GROUP CLASSIFICATION OF A GENERAL BOND-OPTION PRICING EQUATION

by

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Declaration

I, TANKI MOTSEPA, student number 24602825, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ..........................................................

Mr TANKI MOTSEPA

Date: ..........................................................

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..........................................................

PROF C.M. KHALIQUE

Date: ..........................................................
Dedication

To my wife, child and my lovely mother
Acknowledgements

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Abstract

The main purpose of this work is to perform the Lie group classification of a general bond-option pricing equation. The procedure involves finding the Lie symmetries of the said partial differential equation and employing direct method to classify the equation. The optimal systems are then obtained for some cases encountered and group-invariant solutions are found for the optimal systems of one-dimensional subalgebras obtained.
List of Acronyms

PDE: Partial differential equation

ODE: Ordinary differential equation

CIR: Cox-Ingersoll-Ross
Introduction

The theory of option pricing began in 1900 when the French mathematician Louis Jean-Baptiste Alphonse Bachelier [2] deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion. The Black-Scholes equation

\[ u_t + \frac{1}{2} \sigma^2 u_{xx} + r (xu_x - u) = 0 \]  (1)

was introduced by Black and Scholes [7] as the general equilibrium theory of option pricing which is particularly attractive because the final formula is a function of observable variables. Merton [37] extended the Black-Scholes theory of option pricing by introducing more assumptions and found new explicit formulas for pricing both the call and put options as well as the warrants and the down-and-out options. Merton was the first one to refer to equation (1) as the Black-Scholes equation. Equation (1) is sometimes referred to as the Black-Scholes-Merton equation and because of this work they were awarded the Nobel prize in 1997 in economics even though Black did not receive it as he had passed away in 1995. The equation is mainly used to find the fair price of a financial instrument (option or derivative) and to find the implied volatility.

The first bond pricing equation

\[ u_t + \frac{1}{2} \sigma^2 u_{xx} + \kappa (\theta - x) u_x - xu = 0 \]  (2)

was introduced by Vasicek [57] with three assumptions. Firstly, the instantaneous (spot) interest rate follows a diffusion process. Secondly, the price of a discount bond depends only on the spot rate over its term. Lastly, Vasicek assumed that the market is efficient. Many other researchers came up with new one factor models which modelled the term structure of interest rates such as [5, 6, 11, 15, 21–23, 35, 48].
Many differential equations, including financial mathematics equations, involve parameters, arbitrary elements or functions, which need to be determined. Usually, these arbitrary parameters are determined experimentally. However, the Lie symmetry approach through the method of group classification has proven to be a versatile tool in specifying the forms of these parameters systematically (see, for example, [26, 30, 33, 38–42, 52]).

The first group classification problem was investigated by Sophus Lie [34] in 1881 for a linear second-order partial differential equation with two independent variables. The main idea of group classification of a differential equation involving an arbitrary element(s), say, for example, \( g(u) \) and \( f(x) \), consists of finding the Lie point symmetries of the differential equation with arbitrary functions \( g(u) \) and \( f(x) \), and then computing systematically all possible forms of \( g(u) \) and \( f(x) \) for which the principal Lie algebra can be extended.

In the past few decades a considerable amount of development has been made in symmetry methods for differential equations. This is evident by the number of research papers, books and many new symbolic softwares devoted to the subject (see, for example, [3, 4, 12, 14, 19, 20, 49]).

Semi-invariants for the \((1+1)\) linear parabolic equations with two independent variables and one dependent variable were derived by Johnpillai and Mahomed [31]. In addition, joint invariant equation was obtained for the linear parabolic equation and that the \((1+1)\) linear parabolic equation was reducible via a local equivalence transformation to the one-dimensional heat equation. In [36], a necessary and sufficient condition for the parabolic equation to be reducible to the classical heat equation under the equivalence group was provided which improved on work done in [31].

Goard [18] found group invariant solutions of the bond-pricing equation by the use of classical Lie method. The solutions obtained were shown that they satisfy the condition for the bond price, that is, \( P(r, T) = 1 \), where \( P \) is the price of the bond. Here \( r \) is the short-term interest rate which is governed by a stochastic differential equation and \( T \) is time to maturity.

Pooe et al. [47] obtained the fundamental solutions for a number of zero-coupon bond models by transforming the one-factor bond pricing equations corresponding to the bond models to the one-dimensional heat equation whose fundamental solution is well-known. Subsequently,
the transformations were used to construct the fundamental solutions for zero-coupon bond pricing equations.

Sinkala et al. [54] computed the zero-coupon bonds (group invariant solutions satisfying the terminal condition $u(T,T) = 1$) using symmetry analysis for the Vasicek and CIR equations, given by

\begin{align*}
    u_t + \frac{1}{2} \sigma^2 u_{xx} + \kappa (\theta - x) u_x - xu &= 0, \\
    u_t + \frac{1}{2} \sigma^2 x u_{xx} + \kappa (\theta - x) u_x - xu &= 0,
\end{align*}

respectively. In [53] an optimal system of one-dimensional subalgebras was derived and used to construct distinct families of special closed-form solutions of CIR equation. In [52], group classification of the linear second-order parabolic partial differential equation

\begin{equation}
    u_t + \frac{1}{2} \rho^2 x^2 \gamma u_{xx} + (\alpha + \beta x - \lambda \rho x^2) u_x - xu = 0, \tag{3}
\end{equation}

where $\alpha, \beta, \lambda, \rho$ and $\gamma$ are constants was carried out. Lie point symmetries and group invariant solutions were found for certain values of $\gamma$. Also the forms where the equation admitted the maximal seven Lie point symmetry algebra, (3) was transformed into the heat equation. Vasicek, CIR and Longstaff models were recovered from group classification and some other equations were derived which had not been considered before in literature.

Dimas et al. [13] investigated some of the well known equations that arise in mathematics of finance, such as Black-Scholes, Longstaff, Vasicek, CIR and Heath equations. Lie point symmetries of these equations were found and their algebras were compared with that of the heat equation. The equations with seven symmetries were transformed to the heat equation.

In this research, we study a general bond-option pricing equation. The partial differential equation which will be investigated is a generalisation of equations (1) and (2) and is given by

\begin{equation}
    u_t + \alpha x^p u_{xx} + \lambda (\beta - x) u_x + \gamma x^q u = 0, \tag{4}
\end{equation}

where $p$ and $q$ are arbitrary constants.

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations are presented.
In Chapter two, Lie symmetry method is employed to find symmetries of the Black-Scholes equation. The symmetries obtained are then used to compute group invariant solutions.

In Chapter three, we carry out Lie group classification on equation (4), that is, we find values of the constants \( p \) and \( q \) for which the principal Lie algebra extends.

In Chapter four, we obtain optimal systems of one-dimensional subalgebras for two cases of equation (4) and then construct group invariant solutions.

In Chapter five, a summary of the results of the dissertation is presented and future work is discussed.

Bibliography is given at the end of this dissertation.
Chapter 1

Lie symmetry methods for differential equations

In this chapter, some basic methods of Lie symmetry analysis of differential equations including the algorithm to determine the Lie point symmetries of PDEs are given.

1.1 Introduction

Lie group analysis as a method for solving differential equations was developed by Sophus Lie (1842-1899), who showed that the majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory. Lately, many good books have appeared in literature in this field such as, Ovsianikov [44,45], Bluman and Kumei [9], Bluman and Anco [8], Stephani [55], Olver [43], Ibragimov [26–29], Hydon [25], Cantwell [10].

Definitions and results given in this Chapter are taken from the books mentioned above.
1.2 Continuous one-parameter (local) Lie group

In this section a transformation will be understood to mean an invertible transformation, that is, a bijective map. Let $t$ and $x$ be two independent variables and $u$ be a dependent variable. Consider a change of the variables $t$, $x$ and $u$:

\[ T_a: \bar{t} = f(t, x, u, a), \quad \bar{x} = g(t, x, u, a), \quad \bar{u} = h(t, x, u, a) \]  

(1.1)

where $a$ is a real parameter which continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and $f$, $g$ and $h$ are differentiable functions.

**Definition 1.1** A set $G$ of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables $t$, $x$ and $u$ if

(i) For $T_a$, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_a^{-1} = T_{a^{-1}} T_a = T_0$ (Inverse).

It follows from (i) that the associativity property is satisfied. Also, if the identity transformation occurs at $a = a_0 \neq 0$, i.e., $T_{a_0}$ is the identity, then a shift of the parameter $a = \bar{a} + a_0$ will give $T_0$ as above. The group property (i) can be written as

\[ \bar{t} = f(\bar{t}, \bar{x}, \bar{u}, b) = f(t, x, u, \phi(a, b)), \]

\[ \bar{x} = g(\bar{t}, \bar{x}, \bar{u}, b) = g(t, x, u, \phi(a, b)), \]

\[ \bar{u} = h(\bar{t}, \bar{x}, \bar{u}, b) = h(t, x, u, \phi(a, b)). \]  

(1.2)

The function $\phi$ is termed as the *group composition law*. A group parameter $a$ is called *canonical* if $\phi(a, b) = a + b$.

**Theorem 1.1** For any $\phi(a, b)$, there exists the canonical parameter $\bar{a}$ defined by

\[ \bar{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}. \]
Let us now give the definition of a symmetry group for PDEs by considering, for example, evolutionary equations of the second-order, namely

\[ u_t = F(t, x, u, u_x, u_{xx}), \quad \frac{\partial F}{\partial u_{xx}} \neq 0. \]

(1.3)

**Definition 1.2 (Symmetry group)** A one-parameter group \( G \) of transformations (1.1) is called a symmetry group of equation (1.3) if (1.3) is form-invariant (has the same form) in the new variables \( \tilde{t}, \tilde{x} \), and \( \tilde{u} \), i.e.,

\[ \tilde{u}_t = F(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_x, \tilde{u}_{xx}). \]

(1.4)

where the function \( F \) is the same as in equation (1.3).

1.3 Infinitesimal transformations

According to the Lie's theory, the construction of the symmetry group \( G \) is equivalent to the determination of the corresponding infinitesimal transformations:

\[ \tilde{t} \approx t + a \tau(t, x, u), \quad \tilde{x} \approx x + a \xi(t, x, u), \quad \tilde{u} \approx u + a \eta(t, x, u) \]

(1.5)

obtained from (1.1) by expanding the functions \( f, g \), and \( h \) into Taylor series in \( a \) about \( a = 0 \) and also taking into account the initial conditions

\[ f|_{a=0} = t, \quad g|_{a=0} = x, \quad h|_{a=0} = u. \]

Thus, we have

\[ \tau(t, x, u) = \frac{\partial f}{\partial a}|_{a=0}, \quad \xi(t, x, u) = \frac{\partial g}{\partial a}|_{a=0}, \quad \eta(t, x, u) = \frac{\partial h}{\partial a}|_{a=0}. \]

(1.6)

The vector \((\tau, \xi, \eta)\) with components (1.6) is the tangent vector at the point \((t, x, u)\) to the surface curve described by the transformed points \((\tilde{t}, \tilde{x}, \tilde{u})\), and is therefore called the tangent vector field of the group \( G \).

One can now introduce the symbol of the infinitesimal transformations by writing (1.5) as

\[ \tilde{t} \approx (1 + a X)t, \quad \tilde{x} \approx (1 + a X)x, \quad \tilde{u} \approx (1 + a X)u. \]
where
\[ X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}. \] (1.7)

This differential operator \( X \) is known as the infinitesimal operator or generator of the group \( G \). If the group \( G \) is admitted by (1.3), we say that \( X \) is an admitted operator of (1.3) or \( X \) is an infinitesimal symmetry of equation (1.3).

### 1.4 Group invariants

**Definition 1.3** A function \( F(t, x, u) \) is called an invariant of the group of transformation (1.1) if
\[ F(\bar{t}, \bar{x}, \bar{u}) \equiv F(f(t, x, u, a), g(t, x, u, a), h(t, x, u, a)) = F(t, x, u), \] (1.8)
identically in \( t, x, u \) and \( a \).

**Theorem 1.2** (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function \( F(t, x, u) \) to be an invariant is that
\[ X F \equiv \tau(t, x, u) \frac{\partial F}{\partial t} + \xi(t, x, u) \frac{\partial F}{\partial x} + \eta(t, x, u) \frac{\partial F}{\partial u} = 0. \] (1.9)

It follows from the above theorem that every one-parameter group of point transformations (1.1) has two functionally independent invariants, which can be taken to be the left-hand side of any first integrals
\[ J_1(t, x, u) = c_1, \quad J_2(t, x, u) = c_2, \]
of the characteristic equations
\[ \frac{dt}{\tau(t, x, u)} = \frac{dx}{\xi(t, x, u)} = \frac{du}{\eta(t, x, u)}. \]

**Theorem 1.3** Given the infinitesimal transformation (1.5) or its symbol \( X \), the corresponding one-parameter group \( G \) is obtained by solving the Lie equations
\[ \frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}) \] (1.10)
subject to the initial conditions
\[ \bar{t}|_{a=0} = t, \quad \bar{x}|_{a=0} = x, \quad \bar{u}|_{a=0} = u. \]
1.5 Construction of a symmetry group

In this section we briefly describe the algorithm to determine a symmetry group for a given PDE. First we need to give some basic definitions.

1.5.1 Prolongation of point transformations

Consider a second-order PDE

\[ E(t, x, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = 0, \]  

(1.11)

where \( t \) and \( x \) are two independent variables and \( u \) is a dependent variable. Let

\[ X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}, \]  

(1.12)

be the infinitesimal generator of the one-parameter group \( G \) of transformation (1.1). The first prolongation of \( X \) is denoted by \( X^{[1]} \) and is defined by

\[ X^{[1]} = X + \zeta_1(t, x, u, u_t, u_x) \frac{\partial}{\partial u_t} + \zeta_2(t, x, u, u_x) \frac{\partial}{\partial u_x}, \]

where

\[ \zeta_1 = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \]
\[ \zeta_2 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \]

and the total derivatives \( D_t \) and \( D_x \) are given by

\[ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \]  

(1.13)
\[ D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots. \]  

(1.14)

Likewise, the second prolongation of \( X \), denoted by \( X^{[2]} \), is given by

\[ X^{[2]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{12} \frac{\partial}{\partial u_{tt}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}, \]  

(1.15)

where

\[ \zeta_{11} = D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \]
\[
\zeta_{12} = D_x(\zeta_1) - u_t D_x(\tau) - u_{tx} D_x(\xi),
\]
\[
\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi).
\]

Using the definitions of \(D_t\) and \(D_x\), one can write

\[
\zeta_1 = \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u.
\] (1.16)

\[
\zeta_2 = \eta_x + u_x \eta_u - u_x \tau_x - u_t u_x \tau_u - u_x \xi_x - u_t^2 \xi_u.
\] (1.17)

\[
\zeta_{11} = \eta_{tt} + 2u_t \eta_{tu} + u_{tt} \eta_u + u_{tx} \eta_x + u_t \tau_{tt} - u_t \tau_{tx} - 2u_t \tau_{tu} - 3u_t u_t \tau_u - u_t^3 \tau_{uu}
- 2u_{tx} \xi_t - u_x \xi_{tt} - 2u_t u_x \xi_{tu} - u_t^2 u_x \xi_{uu} - (u_x u_t + 2u_t u_{tx}) \xi_u.
\] (1.18)

\[
\zeta_{12} = \eta_{tx} + u_x \eta_u + u_x \eta_{tx} + u_x \eta_{ux} + u_t \eta_{ux} + u_{tx} \eta_{tt} + u_x \tau_{tx} - u_t \tau_{tx} - u_{tx} \xi_t - u_x \xi_{tx} + u_t u_x (\tau_{tu} + \xi_{tx}) - u_t u_x (\tau_{tx} + \xi_{tx}) - u_t \tau_{xx} - u_x \xi_{xx} - u_x \xi_{ux} - u_{xx} \xi_u + u_{xx} \xi_{uu} - (2u_t u_{tx} + u_t u_{xx}) \xi_u + u_t u_x \xi_{uu}.
\] (1.19)

\[
\zeta_{22} = \eta_{xx} + 2u_{xx} \eta_{xx} + u_{xx} \eta_u + u_{xx} \eta_{xx} + u_x \tau_{xx} - u_x \tau_{xx} - 2u_{xx} \tau_{xx} - 2u_{xx} \tau_{xx} - u_{xx} \xi_{xx} - u_{xx} \xi_{xx} - u_{xx} \xi_{xx} - u_{xx} \xi_{xx} - 3u_x u_{xx} \xi_u
- u_{xx}^3 \xi_{uu} - u_{xx} \tau_x - u_{xx} \tau_{xx} - 2u_{xx} \tau_{xx} - (u_t u_{xx} + 2u_x u_{tx}) \tau_u - u_t u_{xx}^2 \tau_{uu}.
\] (1.20)

1.5.2 Group admitted by a PDE

The operator

\[
X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u},
\] (1.21)

is said to be a (generator of) point symmetry of the second-order PDE

\[
E(t, x, u, u_t, u_x, u_{tx}, u_{xx}) = 0
\] (1.22)

if

\[
X^{[2]}(E) = 0
\] (1.23)

whenever \(E = 0\). This can also be written as (symmetry condition)

\[
X^{[2]} E|_{E=0} = 0,
\] (1.24)

where the symbol \(|_{E=0}\) means evaluated on the equation \(E = 0\).
Definition 1.4 Equation (1.24) is called the determining equation of (1.22), because it determines all the infinitesimal symmetries of equation (1.22).

