with $\alpha \neq 0, \beta \neq 0,\ \delta \neq 0, \gamma \neq 0$.

There are two subcases. They are

6.1. If $n = 1$, $\lambda = 0$ and $\sigma = 0$, we obtain $\tau = t$, $\xi = -\dfrac{2}{\gamma}$, $\eta = -\dfrac{2}{\beta}$ and $B = C_3$, C_3 a constant. Therefore we have a single Noether point symmetry

$$X_1 = t\frac{\partial}{\partial t} - \frac{2}{\gamma}\frac{\partial}{\partial u} - \frac{2}{\beta}\frac{\partial}{\partial v} \tag{5.39}$$

and this results in the first integral

$$I = t^2\dot{u}\dot{v} + \frac{\alpha t^2}{\beta}\exp(\beta v) + \frac{\delta t^2}{\gamma}\exp(\gamma u) + \frac{2}{\gamma}t\dot{v} + \frac{2}{\beta}t\dot{u}.$$

6.2. If $n = 0$, $\lambda = 0$ and $\sigma = 0$, we deduce that $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = C_4$, C_4 a constant. The Noether operator is given by

$$X_1 = \frac{\partial}{\partial t}. \tag{5.40}$$

This reduces to Case 2.

Case 7. $f = \alpha \ln v + \beta$, $g = \gamma \ln u + \lambda$, where α, β, γ and λ are constants with $\alpha \neq 0$, $\gamma \neq 0$.
If $n = 0$, we obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = C_5$, C_5 a constant. This reduces to Case 2.

6 Soil Water Redistribution and Extraction Flow Models: Conservation Laws

6.1 Introduction

A mathematical model with three arbitrary elements was proposed by [100] (see also [101]) to simulate soil water infiltration, redistribution, and extraction in a bedded soil profile overlaying a shallow water table and irrigated by a line source drip irrigation system. The partial differential equation that models this system is given by [100, 101]

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)(\psi_z - 1))_z - S(\psi), \tag{6.41}$$

where ψ is soil moisture pressure head, $C(\psi)$ is specific water capacity, $K(\psi)$ is unsaturated hydraulic conductivity, $S(\psi)$ is a sink or source term, t is time, x is the horizontal and z is the vertical axis which is considered positive downward.

The equation (6.41) has attracted a fair amount of attention from various quarters. The symmetry group classification of equation (6.41) with respect to admissible point transformation groups was done by [102] (see, e.g. references [13] and [87] for an account on symmetry methods). This paper also contains some exact solutions, viz., invariant solutions for particular functions in (6.41). Further work on invariant and asymptotic invariant solutions, inspired by [102], was obtained by [103] for certain two-dimensional subalgebras of the symmetry algebra of (6.41).

Special cases of (6.41) were considered for conservation laws in [104].

Here we find all the nontrivial conservation laws for equation (6.41). In the general case, that is for arbitrary elements, equation (6.41) has trivial conservation laws. We also provide all the classes of equations (6.41) that admit point symmetry algebras and for which nontrivial conservation laws exist.

6.2 Conservation laws

We find conservation laws for equation (6.41). The conservation law relation

$$D_t T^1 + D_x T^2 + D_z T^3 = 0 \tag{6.42}$$

which holds on the solutions of (6.41) is invoked. Note that one cannot use Noether's theorem [32] here as there is no Lagrangian for (6.41) as it is a scalar evolution equation.

The solution of (6.42) after some amount of work yields the conserved vector components

$$\begin{aligned}
T^1 &= -a(t,x,z)\psi_x + b(t,x,z)\psi_z + A(t,x,z)\int C(\psi)d\psi + B(t,x,z),\\
T^2 &= a(t,x,z)\psi_t - K(\psi)\psi_x A(t,x,z) - c(t,x,z,\psi)\psi_z + d(t,x,z,\psi),\\
T^3 &= -b(t,x,z)\psi_t - K(\psi)\psi_z A(t,x,z) + c(t,x,z,\psi)\psi_x\\
&\quad +e(t,x,z,\psi),
\end{aligned} \tag{6.43}$$

where the functions a to B satisfy

$$\begin{aligned}
\frac{\partial a}{\partial x} - \frac{\partial b}{\partial z} &= 0,\\
\frac{\partial a}{\partial t} + K(\psi)\frac{\partial A}{\partial x} - \frac{\partial d}{\partial \psi} - \frac{\partial c}{\partial z} &= 0,\\
\frac{\partial b}{\partial t} - K'(\psi)A - \frac{\partial c}{\partial x} - K(\psi)\frac{\partial A}{\partial z} + \frac{\partial e}{\partial \psi} &= 0,\\
\frac{\partial A}{\partial t}\int C(\psi)d\psi + \frac{\partial B}{\partial t} - S(\psi)A + \frac{\partial d}{\partial x} + \frac{\partial e}{\partial z} &= 0.
\end{aligned} \tag{6.44}$$

The equations (6.44) imply the classifying relation

$$K(\psi)\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2}\right) + K'(\psi)\frac{\partial A}{\partial z} - S'(\psi)A + C(\psi)\frac{\partial A}{\partial t} = 0. \tag{6.45}$$

The invocation of (6.45) helps us to classify all conserved vectors for equation (6.41).

It is clear that for arbitrary $C(\psi)$, $K(\psi)$ and $S(\psi)$ we obtain trivial conservation laws since $A = 0$. Now since $K(\psi) \neq 0$, we divide equation (6.45) by $K(\psi)$ and differentiate the resultant equation with respect to ψ. This gives rise to two cases, viz., $K'(\psi)/K(\psi)$ constant and $K'(\psi)/K(\psi)$ not a constant. Each of these two cases further results into three subcases. The calculations are straight forward and are summarized below:

1. For arbitrary $C(\psi), K(\psi)$ and $S(\psi)$ we obtain trivial conservation laws since $A = 0$.

Nontrivial conservation laws are found in the following cases:

2. $C(\psi) = c_0 \exp(\alpha\psi)$, $K(\psi) = k_0 \exp(\alpha\psi)$, $S(\psi) = s_0 + s_1\psi$ if $\alpha = 0$ or $S(\psi) = (s_1/\alpha)\exp(\alpha\psi) + s_0$ if $\alpha \neq 0$, where α, $c_0 \neq 0$, $k_0 \neq 0$, s_0 and s_1 are arbitrary constants. The function A satisfies

$$(A_{xx} + A_{zz})k_0 + k_0\alpha A_z - s_1 A + c_0 A_t = 0$$

together with the system (6.44) being satisfied.

3. $C(\psi)$ arbitrary, $K(\psi) = k_0 \exp(\alpha\psi)$, $S(\psi) = s_0 + s_1\psi$ if $\alpha = 0$ or $S(\psi) = (s_1/\alpha)\exp(\alpha\psi) + s_0$ if $\alpha \neq 0$, where α, $k_0 \neq 0$, s_0 and s_1 are arbitrary constants. The function A satisfies

$$(A_{xx} + A_{zz})k_0 + k_0\alpha A_z - s_1 A = 0, \quad A_t = 0$$

together with the system (6.44).

4. $S(\psi)$ arbitrary, $K(\psi) = k_0 \exp(\alpha\psi)$, $C(\psi) = c_0 S'(\psi) + c_1 \exp(\alpha\psi)$, where $k_0, c_0 \neq 0$, α and c_1 are arbitrary constants and

$$A = \exp(t/c_0) f(x, z),$$

where f solves $(f_{xx} + f_{zz})k_0c_0 + \alpha k_0 c_0 f_z + c_1 f = 0$. The system (6.44) must also hold.

5. $K(\psi)$ arbitrary with $K(\psi) \neq k_0 \exp(\alpha\psi)$, $C(\psi) = c_0 K'(\psi) + c_1 K(\psi)$, $S(\psi) = s_0 + s_1 K(\psi) + s_2 \int K(\psi) d\psi$, where α, k_0, c_0, c_1, s_0, s_1 and s_2 are arbitrary constants and

$$A = \exp(s_1 z) g(t - c_0 z, x),$$

where g solves $g_{xx}+c_0^2g_{\tau\tau}+(c_1-2c_0s_1)g_\tau+(s_1^2-s_2)g=0$, $\tau=t-c_0z$. The system (6.44) needs to be satisfied as well.

6. $K(\psi)$ arbitrary with $K(\psi)\neq k_0\exp(\alpha\psi)$, $C(\psi)\neq c_0K'(\psi)+c_1K(\psi)$, $S(\psi)=s_0+s_1K(\psi)+s_2\int K(\psi)d\psi$, where α, k_0, c_0, c_1, s_0, s_1 and s_2 are arbitrary constants and

$$A=\exp(s_1z)f(x),$$

where f solves $f''+(s_1^2-s_2)f=0$. In addition A must satisfy system (6.44).

7. $K(\psi)$ arbitrary with $K(\psi)\neq k_0\exp(\alpha\psi)$, $C(\psi)=c_0S'(\psi)+c_1K'(\psi)+c_2K(\psi)$, $S(\psi)\neq s_0+s_1K(\psi)+s_2\int K(\psi)d\psi$, where α, k_0, c_0, c_1, s_0, s_1 and s_2 are arbitrary constants and

$$A=g(x)\exp((t-c_1z)/c_0),$$

where g solves $g''+\dfrac{c_1^2+c_0c_2}{c_0^2}g=0$. The system (6.44) needs to hold as well.

In the above, only the Cases **6** and **7** give rise to two nontrivial conservation laws each. The other cases each result in an infinite number of nontrivial conservation laws. We now present details for each case.

For Case **2** with $\alpha=0$, the solution of the system (6.44) with the choices of the functions $a=b=c=B=0$ gives

$$\begin{aligned} T^1 &= c_0\psi A(t,x,z), \\ T^2 &= -k_0\psi_xA+k_0\psi A_x, \\ T^3 &= -k_0\psi_zA+k_0\psi A_z+s_0\int Adz \end{aligned}$$

where A satisfies the second-order PDE $(A_{xx}+A_{zz})k_0-s_1A+c_0A_t=0$. Thus we obtain an infinite number of conservation laws for this case.

When $\alpha\neq 0$, the solution of the system (6.44) with the same choices of the functions a, b, c and B as above, yields

$$\begin{aligned} T^1 &= \frac{c_0}{\alpha}A(t,x,z)\exp(\alpha\psi), \\ T^2 &= -k_0\psi_xA\exp(\alpha\psi)+\frac{k_0}{\alpha}A_x\exp(\alpha\psi)+s_0\int Adx, \\ T^3 &= -k_0\psi_zA\exp(\alpha\psi)+\frac{k_0}{\alpha}A_z\exp(\alpha\psi)+k_0A\exp(\alpha\psi) \end{aligned}$$

where A solves $(A_{xx}+A_{zz})k_0+k_0\alpha A_z-s_1A+c_0A_t=0$ and this results in an infinite number of conservation laws.