The theorem below enables us to construct some solutions of (1.22) from known one.

Theorem 1.4 A symmetry of equation (1.22) transforms any solution of (1.22) into another solution of the same equation.

Proof: It follows from the fact that a symmetry of an equation leaves invariant that equation.

1.6 Lie algebras

Let $X_1$ and $X_2$ be any two operators defined by

$$X_1 = \tau_1(t, x, u) \frac{\partial}{\partial t} + \xi_1(t, x, u) \frac{\partial}{\partial x} + \eta_1(t, x, u) \frac{\partial}{\partial u}$$

and

$$X_2 = \tau_2(t, x, u) \frac{\partial}{\partial t} + \xi_2(t, x, u) \frac{\partial}{\partial x} + \eta_2(t, x, u) \frac{\partial}{\partial u}.$$

Definition 1.5 (Commutator) The commutator of $X_1$ and $X_2$, written as $[X_1, X_2]$, is defined by the formula $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.6 (Lie algebra) A Lie algebra is a vector space $L$ of operators such that, for all $X_1, X_2 \in L$, the commutator $[X_1, X_2] \in L$.

The dimension of a Lie algebra is the dimension of the vector space $L$.

It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$
3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$, 

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$ 

**Theorem 1.5** The set of all solutions of any determining equation forms a Lie algebra.

### 1.7 Conclusion

In this chapter we gave a brief introduction to the Lie group analysis of PDEs and presented some results which will be used throughout this work. We also gave the algorithm to determine the Lie point symmetries of PDEs.
Chapter 2

Symmetry analysis of the Black-Scholes equation

In this section we consider the Black-Scholes equation, which arises in financial mathematics and calculate its symmetry Lie algebra. We also find group-invariant solutions under three symmetry generators of the Black-Scholes equation. Black-Scholes equation (1) was first investigated from the point of view of Lie point symmetry analysis by Gazizov and Ibragimov [17], who found its symmetries and used two different transformations to transform it to the heat equation and the latter was used to solve the initial value problem. The invariance principle was used to construct the fundamental solution that could be used for general analysis of an arbitrary initial value problem.

Pooe et al. [46] obtained two classes of optimal systems of the one-dimensional subalgebras for the Black-Scholes equation using the two transformations by Gazizov and Ibragimov [17] that transform Black-Scholes to the heat equation. Sukhomlin and Ortiz [56] obtained solutions for the Black-Scholes equation and the diffusion equation by ansatz using similarities between the two equations. Also in [56], the equivalence group for the Black-Scholes equation was established and the largest set of transformations each of which converts the Black-Scholes equation to the diffusion equation was obtained.

In [16], two potential symmetries were found and used to obtain new solutions to the Black-Scholes equation. First, the equation was written in conserved form which required the
conservation laws. Conservation laws were found by the method of Kara and Mahomed [32], which uses symmetries to directly compute the conservation laws. Many other researchers also studied the Black-Scholes equation from the point of view of Lie symmetry analysis and pricing of contingent claims, see for example [13, 24, 30, 50, 51, 58].

2.1 Calculation of symmetries of the Black-Scholes equation

Consider the Black-Scholes equation

\[ u_t + \frac{1}{2} A^2 x^2 u_{xx} + B x u_x - C u = 0. \]  

(2.1)

This equation admits the one-parameter Lie group of transformations with infinitesimal generator

\[ X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \]  

(2.2)

if and only if

\[ X^{[2]}(u_t + \frac{1}{2} A^2 x^2 u_{xx} + B x u_x - C u) \bigg|_{(2.1)} = 0. \]  

(2.3)

Using the definition of \( X^{[2]} \) from Chapter one, we obtain

\[ \zeta_1 + A^2 x u_{xx} \xi + \zeta_{22} \frac{1}{2} A^2 x^2 + \xi B u_x + B x \zeta_2 - \eta C \bigg|_{(2.1)} = 0; \]  

(2.4)

where \( \zeta_1, \zeta_2 \) and \( \zeta_{22} \) are given by equations (1.16), (1.17) and (1.20) respectively. Substituting the values of \( \zeta_1, \zeta_2 \) and \( \zeta_{22} \) in equation (2.4) (and replacing \( u_{xx} \) by \( \frac{2}{A^2 x^2} \left[ C u - B x u_x - u_t \right] \)) we obtain

\[
\begin{align*}
&\left[ \eta_t + u_t \eta_u - u_t \tau_t - u_t^2 \tau_u - u_t \xi_t - u_t u_x \xi_u \right] + A^2 x \left[ - \frac{2}{A^2 x^2} \left( C u - B x u_x - u_t \right) \right] + \\
&\frac{1}{2} A^2 x^2 \left[ \eta_{xx} + 2 u_x \eta_{ux} + u_x^2 \eta_{uu} - u_x \xi_{xx} - 2 u_x^2 \xi_{xx} - u_x^2 \xi_{uu} - 2 u_x \xi_{ux} - \\
&u_t \tau_{xx} - 2 u_t u_x \tau_{xx} - 2 u_x u_{xx} \tau_u - u_t u_x^2 \tau_u + \frac{2}{A^2 x^2} \left( C u - B x u_x - u_t \right) \right] \\
\end{align*}
\]
\[
\left( A^2 x \xi + \eta_u - 2 \xi_x - 3 u_x \xi_u - u \tau_u \right) = 0. 
\] (2.5)

Since \( \tau, \xi \) and \( \eta \) depend only on \( t, x \) and \( u \) and are independent of the derivatives of \( u \), the coefficients of like derivatives of \( u \) can be equated to yield the following over determined system of linear PDEs:

\[
\begin{align*}
\tau & = 0, \quad (2.6) \\
\tau_x & = 0, \quad (2.7) \\
\xi & = 0, \quad (2.8) \\
\eta & = 0, \quad (2.9) \\
\tau_t & = 2 \xi_t - \frac{2}{x} \xi - \tau = 0, \quad (2.10) \\
\xi_t & + Bx \xi_x + A^2 x^2 \eta_{xx} - \frac{A^2 x^2}{2} \xi_{xx} - B \xi & = 0, \quad (2.11) \\
\eta_t & + Bx \eta_x - C \eta + \frac{A^2 x^2}{2} \eta_{xx} + C \eta_x + \frac{2C u}{x} \xi - 2 C u \xi_x & = 0. \quad (2.12)
\end{align*}
\]

Equations (2.6) and (2.7) imply that

\[
\tau = a(t), \quad (2.13)
\]

where \( a(t) \) is an arbitrary function of \( t \). Equation (2.8) gives

\[
\xi = b(t, x), \quad (2.14)
\]

where \( b(t, x) \) is an arbitrary function of \( t \) and \( x \). Integrating equation (2.9) twice with respect to \( u \), we obtain

\[
\eta = c(t, x) u + d(t, x), \quad (2.15)
\]

where \( c(t, x) \) and \( d(t, x) \) are arbitrary functions of \( t \) and \( x \). Substituting this value of \( \xi \) in (2.10), we obtain

\[
b_x = \frac{1}{2} a'(t) + \frac{1}{x} b(t, x). \quad (2.16)
\]

Solving the above equation we obtain

\[
b(t, x) = \frac{1}{2} a'(t) \ln x + x e(t), \quad (2.16)
\]

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where \( e(t) \) is an arbitrary function of \( t \). Thus

\[
\xi = \frac{1}{2} a'(t) x \ln x + x e(t). \tag{2.17}
\]

Substituting the values of \( \xi \) and \( \eta \) in (2.11), we obtain

\[
c_x = \frac{1}{2} a''(t) \ln x + \frac{e'(t)}{x A^2} + \frac{a'(t)}{4 x} - \frac{B a'(t)}{2 x A^2}. \tag{2.18}
\]

Integrating the above equation with respect to \( x \), we obtain

\[
c(t, x) = \frac{a''(t) (\ln x)^2}{4 A^2} - \frac{B a'(t) (\ln x)}{2 A^2} + \frac{e'(t) (\ln x)}{A^2} + \frac{a'(t) (\ln x)}{4} + f(t), \tag{2.19}
\]

where \( f(t) \) is an arbitrary function of \( t \). Substituting the values of \( \xi \) and \( \eta \) in (2.12), we obtain

\[
c_t u + d_t + B x(c_x u + d_x) - C d(t, x) + \frac{A^2 x^2}{2} (c_{xx} u + d_{xx}) - 2 C u \left( \frac{a'(t) (\ln x)^2}{2} + e(t) \right) + C u a'(t) \ln x = 0. \tag{2.20}
\]

Separating (2.20) with respect to \( u \), we obtain

\[
u : c_t + B x c_x + \frac{A^2 x^2 c_{xx}}{2} - C a'(t) = 0 \tag{2.21}
\]

\[
u^0 : d_t + B x d_x - C d(t, x) + \frac{A^2 x^2 d_{xx}}{2} = 0. \tag{2.22}
\]

Substituting the value of \( c \) into (2.21), we obtain

\[
\frac{a''(t) (\ln x)^2}{4 A^2} - \frac{B a''(t) (\ln x)}{2 A^2} + \frac{e''(t) (\ln x)}{A^2} + \frac{a''(t) (\ln x)}{4} + f'(t) + B x \left[ \frac{a''(t) \ln x}{2 A^2 x} - \frac{B a'(t)}{2 A^2 x} + \frac{e'(t)}{A^2 x} + \frac{a'(t)}{4 x} \right] + \frac{A^2 x^2}{2} \left[ \frac{a''(t)}{2 A^2 x^2} - \frac{a''(t) \ln x}{2 A^2 x^2} \right] - C a'(t) = 0. \tag{2.23}
\]

Separating (2.23) with respect to \( \ln x \), we obtain

\[
(\ln x)^2 : a'''(t) = 0 \tag{2.24}
\]

\[
(\ln x) : e''(t) = 0 \tag{2.25}
\]

\[
1 : f'(t) - \frac{B^2 a'(t)}{2 A^2} + \frac{B e'(t)}{A^2} + \frac{B a'(t)}{2} + \frac{a''(t)}{4} - \frac{e'(t)}{2} - \frac{A^2 a'(t)}{8} - C a'(t) = 0. \tag{2.26}
\]

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Integrating (2.24) with respect to $t$ three times, we obtain

$$a(t) = \frac{A_1 t^2}{2} + A_2 t + A_3.$$  \hspace{1cm} (2.27)

Now integrating (2.25) gives

$$e(t) = A_4 t + A_5.$$  \hspace{1cm} (2.28)

Substituting the values of $a(t)$ and $e(t)$ in (2.26), and integrating gives

$$f(t) = \frac{B^2 A_1 t^2}{4 A^2} + \frac{B^2 A_2 t}{2 A^2} - \frac{B A_4 t}{4} - \frac{B A_1 t^2}{2} - \frac{B A_2 t}{4} - \frac{A_1 t}{4}$$
$$+ \frac{A_1 t}{2} + \frac{A^2 A_1 t^2}{16} + \frac{A^2 A_2 t}{8} + \frac{C A_1 t^2}{2} + C A_2 t + A_6.$$  \hspace{1cm} (2.29)

Substituting the values of $e(t)$, $a(t)$ and $f(t)$ into (2.19) we obtain

$$c(t, x) = \frac{A_1 (\ln x)^2}{4 A^2} - \frac{B \ln x (A_1 t + A_2)}{2 A^2} + \frac{A_4 \ln x}{A^2} + \frac{\ln x (A_1 t + A_2)}{4}$$
$$+ \frac{B^2 A_1 t^2}{4 A^2} + \frac{B^2 A_2 t}{2 A^2} - \frac{B A_4 t}{4} - \frac{B A_1 t^2}{2} - \frac{B A_2 t}{4} - \frac{A_1 t}{4} + \frac{A_4 t}{2}$$
$$+ \frac{A^2 A_1 t^2}{16} + \frac{A^2 A_2 t}{8} + \frac{C A_1 t^2}{2} + C A_2 t + A_6.$$  \hspace{1cm} (2.30)

Thus

$$\tau = \frac{1}{2} A_1 t^2 + A_2 t + A_3$$  \hspace{1cm} (2.31)

$$\xi = \frac{1}{2} (A_1 t + A_2) x \ln x + x(A_4 t + A_5)$$  \hspace{1cm} (2.32)

$$\eta = \left( \frac{A_1 (\ln x)^2}{4 A^2} - \frac{B \ln x (A_1 t + A_2)}{2 A^2} + \frac{A_4 \ln x}{A^2} + \frac{\ln x (A_1 t + A_2)}{4} \right) u + d(t, x)$$
$$+ \frac{B^2 A_1 t^2}{4 A^2} + \frac{B^2 A_2 t}{2 A^2} - \frac{B A_4 t}{4} - \frac{B A_1 t^2}{2} - \frac{B A_2 t}{4} - \frac{A_1 t}{4} + \frac{A_4 t}{2}$$
$$+ \frac{A^2 A_1 t^2}{16} + \frac{A^2 A_2 t}{8} + \frac{C A_1 t^2}{2} + C A_2 t + A_6.$$  \hspace{1cm} (2.33)

and so the infinitesimal symmetries of the Black-Scholes equation are

$$X_1 = \frac{\partial}{\partial t},$$  \hspace{1cm} (2.34)

$$X_2 = x \frac{\partial}{\partial x}.$$  \hspace{1cm} (2.35)
\[ X_3 = 2A^2t \frac{\partial}{\partial t} + A^2x \ln x \frac{\partial}{\partial x} + (2A^2Ct + D^2t - D \ln x) u \frac{\partial}{\partial u}, \] (2.36)

\[ X_4 = A^2tx \frac{\partial}{\partial x} + (\ln x - Dt) u \frac{\partial}{\partial u}, \] (2.37)

\[ X_5 = 2A^2t^2 \frac{\partial}{\partial t} + 2A^2tx \ln x \frac{\partial}{\partial x} + (2A^2Ct^2 - A^2t + (\ln x - Dt)^2) u \frac{\partial}{\partial u}, \] (2.38)

\[ X_6 = u \frac{\partial}{\partial u} \quad \text{and} \]

\[ X_6 = d(t,x) \frac{\partial}{\partial u}, \] (2.40)

where \( D = B - A^2/2 \) and \( d \) is an arbitrary solution of (2.1). Furthermore, \( X_1, \ldots, X_6 \) are operators which generate six parameter group and \( X_6 \) generates an infinite group.

### 2.2 Invariant solutions of the Black-Scholes equation

In this section, we construct group invariant solutions under some of the symmetry operators of the Black-Scholes equation. We start with the operator \( X_1 \).