Similarly for Case **3** with $\alpha=0$, the solution of the system (6.44) with the choices $a=b=c=B=0$ gives

$$\begin{aligned}
T^1 &= A(x,z)\int C(\psi)d\psi,\\
T^2 &= -k_0\psi_xA+k_0\psi A_x,\\
T^3 &= -k_0\psi_zA+k_0\psi A_z+s_0\int Adz
\end{aligned}$$

where A is a solution of $(A_{xx}+A_{zz})k_0-s_1A(x,z)=0$ and we have an infinite number of conservation laws for this case.

However, when $\alpha\neq 0$, the solution of the system (6.44) with the previous choices of the functions a,b,c and B, yields

$$\begin{aligned}
T^1 &= A(x,z)\int C(\psi)d\psi,\\
T^2 &= -k_0\psi_xA\exp(\alpha\psi)+\frac{k_0}{\alpha}A_x\exp(\alpha\psi)+s_0\int A(x,z)dx,\\
T^3 &= -k_0\psi_zA\exp(\alpha\psi)+\frac{k_0}{\alpha}A_z\exp(\alpha\psi)+k_0A\exp(\alpha\psi)
\end{aligned}$$

where A satisfies $(A_{xx}+A_{zz})k_0+k_0\alpha A_z-s_1A=0,\ A_t=0$ and this again results in an infinite number of conservation laws.

Now for Case **4** with the selection $a=b=c=B=0$, the solution of the system (6.44), yields

$$\begin{aligned}
T^1 &= \left[c_0S(\psi)+\frac{c_1}{\alpha}\exp(\alpha\psi)\right]\exp(t/c_0)f(x,z),\\
T^2 &= k_0\left[\frac{1}{\alpha}f_x-f(x,z)\right]\exp(\alpha\psi+t/c_0),\\
T^3 &= k_0\left[\frac{1}{\alpha}f_z+f(x,z)-f\psi_z\right]\exp(\alpha\psi+t/c_0)
\end{aligned}$$

where f is a solution of $(f_{xx}+f_{zz})k_0c_0+k_0c_0\alpha f_z+c_1f=0$. This results in an infinite number of conservation laws.

For Case **5** we choose the functions $a = b = c = B = 0$ and the solution of the system (6.44) gives

$$
\begin{aligned}
T^1 &= \left[c_0 K(\psi) + c_1 \int K(\psi) d\psi\right] g(x,\tau) \exp(s_1 z), \\
T^2 &= \left[- K(\psi)\psi_x g + g_x \int K(\psi) d\psi + s_0 \int g dx\right] \exp(s_1 z), \\
T^3 &= \left[- K(\psi)\psi_z + K(\psi) + s_1 \int K(\psi) d\psi\right] g \exp(s_1 z)
\end{aligned}
$$

where g solves $g_{xx} + c_0^2 g_{\tau\tau} + (c_1 - 2c_0 s_1) g_\tau + (s_1^2 - s_2) g = 0$, $\tau = t - c_0 z$, which results in an infinite number of conservation laws.

For Case **6** with the selection $a = b = c = B = 0$, the solution of the system (6.44), yields

$$
\begin{aligned}
T^1 &= \left[\int C(\psi) d\psi\right] f(x) \exp(s_1 z), \\
T^2 &= \left[- K(\psi)\psi_x f + f'(x) \int K(\psi) d\psi + s_0 \int f(x) dx\right] \exp(s_1 z), \\
T^3 &= \left[- K(\psi)\psi_z + K(\psi) + s_1 \int K(\psi) d\psi\right] f(x) \exp(s_1 z)
\end{aligned}
$$

where f satisfies $f'' + (s_1^2 - s_2) f = 0$. Thus these components yield two conserved vectors.

For Case **7** we choose $a = b = c = B = 0$ and the solution of the system (6.44) yields

$$
\begin{aligned}
T^1 &= \left[c_0 S(\psi) + c_1 K(\psi) + c_2 \int K(\psi) d\psi\right] g(x) \exp\left(\frac{t - c_1 z}{c_0}\right), \\
T^2 &= \left[- K(\psi)\psi_x g(x) + g'(x) \int K(\psi) d\psi\right] \exp\left(\frac{t - c_1 z}{c_0}\right), \\
T^3 &= \left[- K(\psi)\psi_z + K(\psi) - \frac{c_1}{c_0} \int K(\psi) d\psi\right] g(x) \exp\left(\frac{t - c_1 z}{c_0}\right)
\end{aligned}
$$

where g satisfies $g'' + \dfrac{c_1^2 + c_2 c_0}{c_0^2} g = 0$ and so we obtain two nontrivial conservation laws for this case.

The equations in the group classification of the soil water equations (6.41) which possess nontrivial conservation laws are (the notation utilized in the following corresponds to that used in [102])

I. $K(\psi) = 1$.
1. $C(\psi)$ arbitrary, $S(\psi) = 0$.
2. $C(\psi) = \psi^\sigma$, (i) $S(\psi) = B\psi$, $B = 0$ constant, (ii) (a) $S(\psi) = B\psi^{\sigma+1} + D\psi$, $B \neq 0$, $D \neq 0$, $\sigma \neq -1$ constants, (ii) (b) $S(\psi) = B\psi^{\sigma+1}$, $B \neq 0$, $\sigma \neq -1$ constants, (iii) $S(\psi) = 0$.
3. $C(\psi) = \exp(\psi)$, (ii) $S(\psi) = B\exp(\psi) + D$, $B, D \neq 0$ constants, (iii) $S(\psi) = B\exp(\psi)$, B constant.
4. $C(\psi) = 1$, (iv) $S(\psi) = -\delta\psi$, $\delta = 0, \pm 1$, (v) $S(\psi) = \pm 1$.

II. $K(\psi) = C(\psi)$.
1. $C(\psi) = \sinh^{-2}\psi$, (ii) $S(\psi) = B\coth\psi$, $B \neq 0$ constant, (iii) $S(\psi) = B$, B constant.
2. $C(\psi) = \cosh^{-2}\psi$, (ii) $S(\psi) = -B\tanh\psi$, $B \neq 0$ constant, (iii) $S(\psi) = -B$, B constant.
3. $C(\psi) = M - Au(\ln u - 1)$, $A \neq 0$, M constants and $\psi = \int du/[M - Au(\ln u - 1)]$, (i) $S(\psi) = -u(B\ln u + D)$, B, D constants.

III. $K(\psi) = \exp(-\psi)$, $C(\psi) = \exp(-\psi)(M - \exp(-\psi))^\sigma$, $S(\psi) = -D(M - \exp(-\psi))^{\sigma+1} + B(\exp(-\psi) - M)$ for B, D, M, $\sigma \neq -1$ constants.

IV. $K(\psi) = M - A\exp(u)$, $C(\psi) = M\exp(-u) - A$, $S(\psi) = B\exp u$, $A \neq 0$, B, M constants, and $\psi = \int_{u_0}^{\exp u}(M - Aw)^{-1}dw$.

VI. $K(\psi) = M - Au^{\mu+1}$, $C(\psi) = (M - Au^{\mu+1})u^{-\sigma}$, $S(\psi) = -Bu^{1+2\mu-\sigma}$, $A \neq 0$, B, M, $\sigma \neq 0, -1$, $\mu \neq -1$ constants and for $\sigma = \mu$ or $\sigma = 2\mu$, where $\psi = \frac{1}{\sigma+1}\int_{u_0}^{u^{\sigma+1}}(M - Aw^{(\mu+1)/(\sigma+1)})^{-1}dw$.

7 Solutions and conservation laws of a coupled Kadomtsev-Petviashvili system

7.1 Introduction

Nonlinear partial differential equations (NLPDEs) describe a variety of physical phenomena in the fields such as physics, chemistry, biology, fluid dynamics, etc. Thus finding solutions of such NLPDEs is inevitable. However, determining solutions of NLPDEs is a very difficult task and only in certain cases one can obtain exact solutions. In recent years a number of methods have been proposed for obtaining exact solutions of NLPDEs. Some of the most important methods found in the literature include the inverse scattering transform method, Hirota's bilinear method, the homogeneous balance method, the Bäcklund transformation, the Darboux transformation method, the Fan sub-equation method, the new test function method, the (G'/G)-expansion method, Lie symmetry method, etc. [14–16, 105–113].

The celebrated Korteweg-de Vries (KdV) equation [114]

$$u_t + 6uu_x + u_{xxx} = 0,$$

governs the dynamics of solitary waves. It was derived to describe shallow water waves of long wavelength and small amplitude. It is an important equation from the view point of integrable systems because it has infinite number of conservation laws, gives multiple-soliton solutions, has bi-Hamiltonian structures, a Lax pair, and has many other physical properties [115].

The Kadomtsev-Petviashvili (KP) equation [116] extends the KdV equation and is given by

$$\left(u_t + 6uu_x + u_{xxx}\right)_x + u_{yy} = 0.$$

The KP equation is a model for shallow long waves in the x-direction with some mild dispersion in the y-direction. It is completely integrable by the inverse scattering transform method and gives multiple-soliton solutions.

In [117] a new coupled KdV system

$$u_t + u_{xxx} + 3uu_x + 3ww_x = 0, \tag{7.46a}$$

$$v_t + v_{xxx} + 3vv_x + 3ww_x = 0, \tag{7.46b}$$

$$w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0 \tag{7.46c}$$

was formally derived and examined by using the generalized bi-Hamiltonian structures with the aid of the trace identity. Here we consider the coupled KdV system (7.46) formulated in the KP sense, which is given by [118]

$$\left(u_t + u_{xxx} + 3uu_x + 3ww_x\right)_x + u_{yy} = 0, \tag{7.47a}$$

$$\left(v_t + v_{xxx} + 3vv_x + 3ww_x\right)_x + v_{yy} = 0, \tag{7.47b}$$

$$\left(w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right)_x + w_{yy} = 0 \tag{7.47c}$$

and obtain exact solutions of the system. We also derive conservation laws for (7.47).

7.2 Symmetries and exact solutions of (7.47)

The symmetry group of the coupled KdV system (7.47) will be generated by the vector field of the form

$$\begin{aligned} X \;=\; & \xi^1(t,x,y,u,v,w)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v,w)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v,w)\frac{\partial}{\partial y} \\ & + \eta^1(t,x,y,u,v,w)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v,w)\frac{\partial}{\partial v} + \eta^3(t,x,y,u,v,w)\frac{\partial}{\partial w}. \end{aligned}$$