**Example 2.1** Let us calculate the invariant solution under the symmetry operator \( X_1 \). The operator \( X_1 \) is given by

\[ X_1 = \frac{\partial}{\partial t}. \]

The characteristic equations are

\[ \frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}, \]

which provide the two invariants \( J_1 = x \) and \( J_2 = u \). Thus, the invariant solution is given by \( J_2 = \phi(J_1) \), i.e.,

\[ u = \phi(x). \]

Substituting this value of \( u \) in (2.1), we obtain

\[ \frac{1}{2} A^2x^2 \phi'' + B x \phi' - C \phi = 0. \]

This is a Cauchy-Euler equation and its solution is given by

\[ \phi(x) = c_1 x^\left(\frac{\sqrt{2A^2} + \sqrt{2A^4 - 8A^2B + 16A^2c + 8B^2 - 2\sqrt{2}B}}{2\sqrt{2}A^2}\right) + c_2 x^\left(\frac{\sqrt{2A^2} - \sqrt{2A^4 - 8A^2B + 16A^2c + 8B^2 - 2\sqrt{2}B}}{2\sqrt{2}A^2}\right), \] (2.41)
where \( c_1 \) and \( c_2 \) are arbitrary constants. Hence the invariant solution of (2.1) under \( X_1 \) is
\[
U(t, x) = c_1 x^{\sqrt{2A^2 + \sqrt{2A^4 - 8A^2B + 16A^2e + 8B^2 - 2\sqrt{2B}/(2\sqrt{2A^2})}}}
+ c_2 x^{\sqrt{2A^2 - \sqrt{2A^4 - 8A^2B + 16A^2e + 8B^2 - 2\sqrt{2B}/(2\sqrt{2A^2})}}},
\]
(2.42)

**Example 2.2** Let us calculate the invariant solution under the operator \( X_4 \), namely
\[
X_4 = A^2tx \frac{\partial}{\partial x} + (\ln x - Dt)u \frac{\partial}{\partial u}.
\]
where \( D = B - A^2/2 \).

Now
\[
X_4 J \equiv 0 \frac{\partial J}{\partial t} + A^2tx \frac{\partial J}{\partial x} + (\ln x - Dt)u \frac{\partial J}{\partial u} = 0.
\]
(2.43)

The characteristic equations are
\[
\frac{dt}{0} = \frac{dx}{A^2tx} = \frac{du}{(\ln x - Dt)u}.
\]
Thus, one invariant is \( J_1 = t \). The other is obtained from the equation
\[
\frac{dx}{A^2tx} = \frac{du}{(\ln x - Dt)u},
\]
and is given by \( J_2 = u/\exp \left\{ \frac{(\ln x - Dt)^2}{2A^2t} \right\} \).

Consequently, the invariant solution of (2.1) under \( X_4 \) is \( J_2 = \phi(J_1) \), i.e.,
\[
u = \exp \left\{ \frac{(\ln x - Dt)^2}{2A^2t} \right\} \phi(t),
\]
(2.44)

where \( \phi \) is an arbitrary function of \( t \). Substituting (2.44) into equation (2.1), gives
\[
\phi' + \left( \frac{1}{2t} - C \right) \phi = 0.
\]

This is a first-order variables separable equation and its solutions is given by
\[
\phi(t) = \frac{K}{\sqrt{t}} e^{Ct},
\]
where \( K \) is an arbitrary constant, and hence the invariant solution of the Black-Scholes equation under the operator \( X_4 \) is
\[
u(t, x) = \frac{K}{\sqrt{t}} \exp \left\{ \frac{(\ln x - Dt)^2}{2A^2t} + Ct \right\}.
\]

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Example 2.3 Let us find the invariant solution under the operator $X_5$, namely

$$X_5 = 2A^2 t^2 \frac{\partial}{\partial t} + 2A^2 t x \ln x \frac{\partial}{\partial x} + \left[ (\ln x - Dt)^2 + 2A^2 Ct^2 - A^2 t \right] \frac{\partial}{\partial u}.$$ 

The characteristic equations are

$$\frac{dt}{2A^2 t^2} = \frac{dx}{2A^2 t x \ln x} = \frac{du}{\left[ (\ln x - Dt)^2 + 2A^2 Ct^2 - A^2 t \right] u}.$$ 

By considering

$$\frac{dt}{2A^2 t^2} = \frac{dx}{2A^2 t x \ln x}$$

and integrating, we obtain one invariant as $J_1 = \frac{\ln x}{t}$. The other is obtained from the equation

$$\frac{dt}{2A^2 t^2} = \frac{du}{\left[ (\ln x - Dt)^2 + 2A^2 Ct^2 - A^2 t \right] u},$$

and is given by $J_2 = u \sqrt{t} / \exp \left\{ \frac{(\ln x - Dt)^2}{2A^2 t} + Ct \right\}$.

Consequently, the invariant solution under $X_5$ is $J_2 = \phi(J_1)$, i.e.,

$$u = \frac{1}{\sqrt{t}} \exp \left\{ \frac{(\ln x - Dt)^2}{2A^2 t} + Ct \right\} \phi \left( \frac{\ln x}{t} \right).$$ \hspace{1cm} (2.45)

Substituting $u$, $u_t$, $u_x$ and $u_{xx}$ in (2.1) and simplifying yields

$$\phi'' = 0.$$ \hspace{1cm} (2.46)

Solving equation (2.46) we obtain $\phi(J_1) = K_1 J_1 + K_2$ where $K_1$ and $K_2$ are arbitrary constants of integration. Hence equation (2.45) becomes

$$u(t, x) = \left( K_1 \frac{\ln x}{t^{3/2}} + K_2 \sqrt{t} \right) \exp \left\{ \frac{(\ln x - Dt)^2}{2A^2 t} + Ct \right\}.$$ 

Remark: We note that the operators $X_6$ and $X_d$ do not provide invariant solutions.
2.3 Conclusion

In this chapter we obtained the symmetry Lie algebra for Black-Scholes equation. This equation arises in mathematics of finance. We then constructed group-invariant solutions under some infinitesimal generators of the Black-Scholes equation.
Chapter 3

Group classification of a general bond-option pricing equation

3.1 Introduction

In this chapter we study the general bond-option pricing equation

\[ u_t + \alpha x^p u_{xx} + \lambda (\beta - x) u_x + \gamma x^q u = 0, \quad (3.1) \]

where \( \lambda, \beta, \gamma, \alpha \) are constants with \( \lambda, \gamma, \alpha \) different from zero and \( p \geq 0, q \geq 0 \). Here \( t \) is the time, \( x \) is the stock (share or equity) price or instantaneous short-term interest rate at current time \( t \) and \( u(t, x) \) is the current value of the option or bond depending on the form of (3.1).

Equation (3.1) is a generalisation of the Black-Scholes, the Cox-Ingersoll-Ross and the Vasicek equations because it reduces to the Black-Scholes equation when \( p = 2, q = 0, \beta = 0, \alpha = \sigma^2/2, \lambda = -r \) and \( \gamma = -r \), to Cox-Ingersoll-Ross when \( \alpha = \sigma^2/2, p = 1, q = 1 \) and \( \gamma = -1 \) and to Vasicek when \( \alpha = \sigma^2/2, p = 0, q = 1 \) and \( \gamma = -1 \).

We also note that when \( q = 0 \), equation (3.1) is the option pricing equation and it is the bond pricing equation when \( q = 1 \).

Here we perform Lie group classification of (3.1). We follow the workings of Sinkala et al. [52].
3.2 Determination of classifying equations of (3.1)

The Lie point symmetries for (3.1) are given by the vector field

$$ X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} $$

If and only if

$$ X^{[2]}(u_t + \alpha x^p u_{xx} + \lambda(\beta - x)u_x + \gamma x^q u)|_{(3.1)} = 0, $$

where

$$ X^{[2]} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{22} \frac{\partial}{\partial u_{xx}}. $$

Here \( \zeta_i \)'s are given by

$$ \zeta_1 = D_t(\eta) - u_t D_t(\tau) - u_x D_x(\xi), $$
$$ \zeta_2 = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), $$
$$ \zeta_{22} = D_x(\zeta_2) - u_{xx} D_x(\xi), $$

where the total derivatives \( D_t \) and \( D_x \) are defined as

$$ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, $$
$$ D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots. $$

To perform the group classification of (3.1) it turns out that we need to consider two cases of \( p \) separately; \( p \neq 2 \) and \( p = 2 \).

3.2.1 Classifying equation of (3.1) for \( p \neq 2 \)

Expanding the determining equation (3.3), we obtain

$$ \gamma \xi_q u_{x^{q-1}} - \gamma u_x^q \eta_x + \gamma u_x^q \xi_x u_x + \gamma \eta u_x^q + \eta_x u_x - \lambda \xi u_x + \beta \lambda \eta x - \lambda x_\eta_x + $$
$$ + \gamma u_x^q \xi_x u_x + \gamma \gamma \lambda u_x^q + \eta_x u_x - \lambda \xi u_x + \beta \lambda \eta x - \lambda x_\eta_x + $$
$$ - \gamma \lambda u_x^q \xi_x u_x + \beta \gamma \lambda u_x^q \tau_x u_x + \lambda^2 \lambda \tau_x u_x - \beta \lambda \tau_x u_x + \lambda x \tau_x + $$
$$ - \lambda u_x^q \tau_x - \beta \gamma \lambda u_x^q \tau_x u_x + \beta \gamma \lambda u_x^q \tau_x + \lambda^2 \lambda \tau_x u_x - \beta \lambda \lambda \tau_x u_x - $$
$$ - 2 \alpha \lambda \tau_x u_x - 2 \alpha \lambda \tau_x u_x + \alpha \lambda \lambda x^p u_{xx} - 2 \alpha \lambda \lambda x^p u_{xx} + $$

...
Separating (3.4) with respect to the derivatives of \( u \), since the functions \( \tau \), \( \xi \) and \( \eta \) do not depend on them, leads to the following linear PDEs:

\[
\begin{align*}
\tau_u &= 0, \quad (3.5) \\
\xi_u &= 0, \quad (3.6) \\
\eta_{uu} &= 0, \quad (3.7) \\
\tau_x &= 0, \quad (3.8) \\
p\xi + x(\tau_t - 2\xi_x) &= 0, \quad (3.9) \\
\lambda \xi + \alpha p \xi_{xx} + x\lambda \tau_t + \xi_t + \lambda(\beta - x)\xi_x - 2\alpha p \eta_{xx} - \beta \lambda \tau_t &= 0. \quad (3.11)
\end{align*}
\]

To solve the above system of equations, we first observe from equations (3.5) and (3.8) that \( \tau \) does not depend on \( x \) and \( u \), which means that \( \tau \) is a function of \( t \). Thus

\[\tau = \tau(t). \quad (3.12)\]

Equation (3.6) implies that \( \xi \) depends on both \( t \) and \( x \) but not on \( u \). Thus

\[\xi = \xi(t, x). \quad (3.13)\]

Integration of equation (3.7) with respect to \( u \) twice gives

\[\eta(t, x, u) = A(t, x)u + B(t, x), \quad (3.14)\]

where \( A(t, x) \) and \( B(t, x) \) are arbitrary functions of \( t \) and \( x \). Using the expressions for \( \tau \) and \( \xi \) into (3.9) and integrating with respect to \( x \), leads to

\[\xi(t, x) = c(t)x^{p/2} - \frac{p\tau'(t)}{p - 2}, \quad p \neq 2, \quad (3.15)\]
where $c(t)$ is an arbitrary function of $t$.

Using (3.12), (3.14) and (3.15) in (3.11) yields

$$
8x^{p+1}A_x(t,x)\left(2 - p\right) + c(t)\alpha px^{-1+3p/2}(p - 2)^2 - 2c(t)\lambda x^{1-p/2}(p - 2)^2 - 4x^2\tau''(t) - 2\beta\lambda pC(t)x^{p/2}(p - 2) + 4x^{p/2}x'(t)(p - 2) + \lambda\left(4x^2(p - 2) + 4\beta x(1 - p)\right)\tau'(t) = 0.
$$

Using (3.12), (3.14) and (3.15) in (3.10) gives

$$
16(p - 2)x^{2+p/2}(\alpha px^pB_{xx} + B\gamma x^q + B_t + \beta\lambda B_x - \lambda x B_x) + 8\alpha(p - 2)x^{2+p/2}(2\gamma(p - q - 2)x^q - \lambda(p - 2)(p - 1))\tau_t - 8u\lambda^2(p - 2)^2x^{4-p/2}\tau_t
$$

$$
+8\beta\lambda u(p - 2)\left\{\alpha(p - 1)px^{1+p/2}\tau_t - \beta\lambda(p - 1)x^{2-p/2}\tau_t + \lambda(2p - 3)x^{3-p/2}\tau_t\right\}
$$

$$
+u\left\{8(p - 2)\left(2\alpha(p - 2)dx^{2+p/2} + \alpha(p - 1)x^{2+p/2}\tau_t - 2x^3c_{tt}\right) + 8x^{4-p/2}\tau_{tt}\right\}
$$

$$
+\alpha(p - 2)^2c(t)x^{p-1}u\left\{\alpha p^3x^p - 6\alpha p^2x^p + 8\alpha px^p - 8\beta\lambda px + 16\gamma q x^{q+2}\right\}
$$

$$
+4\lambda^2(p - 2)^2xuc(t)(x - \beta)(-\beta p + px - 2x) = 0.
$$

Since the functions $B$, $c$, $d$ and $\tau$ do not depend on $u$, we split (3.18) with respect to $u$ and get

$$
1 : \quad \alpha px^pB_{xx} + B\gamma x^q + B_t + \beta\lambda B_x - \lambda x B_x = 0
$$

$$
u : \quad 8\alpha(p - 2)x^{2+p/2}(2\gamma(p - q - 2)x^q - \lambda(p - 2)(p - 1))\tau_t - 8\lambda^2(p - 2)^2x^{4-p/2}\tau_t
$$

$$
+8\beta\lambda u(p - 2)\left\{\alpha(p - 1)px^{1+p/2}\tau_t - \beta\lambda(p - 1)x^{2-p/2}\tau_t + \lambda(2p - 3)x^{3-p/2}\tau_t\right\}
$$

$$
+u\left\{8(p - 2)\left(2\alpha(p - 2)dx^{2+p/2} + \alpha(p - 1)x^{2+p/2}\tau_t - 2x^3c_{tt}\right) + 8x^{4-p/2}\tau_{tt}\right\}
$$

$$
+\alpha(p - 2)^2c(t)x^{p-1}u\left\{\alpha p^3x^p - 6\alpha p^2x^p + 8\alpha px^p - 8\beta\lambda px + 16\gamma q x^{q+2}\right\}
$$

$$
+4\lambda^2(p - 2)^2xuc(t)(x - \beta)(-\beta p + px - 2x) = 0.
$$
It is clear that $B$ satisfies the original equation (3.1). Rewriting (3.20) yields our classifying equation as

$$
\begin{align*}
&h_0(t)x^{5p/2} + h_1(t)x^{3+2+3p/2} + h_2(t)x^{1+3p/2} + h_3(t)x^4 + h_4(t)x^3 + h_5(t)x^{p+2} + h_6(t)x^{p+q+3} \\
&+h_7(t)x^{p+3} + h_8(t)x^{3+p/2} + h_9(t)x^{2+p/2} + h_{10}(t)x^{4+p/2} + h_{11}(t)x^5 = 0,
\end{align*}
$$

(3.21)

where

$$
\begin{align*}
h_0(t) &= \alpha^2(p-4)(p-2)^3p(t), \\
h_1(t) &= 16\alpha^2(p-2)^2q\varepsilon(t), \\
h_2(t) &= -8\alpha\beta\lambda(p-2)^2pc(t), \\
h_3(t) &= 8\beta\lambda^2(p-2)(2p-3)r'(t), \\
h_4(t) &= -8\beta^2\lambda^3(p-2)(p-1)r(t), \\
h_5(t) &= 8\alpha\beta\lambda(p-2)(p-1)p\varepsilon(t), \\
h_6(t) &= 16\alpha\gamma(p-2)(p-q-2)r'(t), \\
h_7(t) &= 8\alpha(p-2)((p-1)r''(t) - \lambda(p-2)(p-1)r'(t) + 2(p-2)d'(t)), \\
h_8(t) &= -8\beta\lambda^2(p-2)^2(p-1)c(t), \\
h_9(t) &= 4\beta^2\lambda^2(p-2)^2pc(t), \\
h_{10}(t) &= 4(p-2)(\lambda^2p^2c(t) - 4\lambda^2pc(t) + 4\lambda^2c(t) - 4c''(t)), \\
h_{11}(t) &= -8\left(\lambda^2p^2r'(t) - 4\lambda^2pr'(t) + 4\lambda^2r'(t) - r''(t)\right).
\end{align*}
$$