By applying the fourth prolongation $\mathrm{pr}^{(4)}X$ [15] to (7.47), we obtain an overdetermined system of linear partial differential equations (PDEs). The general solution of the overdetermined system of linear PDEs is given by

$$\xi^1(t,x,y,u,v,w) = 18F_1(t),$$
$$\xi^2(t,x,y,u,v,w) = -3y^2F_1''(t) + 3F_3(t) - 3yF_2'(t) + 6xF_1'(t),$$
$$\xi^3(t,x,y,u,v,w) = 12yF_1'(t) + 6F_2(t),$$
$$\eta^1(t,x,y,u,v,w) = -2C_1w - y^2F_1'''(t) + F_3(t) - yF_2''(t) - 12uF_1'(t) + 2xF_1''(t),$$
$$\eta^2(t,x,y,u,v,w) = -y^2F_1'''(t) + F_3'(t) - yF_2''(t) + 2wC_1 - 12vF_1'(t) + 2xF_1''(t),$$
$$\eta^3(t,x,y,u,v,w) = (u-v)C_1 - 12wF_1'(t),$$

where C_1 is an arbitrary constant and F_1, F_2 and F_3 are arbitrary functions of t. For simplicity we restrict these arbitrary functions to arbitrary constants C_2, C_3 and C_4 respectively. As a result we obtain the 4-dimensional Lie algebra spanned by the following linearly independent operators:

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = -2w\frac{\partial}{\partial u} + 2w\frac{\partial}{\partial v} + (u-v)\frac{\partial}{\partial w}.$$

7.3 Symmetry reduction of (7.47)

We now make use of the symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ and reduce (7.47) to a system of nonlinear PDEs in two independent variables. The symmetry Γ yields the following five invariants:

$$f = t - y, \quad g = x - y, \quad \phi = u, \quad \psi = v, \quad \theta = w. \tag{7.48}$$

Treating ϕ, ψ, θ as the new dependent variables and f, g as new independent variables, (7.47) transforms to

$$3\phi_{fg} + \phi_{gggg} + 3{\phi_g}^2 + \phi_{gg} + 3\theta_{gg}\theta + \phi_{ff} + 3\phi_{gg}\phi + 3{\theta_g}^2 = 0,$$
$$3\psi_{fg} + \psi_{gggg} + 3{\psi_g}^2 + \psi_{gg} + 3\theta_{gg}\theta + \psi_{ff} + 3\psi_{gg}\psi + 3{\theta_g}^2 = 0,$$
$$3\theta_{fg} + \theta_{gggg} + 3\psi_g\theta_g + \frac{3}{2}\phi_{gg}\theta + \theta_{ff} + 3\phi_g\theta_g + \frac{3}{2}\psi_{gg}\theta$$
$$+\frac{3}{2}\theta_{gg}\phi + \frac{3}{2}\theta_{gg}\psi + \theta_{gg} = 0,$$

which is a system of nonlinear PDEs in two independent variables f and g. The above system has two translation symmetries, viz.,

$$\Upsilon_1 = \frac{\partial}{\partial g}, \quad \Upsilon_2 = \frac{\partial}{\partial f}.$$

By taking a linear combination $\Upsilon_1 + \rho\Upsilon_2$ (ρ is a constant) of the above symmetries, we see that it yields the four invariants

$$z = f - \rho g, \quad \phi = E, \quad \psi = F, \quad \theta = H.$$

Now treating E,F,H as new dependent variables and z as the new independent variable the above system transforms to the following system of nonlinear coupled ODEs:

$$\rho^4 E''''(z) + 3\rho^2 E(z)E''(z) + \rho^2 E''(z) - 3\rho E''(z) + E''(z)$$
$$+3\rho^2 E'(z)^2 + 3\rho^2 H(z)H''(z) + 3\rho^2 H'(z)^2 = 0, \tag{7.49a}$$
$$\rho^4 F''''(z) + 3\rho^2 F(z)F''(z) + \rho^2 F''(z) - 3\rho F''(z)$$
$$+F''(z) + 3\rho^2 F'(z)^2 + 3\rho^2 H(z)H''(z) + 3\rho^2 H'(z)^2 = 0, \tag{7.49b}$$
$$\rho^4 H''''(z) + \frac{3}{2}\rho^2 H(z)E''(z) + 3\rho^2 E'(z)H'(z) + \frac{3}{2}\rho^2 E(z)H''(z)$$
$$+\frac{3}{2}\rho^2 H(z)F''(z) + 3\rho^2 F'(z)H'(z) + \frac{3}{2}\rho^2 F(z)H''(z) + \rho^2 H''(z)$$
$$-3\rho H''(z) + H''(z) = 0. \tag{7.49c}$$

In the next two subsections we shall solve the above system of ODEs.

7.4 Solution of (7.47) using (G'/G)-expansion method

In this subsection we use the $(G'/G)-$expansion method [119] and obtain some exact solutions of the system of ODEs (7.49). This will result in the exact solutions of the coupled KP system (7.47).

Let us consider the solution of (7.49) in the form

$$E(z) = \sum_{i=0}^{M} \mathcal{A}_i \left(\frac{G'(z)}{G(z)}\right)^i, F(z) = \sum_{i=0}^{M} \mathcal{B}_i \left(\frac{G'(z)}{G(z)}\right)^i, H(z) = \sum_{i=0}^{M} \mathcal{C}_i \left(\frac{G'(z)}{G(z)}\right)^i, \tag{7.50}$$

where $G(z)$ satisfies the linear second-order ODE with constant coefficients, viz.,

$$G'' + \lambda G' + \mu G = 0, \tag{7.51}$$

where λ and μ are constants. The positive integer M will be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (7.49). $\mathcal{A}_0, \cdots, \mathcal{A}_M$, $\mathcal{B}_0, \cdots, \mathcal{B}_M$ and $\mathcal{C}_0, \cdots, \mathcal{C}_M$, are parameters to be determined.

Application of the balancing procedure to the system of ODEs, yields $M = 2$, so the solutions of (7.49) are of the form

$$E(z) = \mathcal{A}_0 + \mathcal{A}_1\left(\frac{G'(z)}{G(z)}\right) + \mathcal{A}_2\left(\frac{G'(z)}{G(z)}\right)^2, \tag{7.52a}$$

$$F(z) = \mathcal{B}_0 + \mathcal{B}_1\left(\frac{G'(z)}{G(z)}\right) + \mathcal{B}_2\left(\frac{G'(z)}{G(z)}\right)^2, \tag{7.52b}$$

$$H(z) = \mathcal{C}_0 + \mathcal{C}_1\left(\frac{G'(z)}{G(z)}\right) + \mathcal{C}_2\left(\frac{G'(z)}{G(z)}\right)^2. \tag{7.52c}$$

Substituting (7.52) into (7.49) and making use of (7.51) leads to an overdetermined system of algebraic equations. Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\begin{aligned}
\mathcal{A}_1 &= -2\lambda\rho^2, \\
\mathcal{A}_2 &= -2\rho^2, \\
\mathcal{B}_0 &= \mathcal{A}_0, \\
\mathcal{B}_1 &= -2\lambda\rho^2, \\
\mathcal{B}_2 &= -2\rho^2, \\
\mathcal{C}_0 &= \frac{3\mathcal{A}_0\rho^2 + \lambda^2\rho^4 + 8\mu\rho^4 + \rho^2 - 3\rho + 1}{3\rho^2}, \\
\mathcal{C}_1 &= \frac{2\left(3\mathcal{A}_0\lambda\rho^2 + \lambda^3\rho^4 + 8\lambda\mu\rho^4 + \lambda\rho^2 - 3\lambda\rho + \lambda\right)}{3\mathcal{C}_0}, \\
\mathcal{C}_2 &= \frac{\mathcal{C}_1}{\lambda}.
\end{aligned}$$

Now using the general solution of (7.51) in (7.52), we have the following three types of travelling wave solutions of the coupled KP equation (7.47):

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$
\begin{aligned}
u_1(t,x,y) &= \mathcal{A}_0 + \mathcal{A}_1\Big(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\Big) \\
&\quad + \mathcal{A}_2\Big(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\Big)^2, \qquad (7.53a) \\
v_1(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\Big(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\Big) \\
&\quad + \mathcal{B}_2\Big(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\Big)^2, \qquad (7.53b) \\
w_1(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\Big(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\Big) \\
&\quad + \mathcal{B}_2\Big(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\Big)^2, \qquad (7.53c)
\end{aligned}
$$

where $z = t - \rho x + (\rho - 1)y$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$
\begin{aligned}
u_2(t,x,y) &= \mathcal{A}_0 + \mathcal{A}_1\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big) \\
&\quad + \mathcal{A}_2\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big)^2, \\
v_2(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big) \\
&\quad + \mathcal{B}_2\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big)^2, \\
w_2(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big) \\
&\quad + \mathcal{B}_2\Big(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\Big)^2,
\end{aligned}
$$

where $z = t - \rho x + (\rho - 1)y$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$\begin{aligned} u_3(t,x,y) &= \mathcal{A}_0 + \mathcal{A}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathcal{A}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \\ v_3(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathcal{B}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \\ w_3(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathcal{B}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \end{aligned}$$

where $z = t - \rho x + (\rho - 1)y$, C_1 and C_2 are arbitrary constants.

A profile of the solution (7.53) is given in Figures 1-3.

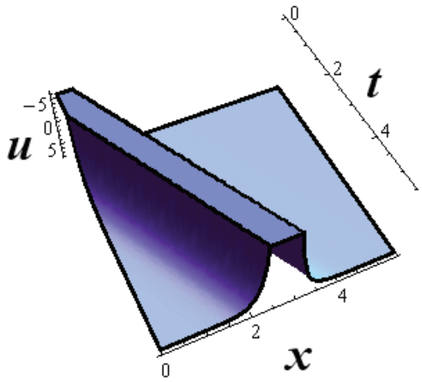

Figure 1: Evolution of travelling wave solution (7.53) with parameters $C_1 = 0$, $C_2 = 1$, $\lambda = 3$, $\mu = 1$, $\rho = 2$, $y = 0$, $\mathcal{A}_0 = 0$.

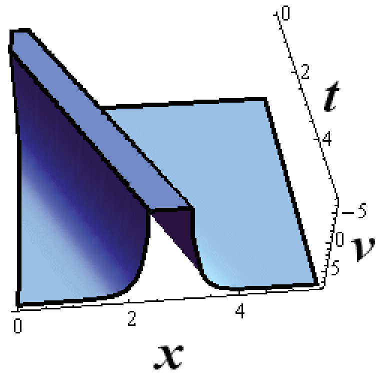

Figure 2: Evolution of travelling wave solution (7.53) with parameters $C_1 = 0$, $C_2 = 1$, $\lambda = 3$, $\mu = 1$, $\rho = 2$, $y = 0$, $\mathcal{A}_0 = 0$.

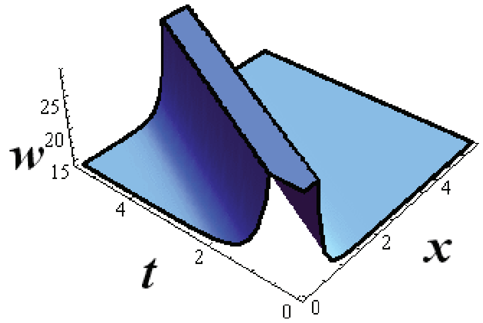

Figure 3: Evolution of travelling wave solution (7.53) with parameters $C_1 = 0$, $C_2 = 1$, $\lambda = 3$, $\mu = 1$, $\rho = 2$, $y = 0$, $\mathcal{A}_0 = 0$.