3.2.2 Classifying equation of (3.1) for $p = 2$

In the case when $p = 2$ in (3.1), we proceed as above and get the determining equation as

$$
\begin{align*}
qu\gamma\xi x^{q-1} + \gamma\eta x^q + u\gamma\tau_x x^q - u\beta\gamma\lambda u_x\tau_u x^q + u\beta\gamma\lambda\tau_x x^q + u\gamma u_x\xi_u x^q \\
-\nu\gamma\eta u x^q - u^2\gamma^2\tau_u x^{2q} + u\gamma\lambda u_x\tau_u x^{q+1} - u\gamma\lambda\tau_x x^{q+1} - u\gamma\tau_u u_x x^{q+2} \\
+u\alpha\gamma u_x^2\xi_u x^{2q+2} + 2u\alpha\gamma u_x\tau_x u x^{q+2} + u\alpha\gamma\tau_x x^{q+2} + \alpha^2 u_x^2\tau_u^{q+2} + \frac{2\alpha^2 u_x u_{xx} x^{q+2}}{2} \\
+2\alpha^2 u_{xx} u_{xx} x^4 + \alpha^2 u_{xx} x^4 - \alpha u_{xx} x^4 - \alpha\lambda\tau_x u_{xx} x^3 - \alpha\lambda u_x^2\tau_u x^3 - 2\alpha\lambda u_x^2\tau_x x^3 \\
-\alpha\lambda u_x\tau_x x^3 + \lambda^2 u_{xx} x^2 - 2\alpha u_x\tau_x u x^2 - 2\alpha u_{xx} x^2 + \alpha\tau u_{xx} x^2 \\
+\alpha\beta\lambda\tau_x u_{xx} x^2 - 2\alpha u_x\xi_u u_{xx} x^2 - 2\alpha\xi u_{xx} x^2 + \alpha\beta\lambda u_x^2\tau_u x^2 + 2\alpha\beta\lambda u_x^2\tau_x x^2.
\end{align*}
$$
\[ +\alpha \beta \lambda u_x \tau_{xx} x^2 - \alpha u_x^3 \xi_{uu} x^2 - 2\alpha u_x^2 \xi_{uu} x^2 - \alpha u_x \xi_{xx} x^2 + \alpha u_x^2 \eta_{uu} x^2 + 2\alpha u_x \eta_{uu} x^2 + \alpha \eta_{xx} x^2 - \lambda u_x \tau_{xx} x - 2\beta \lambda^2 u_x \tau_{xx} x + \lambda u_x \xi_{xx} x - \lambda \eta_x x + 2\alpha \xi_{xx} x - \lambda u_x \xi + \beta \lambda u_x \tau_{xx} - u_x \xi_t - \beta \lambda u_x \xi_x + \eta_t + \beta \lambda \eta_x = 0. \] (3.22)

As before, splitting the above equation (3.22) on derivatives of \( u \) and simplifying leads to

\[ \tau_u = 0, \] (3.23)
\[ \xi_u = 0, \] (3.24)
\[ \eta_{uu} = 0, \] (3.25)
\[ \tau_x = 0, \] (3.26)
\[ 2\alpha \eta_{xx} x^3 - \alpha \xi_{xx} x^3 - \lambda \xi_x x^2 + \lambda \xi_x^2 - \xi_t x + \beta \lambda \xi_t x - 2\beta \lambda \xi = 0, \] (3.27)
\[ 2x \xi_x - 2\xi - x\tau_t = 0, \] (3.28)
\[ x^2 (u \gamma \xi (q - 2) + x (\gamma u - u \gamma u + 2u \gamma \xi_x)) + x^2 \alpha \eta_{xx} x + x \lambda \eta_x (\beta - x) + x \eta_t = 0. \] (3.29)

Equations (3.23) and (3.26), imply that

\[ \tau = \tau(t), \] (3.30)

whereas (3.24) means that \( \xi \) does not depend on \( u \), that is,

\[ \xi = \xi(t, x) \] (3.31)

and (3.25) gives

\[ \eta(t, x, u) = A(t, x) u + B(t, x), \] (3.32)

after integrating twice by \( u \) and for some functions \( A \) and \( B \). Substituting (3.30) and (3.31) into (3.28), we get a linear first order ODE in \( \xi \) which can be easily integrated with respect to \( x \) to give

\[ \xi(t, x) = x e(t) + \frac{1}{2} x \tau'(t) \ln x, \] (3.33)

where \( e(t) \) is an arbitrary function of integration. If we substitute (3.30), (3.33) and (3.32) into (3.27) and solve the resulting differential equation for \( A \), we get

\[ A(t, x) = \frac{1}{8\alpha x} \left[ x [4e'(t) + \ln x \tau''(t)] + 2\tau'(t)[x(\alpha + \lambda) - \beta \lambda]\right] \ln x - 4\beta \lambda e(t) \]
\[ + f(t), \] (3.34)
where $f(t)$ is an arbitrary function of $t$. Substituting (3.34) and (3.33) into (3.29), we obtain

\[
8\alpha^2 x^4 B_{xx} - 8\alpha \lambda x^3 B_x + 8\alpha \beta \lambda x^2 B_x + 8\alpha x^2 B_t + 8\alpha \gamma x^4 B + \\
u (-4\alpha x^2 e'(t) - 4\lambda x^2 e'(t) + 4\epsilon(t) \left(2\alpha \gamma q x^{q+2} + \beta \lambda (\beta \lambda - x(2\alpha + \lambda))\right) + \\
4 x^2 e''(t) \ln x + 8\alpha x^2 f'(t) + 8\alpha \gamma x^{q+2} \tau'(t) + 4\alpha \gamma q x^{q+2} \tau'(t) \ln x - 2\beta^2 \lambda^2 \tau'(t) - \\
2\alpha^2 x^2 \tau'(t) - 4\alpha \lambda x^2 \tau'(t) + 2\alpha x^2 \tau''(t) - 2\lambda^2 x^2 \tau'(t) + x^2 \tau'''(t) \ln^2 x + 4\beta \lambda^2 x \tau'(t) + \\
+8\alpha \beta \lambda x \tau'(t) - 4\alpha \beta \lambda x \tau'(t) \ln x + 2\beta^2 \lambda^2 \tau'(t) \ln x - 2\beta^2 \lambda x \tau'(t) \ln x = 0. \quad (3.35)
\]

Splitting equation (3.35) on $u$ yields

\[
1 : \quad B_t + \gamma x^4 B + \alpha x^2 B_{xx} + (\beta - x) \lambda B_x = 0 \quad (3.36)
\]

\[
u : \quad (4 x^2 e''(t) + 4\alpha \gamma q x^{q+2} \tau'(t) + 2\beta^2 \lambda^2 \tau'(t) - 4\alpha \beta \lambda x \tau'(t) - 2\beta^2 \lambda x \tau'(t) \ln x - \\
-4\alpha x^2 e'(t) - 4\lambda x^2 e'(t) + 4\epsilon(t) \left(2\alpha \gamma q x^{q+2} + \beta \lambda (\beta \lambda - x(2\alpha + \lambda))\right) + 8\alpha x^2 f'(t) + \\
+8\alpha \gamma x^{q+2} \tau'(t) - 2\beta^2 \lambda^2 \tau'(t) - 2\alpha x^2 \tau'(t) - 4\alpha \lambda x^2 \tau'(t) + 2\alpha x^2 \tau''(t) + \\
-2\lambda^2 x^2 \tau'(t) + x^2 \tau'''(t) \ln^2 x + 8\alpha \beta \lambda x \tau'(t) + 4\beta \lambda^2 x \tau'(t) = 0. \quad (3.37)
\]

Rewriting (3.37) we get our classifying equation as

\[
b_0(t) + b_1(t) \ln x + x(b_2(t) + b_3(t) \ln x) + x^2 (b_4(t) + b_5(t) \ln x + b_6(t) \ln^2 x) + \\
x^{q+2} (b_7(t) + b_8(t) \ln x) = 0, \quad (3.38)
\]

where

\[
b_0(t) = 4\beta^2 \lambda^2 e(t) - 2\beta^2 \lambda^2 \tau'(t), \quad b_1(t) = 2\beta^2 \lambda^2 \tau'(t), \quad b_2(t) = -4\beta \lambda (2\alpha + \lambda) (e(t) - \tau'(t)), \quad b_3(t) = -2\beta \lambda (2\alpha + \lambda) \tau'(t),
\]

\[
b_4(t) = 8\alpha f'(t) - 4(\alpha + \lambda) e'(t) - 2(\alpha + \lambda)^2 \tau'(t) + 2\alpha \tau''(t), \quad b_5(t) = 4\epsilon'(t), \quad b_6(t) = \tau'''(t), \quad b_7(t) = 4\alpha \gamma (2\alpha q \epsilon(t) + 2 \tau'(t)), \quad b_8(t) = 4\alpha \gamma q \tau'(t).
\]
3.3 Results of group classification

We note that our classifying equations (3.21) and (3.38) are satisfied if we choose

\[ c(t) = e(t) = 0, \quad d(t) = f(t) = c_2 \quad \text{and} \quad \tau(t) = c_1 \]

for some constants \( c_1 \) and \( c_2 \). Thus using these values, for both cases, the coefficients of the infinitesimal operator are

\[ \tau = c_1, \quad \xi = 0 \quad \text{and} \quad \eta = c_2 u + B(t, x), \]

where \( B(t, x) \) is any solution of equation (3.1).

Case 0. \( \alpha, \gamma, \lambda, \beta, p, q \) arbitrary

We obtain the following Lie point symmetries:

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = u \frac{\partial}{\partial u}, \quad X_B = B(t, x) \frac{\partial}{\partial u}, \quad \text{(3.39)} \]

where the symmetry associated with \( B \) is said to be a solution symmetry. Lie symmetries (3.39) generate what is called the principal Lie algebra.

By equating the powers of \( x \) in equation (3.21) and solving for \( p \) we infer that possible extensions of the principal Lie algebra are possible for the following values of \( p \):

\[ 0, 1, 3, 4, 6, 8, 8, 3, 4, 6. \]

In this project, we consider \( p = 0, 1 \) and 2, as these values of \( p \) provide us with very important equations in mathematics of finance. For example, when \( p = 2 \) and \( q = 0 \), we have the Black-Scholes equation. We obtain the Vasicek equation when \( p = 0 \) and \( q = 1 \) and CIR equation when \( p = 1 \) and \( q = 1 \).

We now discuss equation (3.1) for three different values of \( p \): \( p = 0, p = 1 \) and \( p = 2 \).

Case 1 \( p = 0 \)

The classifying equation (3.21) for \( p = 0 \) gives

\[ g_0(t)x^3 + g_1(t)x^4 + g_2(t)x^5 + g_3x^9 + g_4x^{q+2} = 0, \quad \text{(3.40)} \]
where

\[
\begin{align*}
g_0(t) &= -16 \left( \beta^2 \lambda^2 \tau(t) - 2 \beta \lambda^2 c(t) - 4 \alpha d'(t) - 2 \alpha \lambda \tau'(t) - \alpha \tau''(t) \right), \\
g_1(t) &= 32 c''(t) - 32 \lambda^2 c(t) + 48 \beta \lambda^2 \tau'(t), \\
g_2(t) &= 8 \tau'''(t) - 32 \lambda^2 \tau'(t), \\
g_3(t) &= 32 \alpha \gamma (q + 2) \tau'(t), \\
g_4(t) &= 64 \alpha \gamma q c(t).
\end{align*}
\]

By examining powers of \( x \) in (3.40), we find that the possible values of \( q \) are 0, 1, 2 and 3. Thus, we now look at the following four cases for \( q \):

**Case 1.1** \( q = 0 \)

When \( q = 0 \), (3.40) becomes

\[
(g_0(t) + g_3(t)) x^3 + g_1(t) x^4 + g_2(t) x^5 = 0.
\]

For the above equation to hold, \( g_i \)'s should all identically be zero, that is, we need to solve the following system of equations:

\[
\begin{align*}
x^3 : & \quad 32 \beta \lambda^2 c(t) + 64 \alpha d'(t) + 64 \alpha \gamma \tau'(t) + 32 \alpha \lambda \tau'(t) + 16 \alpha \tau''(t) \\
& \quad -16 \beta^2 \lambda^2 \tau'(t) = 0, \\
x^4 : & \quad 32 c''(t) - 32 \lambda^2 c(t) + 48 \beta \lambda^2 \tau'(t) = 0, \\
x^5 : & \quad 8 \tau^{(3)}(t) - 32 \lambda^2 \tau'(t) = 0.
\end{align*}
\]

Solving (3.41), we obtain

\[
\tau(t) = \frac{c_1 e^{2 \lambda t}}{2 \lambda} - \frac{c_2 e^{-2 \lambda t}}{2 \lambda} + c_3,
\]

where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants of integration. Using (3.44) into (3.42) leads to

\[
32 c''(t) + 48 \beta c_1 \lambda^2 e^{2 \lambda t} + 48 \beta c_2 \lambda^2 e^{-2 \lambda t} - 32 \lambda^2 c(t) = 0.
\]

Integrating (3.45) with respect to \( t \) and solving for \( c(t) \) yields

\[
c(t) = -\frac{1}{2} \beta c_1 e^{2 \lambda t} - \frac{1}{2} \beta c_2 e^{-2 \lambda t} + c_4 e^{\lambda t} + c_5 e^{-\lambda t},
\]

where

\[
\begin{align*}
g_0(t) &= -16 (\beta^2 \lambda^2 \tau(t) - 2 \beta \lambda^2 c(t) - 4 \alpha d'(t) - 2 \alpha \lambda \tau'(t) - \alpha \tau''(t)), \\
g_1(t) &= 32 c''(t) - 32 \lambda^2 c(t) + 48 \beta \lambda^2 \tau'(t), \\
g_2(t) &= 8 \tau'''(t) - 32 \lambda^2 \tau'(t), \\
g_3(t) &= 32 \alpha \gamma (q + 2) \tau'(t), \\
g_4(t) &= 64 \alpha \gamma q c(t).
\end{align*}
\]
where $c_4$ and $c_5$ are arbitrary constants. Substituting (3.46) and (3.44) into (3.41) and integrating with respect to $t$ gives

$$d(t) = \frac{\beta^2 c_1 e^{2\mu t}}{4\alpha} - \frac{\gamma c_1 e^{2\mu t}}{2\lambda} - \frac{1}{2} c_1 e^{2\mu t} + \frac{\gamma c_2 e^{-2\mu t}}{2\lambda} - \frac{\beta^2 c_2 e^{-2\mu t}}{4\alpha}$$

$$= \frac{\beta c_1 e^{-2\mu t}}{2\alpha} + \frac{\beta c_5 e^{-2\mu t}}{2\alpha} + c_6,$$

where $c_6$ is a constant of integration. Substituting expressions for $c$, $d$ and $\tau$ back into (3.14) and (3.15) we get the coefficients of the general Lie symmetry as

$$\tau(t) = \frac{c_1 e^{2\mu t}}{2\lambda} - \frac{c_2 e^{-2\mu t}}{2\lambda} + c_3,$$

$$\xi(t, x) = \frac{1}{2} c_1 e^{2\mu t} (x - \beta) + \frac{1}{2} c_2 e^{-2\mu t} (x - \beta) + c_4 e^{\lambda^2 t} + c_5 e^{-\lambda^2 t},$$

$$\eta(t, u) = \frac{1}{2\alpha} \left\{ c_1 u e^{2\mu t} (-\alpha \gamma + \alpha \lambda + \lambda^2 (\beta - x)^2) \right\} + \frac{\gamma c_2 u e^{-2\mu t}}{2\lambda}$$

$$+ \frac{c_4 \lambda u e^{\lambda^2 t} (x - \beta)}{\alpha} + c_6 u + B(t, x).$$

It follows that the corresponding Lie point symmetries that extend the principal Lie algebra are given by

$$X_3 = \frac{e^{2\mu t}}{2\lambda} \frac{\partial}{\partial t} + e^{2\mu t} \left( \frac{x}{2} - \frac{\beta}{2} \right) \frac{\partial}{\partial x} + u e^{2\mu t} \left( \frac{\beta^2 \lambda}{2\alpha} - \frac{\gamma}{2\lambda} + \frac{\lambda x^2}{2\alpha} - \frac{\beta \lambda x}{\alpha} - \frac{1}{2} \right) \frac{\partial}{\partial u},$$

$$X_4 = \frac{e^{-2\mu t}}{2\lambda} \left[ - \frac{\partial}{\partial t} + \lambda (x - \beta) \frac{\partial}{\partial x} + \gamma u \frac{\partial}{\partial u} \right],$$

$$X_5 = e^{\lambda^2 t} \left[ \frac{\partial}{\partial x} + \frac{\lambda}{\alpha} u (x - \beta) \frac{\partial}{\partial u} \right],$$

$$X_6 = e^{-\lambda^2 t} \frac{\partial}{\partial x},$$

which generates an infinite dimensional Lie algebra.

Case 1.2 $q = 1$

Continuing in the same manner as in Case 1.1, we find that in this case the principal Lie algebra extends by the following Lie point symmetries:

$$X_3 = \frac{e^{2\mu t}}{2\lambda} \frac{\partial}{\partial t} + e^{2\mu t} \left( \frac{-\alpha \gamma + \beta}{\lambda^2} - \frac{x}{2} \right) \frac{\partial}{\partial x} + u e^{2\mu t} \left( \frac{\beta^2 \lambda}{2\alpha} + \frac{\alpha \gamma^2}{2\lambda} + \beta \gamma + \frac{\lambda x^2}{2\alpha} - \frac{\beta \lambda x}{\alpha} - \frac{3 \gamma x}{2\lambda} - \frac{1}{2} \right) \frac{\partial}{\partial u},$$

$$X_4 = \frac{e^{-2\mu t}}{2\lambda} \left[ - \frac{\partial}{\partial t} + \frac{1}{\lambda} \left( \lambda^2 (x - \beta) - 2 \alpha \gamma \right) \frac{\partial}{\partial x} + \frac{\gamma u}{\lambda^2} \left( \lambda^2 x - \alpha \gamma \right) \frac{\partial}{\partial u} \right].$$
\[ X_5 = e^{\lambda t} \left[ \frac{\partial}{\partial x} + \frac{u}{\alpha \lambda} (\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \frac{\partial}{\partial u} \right], \]
\[ X_6 = e^{-\lambda t} \left[ \frac{\partial}{\partial x} + \frac{\gamma u}{\lambda} \frac{\partial}{\partial u} \right]. \]

It should be noted that this case results in the Vasicek equation [54].

**Case 1.3** $q = 2$

This case has two subcases.

1.3.1 $\alpha \neq \frac{\lambda^2}{4\gamma}$

The Lie point symmetries are given by
\[ X_3 = \frac{e^{2\lambda t}}{2\kappa} \frac{\partial}{\partial t} + e^{2\lambda t} \left( \frac{1}{2} x - \frac{\beta \lambda}{2\kappa} \right) \frac{\partial}{\partial x} + \frac{ue^{2\lambda t}}{4\alpha \kappa^3} \left( -2\alpha \beta \gamma \lambda^2 + (\kappa + \lambda) \left( (\kappa x - \beta \lambda) (\kappa^2 x - \beta \lambda^2) - \alpha \kappa x \right) \right) \frac{\partial}{\partial u}, \]
\[ X_4 = -\frac{e^{-2\lambda t}}{2\kappa} \frac{\partial}{\partial t} + e^{-2\lambda t} \left( \kappa^2 x - \beta \lambda^2 \right) \frac{\partial}{\partial x} - \frac{ue^{-2\lambda t}}{4\alpha \kappa^3} \left( \alpha \lambda \left( 2\beta \gamma \lambda + \kappa (\kappa + \lambda) \right) - 4\alpha \gamma \kappa - \beta \lambda^3 (\kappa + \lambda) + \kappa^3 x^2 (\kappa - \lambda) + \beta \kappa \lambda x (\kappa - \lambda)^2 \right) \frac{\partial}{\partial u}, \]
\[ X_5 = e^{\alpha t} \left[ \frac{\partial}{\partial x} + \frac{1}{2\alpha \kappa} u (\kappa + \lambda) (x \kappa + \beta \lambda) \frac{\partial}{\partial u} \right], \]
\[ X_6 = e^{-\alpha t} \left[ \frac{\partial}{\partial x} - \frac{1}{2\alpha \kappa} u (\kappa - \lambda) (x \kappa - \beta \lambda) \frac{\partial}{\partial u} \right], \]

where $\kappa = \sqrt{\lambda^2 - 4\alpha \gamma}$.

1.3.2 $\alpha = \frac{\lambda^2}{4\gamma}$

The Lie point symmetries are
\[ X_3 = t \frac{\partial}{\partial t} + \left( \frac{x}{2} - \frac{3}{4} \beta \lambda^2 t^2 \right) \frac{\partial}{\partial x} + u \left( \frac{1}{2} \beta^2 \gamma \lambda^2 t^3 + \frac{3}{2} \beta^2 \gamma \lambda t^2 - \frac{3}{2} \beta \gamma \lambda t^2 x \right) \frac{\partial}{\partial u}, \]
\[ + \beta^2 \gamma t - \frac{\lambda t}{2} - 3 \beta \gamma tx + \frac{\gamma x^2}{\lambda} - \frac{\beta \gamma}{\lambda} \frac{\partial}{\partial u}, \]
\[ X_4 = t^2 \frac{\partial}{\partial t} + \left( tx - \frac{1}{2} \beta \lambda^2 t^3 \right) \frac{\partial}{\partial x} + u \left( \frac{1}{4} \beta^2 \gamma \lambda^2 t^4 + \beta \gamma \lambda t - \beta \gamma \lambda t^3 x \right) \frac{\partial}{\partial u}, \]
\[ + \beta^2 \gamma t^2 - \frac{\lambda t^2}{2} - 3 \beta \gamma tx^2 + \frac{2 \gamma t x^2}{\lambda} - \frac{2 \beta \gamma t x}{\lambda} - \frac{t}{2} + \frac{\gamma x^2}{\lambda^2} \right) \frac{\partial}{\partial u}, \]
\[ X_5 = \frac{\partial}{\partial x} + u \left( \frac{2 \gamma x}{\lambda} - 2 \beta \gamma t \right) \frac{\partial}{\partial u}, \]
\[ X_6 = t \frac{\partial}{\partial x} + u \left( -\beta \gamma t^2 - 2 \beta \gamma t + 2 \gamma t x + 2 \gamma x \right) \frac{\partial}{\partial u}. \]
Case 1.4 \( q = 3 \)

This case does not extend the principal Lie algebra.

We now consider the case when \( p = 1 \).

Case 2 \( p = 1 \)

Substituting \( p = 1 \) into (3.21), we obtain

\[
 a_0(t)x^{3/2} + a_1(t)x^{7/2} + a_2(t)x^{5/2} + a_3x^4 + a_4x^{q+4} + a_5(t)x^5 = 0,
\]  
(3.47)

where

\[
 a_0(t) = 16c''(t) - 4\lambda^2 c(t),
 a_1(t) = 16\alpha\gamma q c(t),
 a_2(t) = c(t)(\alpha - 2\beta\lambda)(3\alpha - 2\beta\lambda),
 a_3(t) = 16\alpha d'(t) + 8\beta\lambda^2 \tau'(t),
 a_4(t) = 16\alpha\gamma(q + 1)\tau'(t),
 a_5(t) = 8\tau''(t) - 8\lambda^2 \tau'(t).
\]

We deduce from (3.47) that \( q \) takes the values 0, \( \frac{1}{2} \) and \( \frac{3}{2} \). Thus, we need to consider the following four cases for \( q \):

Case 2.1 \( q = 0 \)

This leads to three subcases.

2.1.1 \( \beta \neq \frac{\alpha}{2\lambda} \) and \( \beta \neq \frac{3\alpha}{2\lambda} \)

The principal Lie algebra is extended by

\[
 X_3 = e^{\lambda t} \left[ \frac{1}{\lambda} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \left( \frac{\lambda x}{\alpha} - \beta \lambda - \frac{\gamma}{\lambda} \right) \frac{\partial}{\partial u} \right],
 X_4 = e^{-\lambda t} \left[ - \frac{\partial}{\partial t} + x \frac{\lambda}{\partial x} + \gamma u \frac{\partial}{\partial u} \right].
\]

2.1.2 \( \beta = \frac{\alpha}{2\lambda} \)

The Lie point symmetries are

\[
 X_3 = e^{\lambda t} \left[ \frac{1}{\lambda} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \left( \frac{\lambda x}{\alpha} - \gamma - \frac{1}{2} \right) \frac{\partial}{\partial u} \right],
\]
2.1.3 $\beta = \frac{3\alpha}{2\lambda}$

The Lie point symmetries that extend the principal Lie algebra are

\[
X_3 = \frac{e^{\lambda t}}{\lambda} \left[ -\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \gamma u \frac{\partial}{\partial u} \right] + e^{\lambda t} \left( \frac{\lambda x}{\alpha} - \frac{\gamma}{\lambda} - \frac{3}{2} \right) \frac{\partial}{\partial u},
\]

\[
X_4 = \frac{e^{-\lambda t}}{\lambda} \left[ -\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \gamma u e^{-\lambda t} \frac{\partial}{\partial u} \right] + \frac{\gamma}{\lambda} \frac{\partial}{\partial u},
\]

\[
X_5 = \sqrt{xe^{\frac{\lambda t}{2}} \frac{\partial}{\partial x} + \frac{\lambda u \sqrt{xe^{\frac{\lambda t}{2}}}}{\alpha} \frac{\partial}{\partial u}},
\]

\[
X_6 = \sqrt{xe^{-\frac{\lambda t}{2}} \frac{\partial}{\partial x}}.
\]

Case 2.2 $q = \frac{1}{2}$

This has three subcases.

2.2.1 $\beta \neq \frac{\alpha}{2\lambda}$ and $\beta \neq \frac{3\alpha}{2\lambda}$

This case gives the same symmetries as the principal Lie algebra.

2.2.2 $\beta = \frac{\alpha}{2\lambda}$

This case results in the following extra Lie point symmetries:

\[
X_3 = \frac{e^{\lambda t}}{\lambda} \frac{\partial}{\partial t} + e^{\lambda t} \left( x - \frac{2\alpha \gamma \sqrt{x}}{\lambda^2} \right) \frac{\partial}{\partial x} + ue^{\lambda t} \left( \frac{\lambda x}{\alpha} - \frac{3\gamma \sqrt{x}}{\lambda} + \frac{\alpha}{\lambda^3} - \frac{1}{2} \right) \frac{\partial}{\partial u},
\]

\[
X_4 = -\frac{e^{-\lambda t}}{\lambda} \times \left( x - \frac{2\alpha \gamma \sqrt{x}}{\lambda^2} \right) \frac{\partial}{\partial x} + ue^{-\lambda t} \left( \frac{\gamma \sqrt{x}}{\lambda} - \frac{\alpha}{\lambda^3} \right) \frac{\partial}{\partial u},
\]

\[
X_5 = \sqrt{xe^{\frac{\lambda t}{2}} \frac{\partial}{\partial x} + e^{\frac{\lambda t}{2}} \left( \frac{\lambda \sqrt{x}}{\alpha} - \gamma \right) \frac{\partial}{\partial u}},
\]

\[
X_6 = \sqrt{xe^{-\frac{\lambda t}{2}} \frac{\partial}{\partial x} + \frac{\gamma ue^{-\frac{\lambda t}{2}}}{\lambda} \frac{\partial}{\partial u}}.
\]

2.2.3 $\beta = \frac{3\alpha}{2\lambda}$
The extra Lie point symmetries are

\[
\begin{align*}
X_3 & = e^{\lambda t} \frac{\partial}{\partial t} + e^{\lambda t} \left( x - \frac{2\alpha \gamma \sqrt{x}}{\lambda^2} \right) \frac{\partial}{\partial x} + e^{\lambda t} \left( \frac{\alpha \gamma^2 - 3}{\lambda^2} + \frac{\alpha \gamma}{\lambda} + \frac{\lambda x - 3\gamma \sqrt{x}}{\lambda} \right) \frac{\partial}{\partial u}, \\
X_4 & = -e^{-\lambda t} \frac{\partial}{\partial t} + e^{-\lambda t} \left( x - \frac{2\alpha \gamma \sqrt{x}}{\lambda^2} \right) \frac{\partial}{\partial x} + e^{-\lambda t} \left( -\frac{\alpha \gamma^2}{\lambda^3} + \frac{\alpha \gamma}{\lambda^2} + \frac{\gamma \sqrt{x}}{\lambda} \right) \frac{\partial}{\partial u}, \\
X_5 & = \sqrt{x} e^{-\lambda t} \frac{\partial}{\partial x} + e^{-\lambda t} \left( \frac{\gamma}{\lambda} - \frac{1}{2\sqrt{x}} \right) \frac{\partial}{\partial u}, \\
X_6 & = \sqrt{x} e^{\lambda t} \frac{\partial}{\partial x} + e^{\lambda t} \left( \frac{\alpha \gamma \sqrt{x}}{\lambda} - \frac{\gamma}{\lambda} - \frac{1}{2\sqrt{x}} \right) \frac{\partial}{\partial u}.
\end{align*}
\]

Case 2.3 \( q = 1 \)

This leads to six subcases.

2.3.1 \( \beta \neq \frac{\alpha}{2\lambda} \) and \( \beta \neq \frac{3\alpha}{2\lambda} \) and \( \alpha \neq \frac{\lambda^2}{4\gamma} \)

The principal Lie algebra is extended by

\[
\begin{align*}
X_3 & = e^{\kappa t} \frac{\partial}{\partial t} + xe^{\kappa t} \frac{\partial}{\partial x} + \frac{ue^{\kappa t}}{2\alpha} \left( x\kappa - \frac{\beta \lambda^2}{\kappa} + \lambda x - \beta \lambda \right) \frac{\partial}{\partial u}, \\
X_4 & = -e^{-\kappa t} \frac{\partial}{\partial t} + xe^{-\kappa t} \frac{\partial}{\partial x} + \frac{ue^{-\kappa t}}{2\alpha} \left( \frac{\beta \lambda^2}{\kappa} - \beta \lambda + \lambda x - x\kappa \right) \frac{\partial}{\partial u},
\end{align*}
\]

where \( \kappa = \sqrt{\lambda^2 - 4\alpha \gamma} \). We note that this case gives us the CIR equation [53, 54].

2.3.2 \( \alpha = \frac{\lambda^2}{4\gamma}, \beta \neq \frac{\alpha}{2\lambda} \) and \( \beta \neq \frac{3\alpha}{2\lambda} \)

The extra symmetries that extend the principal Lie algebra are given by

\[
\begin{align*}
X_3 & = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \gamma u \left( \frac{2x}{\lambda} - 2\beta t \right) \frac{\partial}{\partial u}, \\
X_4 & = t^2 \frac{\partial}{\partial t} + 2tx \frac{\partial}{\partial x} + \gamma u \left( \frac{4tx}{\lambda} + \frac{4x}{\lambda^2} - 2\beta t^2 - \frac{4\beta t}{\lambda} \right) \frac{\partial}{\partial u}.
\end{align*}
\]

2.3.3 \( \alpha = \frac{\lambda^2}{4\gamma}, \beta = \frac{\alpha}{2\lambda} \)

Additional Lie point symmetries are

\[
\begin{align*}
X_3 & = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \left( \frac{2\gamma x}{\lambda} - \frac{1}{4} \lambda t \right) \frac{\partial}{\partial u}, \\
X_4 & = t^2 \frac{\partial}{\partial t} + 2tx \frac{\partial}{\partial x} + u \left( \frac{4\gamma tx}{\lambda} - \frac{1}{4} \lambda t^2 - \frac{1}{2} t + \frac{4\gamma x}{\lambda^2} \right) \frac{\partial}{\partial u}, \\
X_5 & = \sqrt{x} \frac{\partial}{\partial x} + \frac{2\gamma u \sqrt{x}}{\lambda} \frac{\partial}{\partial u}.
\end{align*}
\]
\[ X_6 = t \sqrt{x} \frac{\partial}{\partial x} + 2 \gamma u \sqrt{x} \left( \frac{2}{\lambda^2} + \frac{t}{\lambda} \right) \frac{\partial}{\partial u}. \]

2.3.4  \( \alpha = \frac{\lambda^2}{4\gamma}, \beta = \frac{3\alpha}{2\lambda} \)

Additional Lie point symmetries to the principal Lie algebra are given by

\[
\begin{align*}
X_3 &= t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \left( \frac{2\gamma x}{\lambda} - \frac{3}{4} t \lambda \right) \frac{\partial}{\partial u}, \\
X_4 &= t^2 \frac{\partial}{\partial t} + 2tx \frac{\partial}{\partial x} + u \left( \frac{4\gamma tx}{\lambda^2} - \frac{3}{2} t^2 + \frac{4\gamma x}{\lambda^2} - \frac{3\lambda t^2}{4} \right) \frac{\partial}{\partial u}, \\
X_5 &= \sqrt{x} \frac{\partial}{\partial x} + u \left( \frac{2\gamma \sqrt{x}}{\lambda} - \frac{1}{2\sqrt{x}} \right) \frac{\partial}{\partial u}, \\
X_6 &= t \sqrt{x} \frac{\partial}{\partial x} + u \left( \frac{2\gamma t \sqrt{x}}{\lambda} - \frac{t}{2\sqrt{x}} + \frac{4\gamma \sqrt{x}}{\lambda^2} \right) \frac{\partial}{\partial u}.
\end{align*}
\]

2.3.5  \( \beta = \frac{\alpha}{2\lambda}, \alpha \neq \frac{\lambda^2}{4\gamma} \)

The extra Lie point symmetries are

\[
\begin{align*}
X_3 &= \frac{e^{t\kappa}}{\kappa} \frac{\partial}{\partial t} + xe^{t\kappa} \frac{\partial}{\partial x} + ue^{t\kappa} \left( \frac{x\kappa}{2\alpha} - \frac{1}{4} - \frac{\lambda}{4\kappa} + \frac{\lambda x}{2a} \right) \frac{\partial}{\partial u}, \\
X_4 &= -\frac{e^{-t\kappa}}{\kappa} \frac{\partial}{\partial t} + xe^{-t\kappa} \frac{\partial}{\partial x} + ue^{-t\kappa} \left( \frac{\lambda}{4\kappa} - \frac{1}{4} + \frac{\lambda x}{2a} - \frac{x\kappa}{2a} \right) \frac{\partial}{\partial u}, \\
X_5 &= \sqrt{x} e^{\frac{1}{2} t\kappa} \frac{\partial}{\partial x} + u \sqrt{x} e^{\frac{1}{2} t\kappa} \frac{\lambda + \kappa \partial}{2\alpha} \frac{\partial}{\partial u}, \\
X_6 &= \sqrt{x} e^{-\frac{3}{2} t\kappa} \frac{\partial}{\partial x} + u \sqrt{x} e^{-\frac{3}{2} t\kappa} \frac{\lambda - \kappa \partial}{2\alpha} \frac{\partial}{\partial u},
\end{align*}
\]

where \( \kappa = \sqrt{\lambda^2 - 4\alpha \gamma} \).