7.5 Conservation laws of (7.47)

In this section we construct conservation laws for (7.47) using the multiplier method [120].

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{7.54}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total derivative operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n, \tag{7.55}$$

with the summation convention used whenever appropriate [16].

The Euler-Lagrange operator, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{7.56}$$

The n-tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, ($\mathcal{A}$ is the space of differential functions) is a conserved vector of (7.54) if T^i satisfies

$$D_i T^i|_{(7.54)} = 0. \tag{7.57}$$

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that [15]

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{7.58}$$

holds identically. Here we will consider multipliers of the zeroth order, i.e., Λ_α will depend only on t, x, y, u, v, w. The determining equation for the multiplier Λ_α is [15]

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{7.59}$$

Once the multipliers are obtained, the conserved vectors are calculated via a homotopy formula [15, 120].

For the coupled KP system (7.47), we see that the three zeroth-order multipliers, namely $\Lambda_1 = \Lambda_1(t, x, y, u, v, w)$, $\Lambda_2 = \Lambda_2(t, x, y, u, v, w)$ and $\Lambda_3 = \Lambda_3(t, x, y, u, v, w)$ are given by

$$\begin{aligned}
\Lambda_1 &= xyf_1(t) + xf_2(t) - \frac{1}{6}y^3 f_1'(t) - \frac{1}{2}y^2 f_2'(t) + yf_7(t) + f_8(t), \\
\Lambda_2 &= xyf_3(t) + xf_4(t) - \frac{1}{6}y^3 f_3'(t) - \frac{1}{2}y^2 f_4'(t) + yf_{11}(t) + f_{12}(t), \\
\Lambda_3 &= xyf_5(t) + xf_6(t) - \frac{1}{6}y^3 f_5'(t) - \frac{1}{2}y^2 f_6'(t) + yf_9(t) + f_{10}(t),
\end{aligned}$$

where f_i, $i = 1, \cdots, 12$ are arbitrary functions of t.

Therefore, corresponding to the above multipliers we have the following twelve conserved vectors of (7.47):

$$\begin{aligned}
T_1^t &= \frac{1}{12}\Big\{ -6yf_1(t)u + 6xyf_1(t)u_x + y^3\left(-f_1'(t)\right)u_x \Big\},\\
T_1^x &= \frac{1}{12}\Big\{ -6y^3 f_1'(t)u_x u + 36xyf_1(t)u_x u - 6y^3 f_1'(t)w_x w + 36xyf_1(t)w_x w + y^3 f_1{}''(t)u\\
&\quad -6xyf_1'(t)u - 18yf_1(t)u^2 - 18yf_1(t)w^2 - 12yf_1(t)u_{xx} + 12xyf_1(t)u_{xxx} + 6xyf_1(t)u_t\\
&\quad -y^3 f_1'(t)u_t - 2y^3 f_1'(t)u_{xxx} \Big\},\\
T_1^y &= \frac{1}{6}\Big\{3y^2 f_1'(t)u - 6xf_1(t)u + 6xyf_1(t)u_y + y^3\left(-f_1'(t)\right)u_y\Big\};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{4}\Big\{ -2f_2(t)u + 2xf_2(t)u_x + y^2\left(-f_2'(t)\right)u_x \Big\},\\
T_2^x &= \frac{1}{4}\Big\{ -6y^2 f_2'(t)u_x u + 12xf_2(t)u_x u - 6y^2 f_2'(t)w_x w + 12xf_2(t)w_x w\\
&\quad +y^2 f_2{}''(t)u - 2xf_2'(t)u - 6f_2(t)u^2 - 6f_2(t)w^2 - 4f_2(t)u_{xx} + 4xf_2(t)u_{xxx}\\
&\quad +2xf_2(t)u_t - y^2 f_2'(t)u_t - 2y^2 f_2'(t)u_{xxx} \Big\},\\
T_2^y &= \frac{1}{2}\Big\{2yf_2'(t)u + 2xf_2(t)u_y + y^2\left(-f_2'(t)\right)u_y\Big\};
\end{aligned}$$

$$\begin{aligned}
T_3^t &= \frac{1}{12}\Big\{ -6yf_3(t)v + 6xyf_3(t)v_x + y^3\left(-f_3'(t)\right)v_x \Big\},\\
T_3^x &= \frac{1}{12}\Big\{ -6y^3 f_3'(t)v_x v + 36xyf_3(t)v_x v - 6y^3 f_3'(t)w_x w + 36xyf_3(t)w_x w\\
&\quad +y^3 f_3{}''(t)v - 6xyf_3'(t)v - 18yf_3(t)v^2 - 18yf_3(t)w^2 - 12yf_3(t)v_{xx}\\
&\quad +12xyf_3(t)v_{xxx} + 6xyf_3(t)v_t - y^3 f_3'(t)v_t - 2y^3 f_3'(t)v_{xxx} \Big\},\\
T_3^y &= \frac{1}{6}\Big\{3y^2 f_3'(t)v - 6xf_3(t)v + 6xyf_3(t)v_y + y^3\left(-f_3'(t)\right)v_y\Big\};
\end{aligned}$$