2.3.6  \( \beta = \frac{3\alpha}{2\lambda}, \alpha \neq \frac{\lambda^2}{4\gamma} \)

The principal Lie algebra is extended by

\[
\begin{align*}
X_3 &= \frac{e^{\kappa t}}{\kappa} \frac{\partial}{\partial t} + xe^{\kappa t} \frac{\partial}{\partial x} + e^{\kappa t} \left( \frac{\kappa x}{2\alpha} - \frac{3\lambda}{4\kappa} - \frac{3}{4} + \frac{\lambda x}{2\alpha} \right) \frac{\partial}{\partial u}, \\
X_4 &= -\frac{e^{-\kappa t}}{\kappa} \frac{\partial}{\partial t} + xe^{-\kappa t} \frac{\partial}{\partial x} + e^{-\kappa t} \left( \frac{3\lambda}{4\kappa} - \frac{3}{4} + \frac{\lambda x}{2\alpha} - \frac{\kappa x}{2\alpha} \right) \frac{\partial}{\partial u}, \\
X_5 &= \sqrt{x} e^{\frac{3}{2} \kappa t} \frac{\partial}{\partial x} + e^{\frac{3}{2} \kappa t} \left( \frac{\lambda \sqrt{\lambda}}{2\alpha} + \frac{\kappa \sqrt{\lambda}}{2\alpha} - \frac{1}{2\sqrt{\lambda}} \right) \frac{\partial}{\partial u}, \\
X_6 &= \sqrt{x} e^{-\frac{3}{2} \kappa t} \frac{\partial}{\partial x} + e^{-\frac{3}{2} \kappa t} \left( \frac{\lambda \sqrt{\lambda}}{2\alpha} - \frac{\kappa \sqrt{\lambda}}{2\alpha} - \frac{1}{2\sqrt{\lambda}} \right) \frac{\partial}{\partial u}.
\end{align*}
\]
where $\kappa = \sqrt{\lambda^2 - 4\alpha \gamma}$.

**Case 2.4** $q = \frac{3}{2}$

This does not lead to any extension of the principal Lie algebra.

**Case 3** $p = 2$

We can conclude from (3.38) that $q$ can only take the value 0. Proceeding as before, we find that the principal Lie algebra extends for the case when $\beta = 0$ by the following symmetry operators:

\[
X_3 = t \frac{\partial}{\partial t} + \frac{1}{2} x \ln x \frac{\partial}{\partial x} + u \left( \frac{\lambda t}{4\alpha} + \frac{\lambda t^2}{4\alpha} - \gamma t + \frac{\lambda t}{2} - \frac{\gamma t^2}{4\alpha} + \frac{\lambda t}{4\alpha} + \frac{\gamma t}{4\alpha} \right) \frac{\partial}{\partial u},
\]

\[
X_4 = t^2 \frac{\partial}{\partial t} + t x \ln x \frac{\partial}{\partial x} + u \left( \frac{\lambda t^2}{4\alpha} + \frac{\gamma t^2}{4\alpha} - \gamma t^2 + \frac{\lambda t^2}{2} - \frac{\gamma t^2}{4\alpha} \right) \frac{\partial}{\partial u},
\]

\[
X_5 = x \frac{\partial}{\partial x},
\]

\[
X_6 = t x \frac{\partial}{\partial x} + u \left( \frac{\lambda t^2}{2\alpha} + \frac{\lambda t}{2} + \frac{\gamma t^2}{2\alpha} + \frac{\gamma t}{2\alpha} \right) \frac{\partial}{\partial u}.
\]

This case gives us the Black-Scholes equation [17].

### 3.4 Conclusions

Group classification of the general bond-option pricing PDE (3.1) was performed for the values $p = 0, 1$ and 2. The principal Lie algebra was found to be $X_1 = \partial_t$, $X_2 = u \partial_u$ and $X_B = B(t, x) \partial_u$. These values of $p$ resulted in 16 cases, which extended the principal Lie algebra. We presented the Lie point symmetries for each case. Three cases gave us the option pricing equations, which were given by Cases 1.1, 2.1.1 and 3. In the last case, Black-Scholes equation was recovered. Seven bond pricing equations were obtained and they were Case 1.2 and Cases 2.3.1-2.3.6. Cases 1.2 and 2.3.1 were found to be the Vasicek and CIR equations, respectively.
Chapter 4

Optimal systems and classification of group invariant solutions for some bond-option pricing equations

4.1 Introduction

In this chapter, we first obtain optimal systems of one-dimensional subalgebras for Cases 1.2 and 2.1.1 of Chapter three. Subsequently, we perform symmetry reductions and construct group invariant solutions using each element of the optimal systems of the corresponding case.

4.2 Classification of group invariant solutions of Case 1.2

To find the optimal system of one-dimensional subalgebras, we first, compute the commutator and adjoint representation tables.
4.2.1 Computation of commutators

For the two symmetries $X_1$ and $X_2$, its commutator is given by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

We now compute the commutator table for the Case 1.2. Recall that the symmetries for Case 1.2 are (omitting the solution symmetry $X_B$)

$$X_1 = \frac{e^{2\lambda t}}{2\lambda} \frac{\partial}{\partial t} + \frac{e^{2\lambda t}}{2\lambda^2} (\lambda^2 x - 2\alpha \gamma - \beta \lambda^2) \frac{\partial}{\partial x} + \frac{ue^{2\lambda t}}{2\alpha \lambda^3} (\alpha^2 \gamma^2 + 2\alpha \beta \gamma \lambda^2 - \alpha \lambda^3 - 3\alpha \gamma \lambda^2 x + \lambda^4 (\beta - x)^2) \frac{\partial}{\partial u},$$

$$X_2 = \frac{e^{-2\lambda t}}{2\lambda} \left[ -\frac{\partial}{\partial t} + \frac{1}{\lambda} (\lambda^2 (x - \beta) - 2\alpha \gamma) \frac{\partial}{\partial x} + \frac{\gamma u}{\lambda^2} (\lambda^2 x - \alpha \gamma) \frac{\partial}{\partial u} \right],$$

$$X_3 = \frac{\partial}{\partial t},$$

$$X_4 = e^{\lambda t} \left[ \frac{\partial}{\partial x} + \frac{u}{\alpha \lambda} (-\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \frac{\partial}{\partial u} \right],$$

$$X_5 = e^{-\lambda t} \left[ \frac{\partial}{\partial x} + \frac{\gamma u}{\lambda} \frac{\partial}{\partial u} \right],$$

$$X_6 = u \frac{\partial}{\partial u}. \quad (4.1)$$

We show detailed work of computing nonzero commutators and later write them in a tabulated form. Bear in mind also that the commutator is skew-symmetric, $[X_i, X_j] = -[X_j, X_i]$ and that the diagonal elements in the commutator table are all zero, that is, $[X_i, X_i] = 0$.

As an illustration, we compute one nonzero commutator $[X_3, X_4]$.

$$[X_3, X_4] = X_3(X_4) - X_4(X_3)$$

$$= \frac{\partial}{\partial t} (e^{\lambda t}) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \left( e^{\lambda t} \frac{u}{\alpha \lambda} (-\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \right) \frac{\partial}{\partial u}$$

$$- e^{\lambda t} \left[ \frac{\partial}{\partial x} (1) + \frac{u}{\alpha \lambda} (-\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \frac{\partial}{\partial u} (1) \right]$$

$$= \lambda e^{\lambda t} \frac{\partial}{\partial x} + \lambda e^{\lambda t} \frac{u}{\alpha \lambda} (-\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \frac{\partial}{\partial u}$$

$$= \lambda \left( e^{\lambda t} \frac{\partial}{\partial x} + e^{\lambda t} \frac{u}{\alpha \lambda} (-\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \frac{\partial}{\partial u} \right)$$

$$= \lambda X_4.$$ 

The full commutator table is given in Table 4.1.
Table 4.1: The commutator table of subalgebras

<table>
<thead>
<tr>
<th>([X_i, X_j])</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_3)</th>
<th>(X_4)</th>
<th>(X_5)</th>
<th>(X_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_1)</td>
<td>0</td>
<td>(\frac{1}{\lambda} X_3 - \nu X_6)</td>
<td>(-2\lambda X_1)</td>
<td>0</td>
<td>(-X_4)</td>
<td>0</td>
</tr>
<tr>
<td>(X_2)</td>
<td>(-\frac{1}{\lambda} X_3 + \nu X_6)</td>
<td>0</td>
<td>(2\lambda X_2)</td>
<td>(-X_5)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(X_3)</td>
<td>(2\lambda X_1)</td>
<td>(-2\lambda X_2)</td>
<td>0</td>
<td>(\lambda X_4)</td>
<td>(-\lambda X_5)</td>
<td>0</td>
</tr>
<tr>
<td>(X_4)</td>
<td>0</td>
<td>(X_5)</td>
<td>(-\lambda X_4)</td>
<td>0</td>
<td>(-\frac{\lambda}{\alpha} X_5)</td>
<td>0</td>
</tr>
<tr>
<td>(X_5)</td>
<td>(X_4)</td>
<td>0</td>
<td>(\lambda X_5)</td>
<td>(\frac{\lambda}{\alpha} X_6)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(X_6)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where \(\nu = \frac{(2\alpha \gamma^2 + 2\beta \gamma \lambda^2 + \lambda^3)}{2\lambda^3}\).

4.2.2 Adjoint representation

To compute all the entries in the adjoint table we give an illustration by computing the adjoint representation of \(X_1\) and \(X_2\). With assistance from Table 4.1, we proceed as follows:

\[
\text{Ad}(\exp(\varepsilon X_1))X_2 = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\text{ad} X_1)^n (X_2)
\]

\[
= X_2 - \varepsilon [X_1, X_2] + \frac{\varepsilon^2}{2!} [X_1, [X_1, X_2]] - \cdots
\]

\[
= X_2 - \varepsilon \left(\frac{1}{\lambda} X_3 - \nu X_6\right) + \frac{\varepsilon^2}{2!} [X_1, \frac{1}{\lambda} X_3 - \nu X_6] - \cdots
\]

\[
= X_2 - \varepsilon \left(\frac{1}{\lambda} X_3 - \nu X_6\right) - \varepsilon^2 X_1 - 0 \cdots
\]

\[
= -\varepsilon^2 X_1 + X_2 - \frac{\varepsilon}{\lambda} X_3 + \varepsilon \nu X_6.
\]

The remaining adjoint representation table entries are obtained in the same manner and are tabulated in Table 4.2.
<table>
<thead>
<tr>
<th>( \text{Ad} )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
<th>( X_5 )</th>
<th>( X_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>( X_1 )</td>
<td>( -\varepsilon^2 X_1 + X_2 - \frac{\varepsilon}{\lambda} X_3 + \varepsilon \nu X_6 )</td>
<td>( 2\varepsilon \lambda X_1 + X_3 )</td>
<td>( X_4 )</td>
<td>( \varepsilon X_4 + X_5 )</td>
<td>( X_6 )</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>( X_1 - \varepsilon^2 X_2 + \frac{\varepsilon}{\lambda} X_3 - \nu \varepsilon X_6 )</td>
<td>( X_2 )</td>
<td>( -2\varepsilon \lambda X_2 + X_3 )</td>
<td>( X_4 + \varepsilon X_5 )</td>
<td>( X_5 )</td>
<td>( X_6 )</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>( e^{-2\varepsilon \lambda} X_1 )</td>
<td>( e^{2\varepsilon \lambda} X_2 )</td>
<td>( X_3 )</td>
<td>( e^{-\varepsilon \lambda} X_4 )</td>
<td>( e^{\varepsilon \lambda} X_5 )</td>
<td>( X_6 )</td>
</tr>
<tr>
<td>( X_4 )</td>
<td>( X_1 )</td>
<td>( X_2 - \varepsilon X_5 - \frac{\lambda \varepsilon^2}{2 \alpha} X_6 )</td>
<td>( X_3 + \varepsilon \lambda X_4 )</td>
<td>( X_4 )</td>
<td>( X_5 + \frac{\varepsilon \lambda}{\alpha} X_6 )</td>
<td>( X_6 )</td>
</tr>
<tr>
<td>( X_5 )</td>
<td>( X_1 - \varepsilon X_4 + \frac{\lambda \varepsilon^3}{2 \alpha} X_6 )</td>
<td>( X_2 )</td>
<td>( X_3 - \varepsilon \lambda X_5 )</td>
<td>( X_4 - \frac{\varepsilon \lambda}{\alpha} X_6 )</td>
<td>( X_5 )</td>
<td>( X_6 )</td>
</tr>
<tr>
<td>( X_6 )</td>
<td>( X_1 )</td>
<td>( X_2 )</td>
<td>( X_3 )</td>
<td>( X_4 )</td>
<td>( X_5 )</td>
<td>( X_6 )</td>
</tr>
</tbody>
</table>

Table 4.2: Adjoint representation of subalgebras
4.2.3 Optimal system of one dimensional subalgebras

The Lie algebra $L_6$ spanned by symmetries (4.1) provides a possibility to find invariant solutions of Eq. (3.1) with $p = 0$ and $q = 1$ which is based on any one-dimensional subalgebra of $L_6$. In light of these we can write an arbitrary operator from $L_6$ as

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5 + \alpha_6 X_6,$$

which depends on the six arbitrary constants $\alpha_1, \ldots, \alpha_6$.

To construct the optimal system of one-dimensional subalgebra, we follow the method by Olver [43]. First, we need to find the invariant of the full adjoint map as it places limits on how far we can expect to simplify operator $X$. The composition of adjoint $X_1$ and $X_2$ on $X$, is given by

$$\tilde{X} = \sum_{j=1}^{6} \tilde{\alpha}_j X_j = \text{Ad} \left( e^{\varepsilon_1 X_1} \right) \circ \text{Ad} \left( e^{\varepsilon_2 X_2} \right) X.$$

Let us first start by computing $\text{Ad} \left( e^{\varepsilon_2 X_2} \right) X$ using the help of the adjoint Table 4.2.

$$\text{Ad} \left( e^{\varepsilon_2 X_2} \right) X = \text{Ad} \left( e^{\varepsilon_2 X_2} \right) \left( \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5 + \alpha_6 X_6 \right)$$

$$= \alpha_1 \left( X_1 - \varepsilon_2^2 X_2 + \frac{\varepsilon_2}{\lambda} X_3 - \nu \varepsilon_2 X_6 \right) + \alpha_2 X_2 + \alpha_3 (-2 \varepsilon_2 \lambda X_2 + X_3)$$

$$+ \alpha_4 (X_4 + \varepsilon_2 X_5) + \alpha_5 X_5 + \alpha_6 X_6$$

$$= \alpha_1 X_1 + (\alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda) X_2 + \left( \alpha_3 + \frac{\alpha_1 \varepsilon_2}{\lambda} \right) X_3$$

$$+ \alpha_4 X_4 + (\alpha_5 + \alpha_4 \varepsilon_2) X_5 + (\alpha_6 - \nu \alpha_1 \varepsilon_2) X_6.$$ 

By letting $X' = \text{Ad} \left( e^{\varepsilon_2 X_2} \right) X$ and then acting by $\text{Ad} \left( e^{\varepsilon_1 X_1} \right)$ on $X'$ yields

$$\text{Ad} \left( e^{\varepsilon_1 X_1} \right) X' = \tilde{\alpha}_1 X_1 + \tilde{\alpha}_2 X_2 + \tilde{\alpha}_3 X_3 + \tilde{\alpha}_4 X_4 + \tilde{\alpha}_5 X_5 + \tilde{\alpha}_6 X_6,$$

where

$$\tilde{\alpha}_1 = \alpha_1 - \varepsilon_1^2 (\alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda) + 2 \varepsilon_1 \lambda \left( \alpha_3 + \frac{\alpha_1 \varepsilon_2}{\lambda} \right),$$

$$\tilde{\alpha}_2 = \alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda,$$

$$\tilde{\alpha}_3 = \alpha_3 + \frac{\alpha_1 \varepsilon_2}{\lambda} - \frac{\varepsilon_1}{\lambda} \left( \alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda \right),$$

$$\tilde{\alpha}_4 = \alpha_4 + \varepsilon_1 (\alpha_5 + \alpha_4 \varepsilon_2),$$

$$\tilde{\alpha}_5 = \alpha_5 + \varepsilon_1 \lambda (\alpha_4 + \varepsilon_1) + \frac{\varepsilon_1}{\lambda} \left( \alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda \right),$$

$$\tilde{\alpha}_6 = \alpha_6 + \varepsilon_1 \lambda (\alpha_4 + \varepsilon_1) + \frac{\varepsilon_1}{\lambda} \left( \alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda \right).$$
\[ \tilde{\alpha}_5 = \alpha_5 + \alpha_4 \varepsilon_2, \]  
\[ \tilde{\alpha}_6 = \alpha_6 - \nu \alpha_1 \varepsilon_2 + \varepsilon_1 \nu (\alpha_2 - \alpha_1 \varepsilon_2^2 - 2 \alpha_3 \varepsilon_2 \lambda). \]

We now focus on (4.2), (4.3) and (4.4).

It can be seen that the invariant of the full adjoint action is given by \( \eta = \alpha_3^2 \lambda^2 + \alpha_1 \alpha_2 \). To find the optimal system of (4.1), we look at three cases based on the value of \( \eta \), that is, when \( \eta > 0 \), \( \eta < 0 \) and finally, when \( \eta = 0 \).

**Case 1.** If \( \eta > 0 \), we can make \( \tilde{\alpha}_2 = 0 \) and \( \tilde{\alpha}_1 = 0 \) by solving for \( \varepsilon_2 \) and \( \varepsilon_1 \) respectively, where \( \varepsilon_2 \) is the real root of (4.3). Since \( \eta > 0 \), we take \( \varepsilon_2 = -(\alpha_3 \lambda + \sqrt{\alpha_3^2 \lambda^2 + \alpha_1 \alpha_2})/\alpha_1 \) and \( \varepsilon_1 = \alpha_1/(2 \sqrt{\alpha_3^2 \lambda^2 + \alpha_1 \alpha_2}) \). The substitution of the expressions for \( \varepsilon_1 \) and \( \varepsilon_2 \) in (4.2) and (4.3), ensures that \( \tilde{\alpha}_1 = 0 \) and \( \tilde{\alpha}_2 = 0 \) while \( \tilde{\alpha}_3 = -\sqrt{\alpha_3^2 \lambda^2 + \alpha_1 \alpha_2}/\lambda \neq 0 \). By taking \( \tilde{\alpha}_3 = 1 \), then \( X \) is equivalent to a scalar multiple of \( \bar{X} = X_3 + \tilde{\alpha}_4 X_4 + \tilde{\alpha}_5 X_5 + \tilde{\alpha}_6 X_6 \). Now acting by \( \text{Ad}(\exp(-\tilde{\alpha}_4)) \) on \( \bar{X} \), we can make the coefficient of \( X_4 \) to vanish. Then \( X \) is a scalar multiple of \( \bar{X}' = X_3 + \tilde{\alpha}_5' X_5 + \tilde{\alpha}_6' X_6 \). Acting further by adjoint map generated by \( X_5 \) gives

\[ \text{Ad}(e^{\tilde{\alpha}_5}) \bar{X}' = X_3 + (\tilde{\alpha}_5' - \varepsilon \lambda) X_5 + \tilde{\alpha}_6' X_6. \]

If we let \( \varepsilon = \tilde{\alpha}_5'/\lambda \), then \( \bar{X}' \) becomes a scalar multiple of \( X_3 + a X_6 \), where \( a \in \mathbb{R} \) and no further simplifications are possible.

**Case 2.** If \( \eta < 0 \), we let \( \varepsilon_2 = 0 \), \( \varepsilon_1 = -(\alpha_3 \lambda)/\alpha_2 \) to make \( \tilde{\alpha}_3 = 0 \) while \( \tilde{\alpha}_1 \neq 0 \) and \( \tilde{\alpha}_2 \neq 0 \). We can scale both coefficients of \( X_1 \) and \( X_2 \) to one and, we also observe that \( X \) is equivalent to a scalar multiple of \( \bar{X} = (X_1 + X_2) + \tilde{\alpha}_4 X_4 + \tilde{\alpha}_5 X_5 + \tilde{\alpha}_6 X_6 \). Acting on \( \bar{X} \) by adjoint map generated by \( X_4 \), yields

\[ \text{Ad}(e^{\tilde{\alpha}_5}) \bar{X} = X_1 + X_2 + \tilde{\alpha}_4 X_4 + (\tilde{\alpha}_5 - \varepsilon) X_5 + \left( \tilde{\alpha}_6 + \frac{\varepsilon \lambda}{\alpha} \tilde{\alpha}_5 - \frac{\lambda \varepsilon^2}{2\alpha} \right) X_6. \]

By setting \( \varepsilon = \tilde{\alpha}_5 \), then \( \bar{X} \) is equivalent to a scalar multiple of \( \bar{X}' = X_1 + X_2 + \tilde{\alpha}_4' X_4 + \tilde{\alpha}_6' X_6 \). Acting by \( \text{Ad}(e^{\tilde{\alpha}_5'}) \) on \( \bar{X}' \), the coefficient of \( X_4 \) vanishes and then we observe that \( \bar{X}' \) is equivalent to a scalar multiple of \( X_1 + X_2 + a X_6 \), where \( a \) is a real constant. No further simplifications are possible at this stage.

**Case 3.** If \( \eta = 0 \), there are two possible subcases, either \( \tilde{\alpha}_1 = \tilde{\alpha}_2 = \tilde{\alpha}_3 = 0 \) or not all are zero. If we set \( \tilde{\alpha}_2 \neq 0 \) and \( \tilde{\alpha}_1 = \tilde{\alpha}_3 = 0 \), then \( X \) is equivalent to a scalar multiple of
Now suppose that $\alpha_4 \neq 0$, then we can make the coefficients of $X_6$ and $X_5$ vanish using groups generated by $X_5$ and $X_2$, respectively.

Furthermore, acting by group generated by $X_3$ independently, scales the coefficients of $X_2$ and $X_4$, that is, $e^{2\lambda t}X_2 + \tilde{\alpha}_4 e^{-3\lambda t}X_4$. This can be rearranged as $X_2 + \tilde{\alpha}_4 e^{-3\lambda t}X_4$ and depending on the sign of $\tilde{\alpha}_4$ we take the coefficient of $X_4$ as ±1. Thus, we get either $X_2 + X_4$ or $X_2 - X_4$. If $\alpha_4 = 0$, then the group generated by $X_4$ reduces $X$ to a vector of the form $X_2 + aX_6$, where $a$ is a constant.

The last subcase occurs when $\alpha_1 = \tilde{\alpha}_2 = \tilde{\alpha}_3 = 0$, which results in $\tilde{\alpha}_4 X_4 + \tilde{\alpha}_5 X_5 + \tilde{\alpha}_6 X_6$. Assuming that $\tilde{\alpha}_4 \neq 0$, then we can use groups generated by $X_5$ and $X_2$ to eliminate coefficients of $X_6$ and $X_5$, respectively. Hence, $X$ becomes a scalar multiple of $X_4$. If $\alpha_4 = 0$, then $X$ becomes a multiple of $\tilde{\alpha}_5 X_5 + \tilde{\alpha}_6 X_6$. If we let $\tilde{\alpha}_5 \neq 0$, then acting by a group generated by $X_4$, we reduce $X$ to a scalar multiple of $X_5$. Finally, if $\tilde{\alpha}_5 = 0$, we are left with multiples of $X_6$.

In summary, we get the following optimal system of one-dimensional subalgebras

\begin{align}
X_4, & \quad \eta = 0, \\
X_5, & \quad \eta = 0, \\
X_6, & \quad \eta = 0, \\
X_2 + X_4, & \quad \eta = 0, \\
X_2 - X_4, & \quad \eta = 0, \\
X_2 + aX_6, & \quad \eta = 0, \quad a \in \mathbb{R}, \\
X_1 + X_2 + aX_6, & \quad \eta < 0, \quad a \in \mathbb{R}, \\
X_3 + aX_6, & \quad \eta > 0, \quad a \in \mathbb{R}.
\end{align}

### 4.2.4 Symmetry reductions and group invariant solutions

We use the above system (4.8) to construct group invariant solutions of equation (3.1) when $p = 0$ and $q = 1$, which is given by

\[ u_t + \alpha u_{xx} + \lambda (\beta - x) u_x + \gamma xu = 0. \]
The system of one-dimensional subalgebras (4.8) gives seven cases of group invariant solutions. We note that operator $X_6$ does not have invariants, hence it is omitted when finding the invariant solutions.

**Case 1.1 $X_4$**

The symmetry operator $X_4$, namely,

$$X_4 = e^{\mu} \frac{\partial}{\partial x} + e^{\mu} \frac{u}{\alpha \lambda} (-\alpha \gamma - \beta \lambda^2 + \lambda^2 x) \frac{\partial}{\partial u},$$

whose characteristic equations are

$$\frac{dt}{0} = \frac{dx}{e^{u}} = \frac{du}{we^{\mu} (\lambda^2 x - \beta \lambda^2 - \alpha \gamma) / (\alpha \lambda)},$$

provides the two invariants

$$J_1 = t,$$

$$J_2 = u \exp \left\{ \frac{x(\lambda^2(-x + 2\beta) + 2\alpha \gamma)}{2\alpha \lambda} \right\}.$$  

Hence $J_2 = f(J_1)$ is an invariant solution of (4.9). This gives

$$u = f(t) \exp \left\{ \frac{x(\lambda^2(-x + 2\beta) - 2\alpha \gamma)}{2\alpha \lambda} \right\}. \quad (4.10)$$

Substituting (4.10) into (4.9) and simplifying we obtain an ODE in $t$ given by

$$\lambda^2 f'(t) + f(t) (\alpha \gamma^2 + \lambda^2 (\beta \gamma + \lambda)) = 0. \quad (4.11)$$

In order to integrate equation (4.11), we first rearrange and obtain

$$\frac{-\alpha \gamma^2 + \lambda^2 (\beta \gamma + \lambda)}{\lambda^2} = \frac{f'(t)}{f(t)},$$

which is now a variable separable ODE and its solution is

$$f(t) = k_1 \exp \left\{ \frac{t (-\alpha \gamma^2 - \beta \gamma \lambda^2 - \lambda^3)}{\lambda^2} \right\}, \quad (4.12)$$

where $k_1$ is a constant of integration. Substituting Eq. (4.12) into Eq. (4.10), we get the invariant solution of equation (4.9), under $X_4$, as

$$u(t, x) = k_1 \exp \left\{ \frac{\lambda x (\lambda^2(x - 2\beta) - 2\alpha \gamma) - 2\alpha t (\alpha \gamma^2 + \lambda^2 (\beta \gamma + \lambda))}{2\alpha \lambda^2} \right\}.$$
Case 1.2 $X_5$

The symmetry operator $X_5$ given by

$$X_5 = e^{-\lambda t} \frac{\partial}{\partial x} + \frac{\gamma u e^{-\lambda t}}{\lambda} \frac{\partial}{\partial u}$$

has the characteristic equations

$$\frac{dt}{0} = \frac{dx}{e^{-\lambda t}} = \frac{du}{\gamma u e^{-\lambda t}/\lambda},$$

which provide the two invariants $J_1 = t$ and $J_2 = u e^{-\gamma x/\lambda}$. Thus $J_2 = f(J_1)$ is an invariant solution of (4.9). That is, $u = e^{\gamma x/\lambda} f(t)$. Substituting this $u$ in equation (4.9), we obtain the first order ODE

$$\lambda^2 f'(t) + \gamma f(t) (\alpha \gamma + \beta \lambda^2) = 0,$$

whose solution is given by

$$f(t) = k_2 \exp \left\{ -\frac{\alpha \gamma^2 t}{\lambda^2} - \beta \gamma t \right\},$$

where $k_2$ is a constant of integration. Hence, the invariant solution of (4.9), under $X_5$ is

$$u(t, x) = k_2 \exp \left\{ -\frac{\gamma (\alpha \gamma t + \beta \lambda^2 t - \lambda x)}{\lambda^2} \right\}.$$

Case 1.3 $X_2 + aX_4$, $a = \pm 1$

The symmetry generator $X_2 + aX_4$ leads to the Lagrange system

$$dt \quad dx$$

$$-e^{-2\lambda t}/(2\lambda) = \frac{dx}{e^{-2\lambda t}(\lambda^3(x - \beta) - 2\alpha \gamma)/(2\lambda^2) + ae^{\lambda t}} = \frac{du}{\gamma u e^{-2\lambda t}(\lambda^2 x - \alpha \gamma)/(2\lambda^3) + au e^{\lambda t}(\alpha \gamma + \lambda^2(\beta - x))/(\alpha \lambda)}.$$

(4.13)

Using the first equation of (4.13) and simplifying we get a first order linear ODE which gives the invariant

$$J_1 = xe^{\lambda t} - \frac{2\alpha \gamma}{\lambda^2} e^{\lambda t} + \frac{1}{2} ae^{4\lambda t} - \beta e^{\lambda t}.$$  

(4.14)

By considering

$$\frac{-2\lambda dt}{e^{-2\lambda t}} = \frac{du}{\gamma u e^{-2\lambda t}(\lambda^2 x - \alpha \gamma)/(2\lambda^3) - au e^{\lambda t}(\alpha \gamma + \lambda^2(\beta - x))/(\alpha \lambda)}.$$  

(4.15)
and substituting $x$ from (4.14) into (4.15), we get a variables separable ODE in $u$ and $t$, which on solving gives the following invariant:

$$J_2 = u \exp \left\{ \frac{a\lambda e^{3\lambda t} (ae^{3\lambda t} - 3 \beta + 3x)}{3\alpha} - \gamma \left( \frac{2ae^{3\lambda t} - \beta(\lambda t + 1) + x}{\lambda} + \frac{\alpha \gamma^2 (\lambda t + 2)}{\lambda^3} \right) \right\}.$$ 

Thus $J_2 = f(J_1)$ is an invariant solution of (4.9), which gives

$$u = f(z) \exp \left\{ -\frac{a\lambda e^{3\lambda t} (ae^{3\lambda t} - 3 \beta + 3x)}{3\alpha} + \gamma \left( \frac{2ae^{3\lambda t} - \beta(\lambda t + 1) + x}{\lambda} \right) - \frac{\alpha \gamma^2 (\lambda t + 2)}{\lambda^3} \right\}, \quad z = xe^{\lambda t} - \frac{2\alpha \gamma}{\lambda^2} e^{\lambda t} + \frac{1}{2} a e^{4\lambda t} - \beta e^{\lambda t}. \quad (4.16)$$

Substitution of (4.16) into (4.9) and simplifying yields the second order nonlinear ODE

$$2a\lambda^2 z f'(z) - \alpha^2 f''(z) = 0,$$

whose solution, with the help of Mathematica, is given by

$$f(z) = \text{AiryAi} \left( \sqrt[3]{\frac{\alpha^2}{a}} \frac{\lambda^2 z}{2} \right) c_1 + \text{AiryAi} \left( \sqrt[3]{\frac{\alpha^2}{a}} \frac{\lambda^2}{2} \right) c_2, \quad (4.17)$$

where AiryAi is the Airy function $\text{Ai}(z)$ [1] and $c_1$ and $c_2$ are arbitrary constants. Hence, the invariant solution is given by

$$u(t,x) = f \left( e^{\lambda t} \left( \frac{\lambda^2 (ae^{3\lambda t} - 2\beta + 2x) - 4\alpha \gamma}{2\lambda^2} \right) \right) \exp \left\{ -\frac{a\lambda e^{3\lambda t} (ae^{3\lambda t} - 3 \beta + 3x)}{3\alpha} + \gamma \left( \frac{2ae^{3\lambda t} - \beta(\lambda t + 1) + x}{\lambda} \right) - \frac{\alpha \gamma^2 (\lambda t + 2)}{\lambda^3} \right\},$$

where $f$ is given by (4.17).