$$
\begin{aligned}
T_4^t &= \frac{1}{4}\Big\{-2f_4(t)v + 2xf_4(t)v_x + y^2\left(-f_4'(t)\right)v_x\Big\},\\
T_4^x &= \frac{1}{4}\Big\{-6y^2f_4'(t)v_xv + 12xf_4(t)v_xv - 6y^2f_4'(t)w_xw + 12xf_4(t)w_xw\\
&\quad +y^2f_4''(t)v - 2xf_4'(t)v - 6f_4(t)v^2 - 6f_4(t)w^2 - 4f_4(t)v_{xx} + 4xf_4(t)v_{xxx}\\
&\quad +2xf_4(t)v_t - y^2f_4'(t)v_t - 2y^2f_4'(t)v_{xxx}\Big\},\\
T_4^y &= \frac{1}{2}\Big\{2yf_4'(t)v + 2xf_4(t)v_y + y^2\left(-f_4'(t)\right)v_y\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_5^t &= \frac{1}{12}\Big\{-6yf_5(t)w + 6xyf_5(t)w_x + y^3\left(-f_5'(t)\right)w_x\Big\},\\
T_5^x &= \frac{1}{12}\Big\{-3y^3f_5'(t)u_xw - 3y^3f_5'(t)w_xu + 18xyf_5(t)u_xw + 18xyf_5(t)w_xu\\
&\quad -3y^3f_5'(t)v_xw - 3y^3f_5'(t)w_xv + 18xyf_5(t)v_xw + 18xyf_5(t)w_xv - 18yf_5(t)uw\\
&\quad -18yf_5(t)vw + y^3f_5''(t)w - 6xyf_5'(t)w - 12yf_5(t)w_{xx} + 12xyf_5(t)w_{xxx}\\
&\quad +6xyf_5(t)w_t - y^3f_5'(t)w_t - 2y^3f_5'(t)w_{xxx}\Big\},\\
T_5^y &= \frac{1}{6}\Big\{3y^2f_5'(t)w - 6xf_5(t)w + 6xyf_5(t)w_y + y^3\left(-f_5'(t)\right)w_y\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_6^t &= \frac{1}{4}\Big\{-2f_6(t)w + 2xf_6(t)w_x + y^2\left(-f_6'(t)\right)w_x\Big\},\\
T_6^x &= \frac{1}{4}\Big\{-3y^2f_6'(t)u_xw - 3y^2f_6'(t)w_xu + 6xf_6(t)u_xw + 6xf_6(t)w_xu\\
&\quad -3y^2f_6'(t)v_xw - 3y^2f_6'(t)w_xv + 6xf_6(t)v_xw + 6xf_6(t)w_xv - 6f_6(t)uw\\
&\quad -6f_6(t)vw + y^2f_6''w - 2xf_6'(t)w - 4f_6(t)w_{xx} + 4xf_6(t)w_{xxx} + 2xf_6(t)w_t\\
&\quad -y^2f_6'(t)w_t - 2y^2f_6'(t)w_{xxx}\Big\},\\
T_6^y &= \frac{1}{2}\Big\{2yf_6'(t)w + 2xf_6(t)w_y + y^2\left(-f_6'(t)\right)w_y\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_7^t &= \frac{1}{2} y f_7(t) u_x,\\
T_7^x &= \frac{1}{2}\left\{6yf_7(t)u_xu + 6yf_7(t)w_xw - yf_7'(t)u + 2yf_7(t)u_{xxx} + yf_7(t)u_t\right\},\\
T_7^y &= yf_7(t)u_y - f_7(t)u;
\end{aligned}
$$

$$
\begin{aligned}
T_8^t &= \frac{1}{2} f_8(t) u_x,\\
T_8^x &= \frac{1}{2}\left\{6f_8(t)u_xu + 6f_8(t)w_xw - f_8'(t)u + 2f_8(t)u_{xxx} + f_8(t)u_t\right\},\\
T_8^y &= f_8(t)u_y;
\end{aligned}
$$

$$
\begin{aligned}
T_9^t &= \frac{1}{2} y f_9(t) w_x,\\
T_9^x &= \frac{1}{2}\left\{3yf_9(t)u_xw + 3yf_9(t)w_xu + 3yf_9(t)v_xw + 3yf_9(t)w_xv - yf_9'(t)w\right.\\
&\quad \left.+2yf_9(t)w_{xxx} + yf_9(t)w_t\right\},\\
T_9^y &= yf_9(t)w_y - f_9(t)w;
\end{aligned}
$$

$$
\begin{aligned}
T_{10}^t &= \frac{1}{2} f_{10}(t) w_x,\\
T_{10}^x &= \frac{1}{2}\left\{3f_{10}(t)u_xw + 3f_{10}(t)w_xu + 3f_{10}(t)v_xw + 3f_{10}(t)w_xv - f_{10}'(t)w\right.\\
&\quad \left.+2f_{10}(t)w_{xxx} + f_{10}(t)w_t\right\},\\
T_{10}^y &= f_{10}(t)w_y;
\end{aligned}
$$

$$
\begin{aligned}
T_{11}^t &= \frac{1}{2} y f_{11}(t) v_x,\\
T_{11}^x &= \frac{1}{2}\left\{6yf_{11}(t)v_xv + 6yf_{11}(t)w_xw - yf_{11}'(t)v + 2yf_{11}(t)v_{xxx} + yf_{11}(t)v_t\right\},\\
T_{11}^y &= yf_{11}(t)v_y - f_{11}(t)v;
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^t &= \frac{1}{2} f_{12}(t) v_x,\\
T_{12}^x &= \frac{1}{2}\left\{6f_{12}(t)v_xv + 6f_{12}(t)w_xw - f_{12}'(t)v + 2f_{12}(t)v_{xxx} + f_{12}(t)v_t\right\},\\
T_{12}^y &= f_{12}(t)v_y.
\end{aligned}
$$

Remark Due to the presence of the arbitrary functions, f_i, $i = 1, \cdots, 12$, in the multipliers, one can obtain an infinitely many conservation laws for the coupled KP system.

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