**Case 1.4 $X_2 + aX_6$**

We have two subcases.

**1.4.1 $a \neq 0$**

The symmetry $X_2 + aX_6$ gives rise to the following two invariants:

$$J_1 = xe^{\lambda t} - \frac{e^{\lambda t} (2\alpha \gamma + \beta \lambda^2)}{\lambda^2}, \quad (4.18)$$

$$J_2 = u \exp \left\{ ae^{2\lambda t} - \gamma \left( x - \frac{2\alpha \gamma + \beta \lambda^2}{\lambda^2} \right) + \frac{\alpha \gamma^2 t}{\lambda^2} + \beta \gamma t \right\}. \quad (4.19)$$
Equations (4.18) and (4.19) lead to the invariant solution

\[ u = f(z) \exp \left\{ -\frac{\lambda^2 (a \lambda e^{2\lambda t} + \beta \gamma + \beta \gamma \lambda t - \gamma x) + \alpha \gamma^2 (\lambda t + 2)}{\lambda^3} \right\}, \quad (4.20) \]

where \( z = x e^{\lambda t} - e^{\lambda t} (2 \alpha \gamma + \beta \lambda^2) / \lambda^2 \). Substituting (4.20) into (4.9) and simplifying we obtain

\[ \alpha f''(z) - 2a \lambda f(z) = 0, \]

whose solution is given by

\[ f(z) = a_1 e^{\sqrt{2a \lambda} \sqrt{z}} + a_2 e^{-\sqrt{2a \lambda} \sqrt{z}}, \]

where \( a_1 \) and \( a_2 \) are arbitrary constants. Hence, we obtain the invariant solution of equation (4.9) as

\[ u(t, x) = \left[ a_1 e^{\sqrt{2a \lambda} (2 \alpha \gamma + \beta \lambda^2 - x \lambda^2) / (\lambda^2 \sqrt{z})} + a_2 e^{-\sqrt{2a \lambda} (2 \alpha \gamma + \beta \lambda^2 - x \lambda^2) / (\lambda^2 \sqrt{z})} \right] \times \exp \left\{ -\left( a \lambda e^{2\lambda t} + \beta \gamma + \beta \gamma \lambda t - \gamma x \right) / \lambda - \alpha \gamma^2 (\lambda t + 2) / \lambda^3 \right\}. \]

1.4.2 \( a = 0 \)

The symmetry operator \( X_2 \) gives invariants

\[ J_1 = e^{\lambda t} (-2 \alpha \gamma - \beta \lambda^2 + \lambda^2 x), \]
\[ J_2 = u \exp \left\{ \gamma \left( \frac{\alpha \gamma t}{\lambda^2} + \beta t + \frac{2 \alpha \gamma + \beta \lambda^2 - \lambda^2 x}{\lambda^3} \right) \right\}. \]

The invariant solution is given by setting \( J_2 = f(J_1) \), that is,

\[ u = \exp \left\{ \gamma \left( -\frac{\alpha \gamma t}{\lambda^2} - \beta t - \frac{2 \alpha \gamma - \beta \lambda^2 + \lambda^2 x}{\lambda^3} \right) \right\} f \left( \frac{e^{\lambda t} (-2 \alpha \gamma - \beta \lambda^2 + \lambda^2 x)}{\lambda^2} \right). \]

Substituting this value of \( u \) into (4.9) and simplifying we obtain

\[ f''(z) = 0, \quad z = e^{\lambda t} (-2 \alpha \gamma - \beta \lambda^2 + \lambda^2 x) / \lambda^2. \quad (4.21) \]

The solution of equation (4.21) is

\[ f = c_1 + z c_2, \]

where \( c_1, c_2 \) are constants of integration. Hence, the invariant solution is given by

\[ u(t, x) = \left[ c_1 + c_2 e^{\lambda t} \left( -\frac{2 \alpha \gamma}{\lambda^2} - \beta + x \right) \right] \exp \left\{ -\frac{2 \alpha \gamma^2}{\lambda^3} - \frac{\beta \gamma}{\lambda} - \frac{\alpha \gamma^2 t}{\lambda^2} - \beta \gamma t + \frac{\gamma x}{\lambda} \right\}. \]
Case 1.5 \( X_1 + X_2 + aX_6 \)

We have to consider two subcases.

1.5.1 \( a \neq 0 \)

The operator \( X_1 + X_2 + aX_6 \) yields the two invariants

\[
J_1 = \frac{e^{\lambda t} (\lambda^2 (x - \beta) - 2\alpha \gamma)}{\lambda^2 \sqrt{1 - e^{4\lambda t}}},
\]

\[
J_2 = u (1 - e^{-2\lambda t})^{\frac{1}{4}(1-2a)} (e^{-2\lambda t} + 1)^{\frac{1}{4}(2a+1)} \exp \left\{ t \left( \frac{\alpha \gamma^2}{\lambda^2} + \beta \gamma + \lambda \right) \right. 
- \frac{e^{4\lambda t} (2\alpha \gamma + \lambda^2 (\beta - x))^2}{2\alpha \lambda^3 (e^{4\lambda t} - 1)} + \frac{\gamma (2\alpha \gamma + \lambda^2 (\beta - x))}{\lambda^3} \right\}.
\]

The above invariants give the invariant solution as

\[
u = f(z) \left( 1 - e^{-2\lambda t} \right)^{\frac{1}{4}(2a-1)} (e^{-2\lambda t} + 1)^{\frac{1}{4}(2a+1)} \exp \left\{ -t \left( \frac{\alpha \gamma^2}{\lambda^2} + \beta \gamma + \lambda \right) \right. 
+ \frac{e^{4\lambda t} (2\alpha \gamma + \lambda^2 (\beta - x))^2}{2\alpha \lambda^3 (e^{4\lambda t} - 1)} - \frac{\gamma (2\alpha \gamma + \lambda^2 (\beta - x))}{\lambda^3} \right\},
\]

where \( z = e^{\lambda t} (\lambda^2 (x - \beta) - 2\alpha \gamma)/(\lambda^2 \sqrt{1 - e^{4\lambda t}}) \) and \( f(z) \) satisfies

\[(2\alpha \gamma + \lambda^2 z^2) f(z) - \alpha^2 f''(z) = 0.\]

1.5.2 \( a = 0 \)

The symmetry operator \( X_1 + X_2 \) gives the following invariants:

\[
J_1 = -\frac{e^{\lambda t} (2\alpha \gamma + \beta \lambda^2 - \lambda^2 x)}{\lambda^2 \sqrt{1 - e^{4\lambda t}}},
\]

\[
J_2 = u \sqrt{1 - e^{-4\lambda t}} \exp \left\{ t \left( \frac{\alpha \gamma^2}{\lambda^2} + \beta \gamma + \lambda \right) - \frac{e^{4\lambda t} (2\alpha \gamma + \lambda^2 (\beta - x))^2}{2\alpha \lambda^3 (e^{4\lambda t} - 1)} + \frac{\gamma (2\alpha \gamma + \lambda^2 (\beta - x))}{\lambda^3} \right\}.
\]

The invariant solution is given by

\[
u(t, x) = f(z) = c_2 \exp \left\{ -t \left( \frac{\alpha \gamma^2}{\lambda^2} + \beta \gamma + \lambda \right) + \frac{e^{4\lambda t} (2\alpha \gamma + \lambda^2 (\beta - x))^2}{2\alpha \lambda^3 (e^{4\lambda t} - 1)} - \frac{\gamma (2\alpha \gamma + \lambda^2 (\beta - x))}{\lambda^3} \right\},
\]

where \( z = e^{\lambda t} (\lambda^2 (x - \beta) - 2\alpha \gamma)/(\lambda^2 \sqrt{1 - e^{4\lambda t}}) \) and \( f(z) \) solves the nonlinear second order ODE

\[
\alpha^2 f''(z) - \lambda^2 z^2 f(z) = 0.
\]
Case 1.6 $X_3 + aX_6$

The operator $X_3 + aX_6$, results in the following characteristic equations:

$$
\frac{dt}{1} = \frac{dx}{0} = \frac{du}{au}.
$$

Solving the above characteristic equations we find the two invariants $J_1 = x$, $J_2 = u e^{-at}$.

Therefore, the corresponding group invariant solution is given by

$$
u(t, x) = e^{at} f(x),$$

where $f$ is the solution of

$$
\alpha f''(x) + \lambda (\beta - x) f'(x) + (\gamma x + a) f(x) = 0.
$$

4.3 Classification of group invariant solutions of Case 2.1.1

4.3.1 Commutator of subalgebras

In this case $p = 1$ and $q = 0$, and (3.1) then becomes

$$
u_t + \alpha xu_{xx} + \lambda(\beta - x)u_x + \gamma u = 0,
$$

whose symmetries are given by (omitting the solution symmetry)

$$
X_1 = e^{\lambda t} \left[ \frac{1}{\lambda} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \left( \frac{\lambda x}{\alpha} - \frac{\beta \lambda}{\alpha} - \frac{\gamma}{\lambda} \right) \frac{\partial}{\partial u} \right],
$$

$$
X_2 = e^{-\lambda t} \left[ - \frac{\partial}{\partial t} + x \lambda \frac{\partial}{\partial x} + \gamma u \frac{\partial}{\partial u} \right],
$$

$$
X_3 = \frac{\partial}{\partial t},
$$

$$
X_4 = u \frac{\partial}{\partial u}.
$$
The commutator table is given by Table 4.3.

Table 4.3: The commutator table of subalgebras

<table>
<thead>
<tr>
<th>$[X_i, X_j]$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0</td>
<td>$\frac{2}{\lambda} X_3 - \chi X_4$</td>
<td>$-\lambda X_1$</td>
<td>0</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$-\frac{2}{\lambda} X_3 + \chi X_4$</td>
<td>0</td>
<td>$\lambda X_2$</td>
<td>0</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$\lambda X_1$</td>
<td>$-\lambda X_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $\chi = \frac{(2\alpha \gamma + \beta \lambda^2)}{(\alpha \lambda)}$.

4.3.2 Adjoint representation of subalgebras

As before, we compute the adjoint table entries using Table 4.3 and its results are tabulated in Table 4.4.

Table 4.4: The adjoint table of subalgebras

<table>
<thead>
<tr>
<th>Ad</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$X_1$</td>
<td>$-\varepsilon^2 X_1 + X_2 - \frac{2\varepsilon}{\lambda} X_3 + \varepsilon \chi X_4$</td>
<td>$\varepsilon \lambda X_1 + X_3$</td>
<td>$X_4$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$X_1 - \varepsilon^2 X_2 + \frac{2\varepsilon}{\lambda} X_3 - \varepsilon \chi X_4$</td>
<td>$X_2$</td>
<td>$-\varepsilon \lambda X_2 + X_3$</td>
<td>$X_4$</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$e^{-\varepsilon \lambda} X_1$</td>
<td>$e^{\varepsilon \lambda} X_2$</td>
<td>$X_3$</td>
<td>$X_4$</td>
</tr>
<tr>
<td>$X_4$</td>
<td>$X_1$</td>
<td>$X_2$</td>
<td>$X_3$</td>
<td>$X_4$</td>
</tr>
</tbody>
</table>

where $\chi = \frac{(2\alpha \gamma + \beta \lambda^2)}{(\alpha \lambda)}$. 

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4.3.3 One-dimensional optimal system of subalgebras

Let

\[ X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4, \quad a_1, \ldots, a_4 \in \mathbb{R}, \]

then the composition of adjoint maps generated by \( X_1 \) and \( X_2 \) on \( X \) is given by

\[ \tilde{X} = \sum_{j=1}^{4} \tilde{\alpha}_j X_j = \text{Ad}(e^{\epsilon_1 X_1}) \circ \text{Ad}(e^{\epsilon_2 X_2}) X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4, \]

where

\[ \begin{align*}
\alpha_1 &= a_1 - \frac{\epsilon_1^2}{\lambda} (a_2 - a_1 \epsilon_2 - a_3 \lambda \epsilon_2) + \epsilon_1 \lambda \left( a_3 + 2 \frac{a_1 \epsilon_2}{\lambda} \right), \\
\alpha_2 &= a_2 - a_1 \epsilon_2 - a_3 \lambda \epsilon_2, \\
\alpha_3 &= \frac{2 \epsilon_1}{\lambda} (a_2 - a_1 \epsilon_2 - a_3 \lambda \epsilon_2) + a_3 + 2 \frac{a_1 \epsilon_2}{\lambda}, \\
\alpha_4 &= a_4 + \frac{(2 \alpha \gamma + \beta \lambda^2)}{\alpha \lambda} \left[ \epsilon_1 (a_2 - a_1 \epsilon_2 - a_3 \lambda \epsilon_2) - a_1 \epsilon_2 \right].
\end{align*} \]

Focusing only on (4.24), (4.25) and (4.26) we can observe that the invariant is given by \( \eta = 4a_1 a_2 + a_3^2 \lambda^2 \). Depending on the value of the invariant we have three cases, viz., \( \eta > 0 \), \( \eta < 0 \) and lastly \( \eta = 0 \). We proceed as follows:

**Case 1.** If \( \eta > 0 \), we can make \( \alpha_1 \) and \( \alpha_2 \) to be zero by solving for \( \epsilon_1 \) and \( \epsilon_2 \) in (4.24) and (4.25), simultaneously, while \( \alpha_3 \neq 0 \). After scaling the coefficient of \( X_3 \) to one, we observe that \( X \) is equivalent to a scalar multiple of \( X_3 + aX_4 \) for some constant \( a \). No further simplifications are possible.

**Case 2.** If \( \eta < 0 \), then by setting \( \epsilon_2 = 0 \) and \( \epsilon_1 = -a_3 \lambda/(2a_2) \) where \( a_2 \neq 0 \), we make \( \alpha_3 = 0 \) while \( \alpha_1 \) and \( \alpha_2 \) are both different from zero. After scaling the coefficients of both \( X_1 \) and \( X_2 \) to one, we get the vector \( X_1 + X_2 + aX_4 \), where \( a \in \mathbb{R} \). There are no further simplifications possible at this stage.

**Case 3.** If \( \eta = 0 \), we can either have all of \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) or choose \( \epsilon_1 \) and \( \epsilon_2 \) such that \( \alpha_2 \neq 0 \) but \( \alpha_1 = \alpha_3 = 0 \). In the latter case, \( X \) becomes equivalent to a scalar multiple of \( X_2 + a_4 X_4 \). Acting by adjoint map generated by \( X_3 \), we can rescale the coefficients of \( X_2 \) and \( X_4 \) and depending on the sign of \( a_4 \) we get either \( X_2 + X_4 \) or \( X_2 - X_4 \).
The last subcase occurs when \( a_1 = a_2 = a_3 = 0 \), then we get multiples of vector \( X_4 \).

Hence, the optimal system of Case 3.2.1 is given by

\[
\begin{align*}
X_4, & \quad \eta = 0 \\
X_2 + aX_4, & \quad \eta = 0, \quad a \in \mathbb{R} \\
X_1 + X_2 + aX_4, & \quad \eta < 0, \quad a \in \mathbb{R} \\
X_3 + X_4, & \quad \eta > 0, \\
X_3 - X_4, & \quad \eta > 0.
\end{align*}
\]

(4.28)

### 4.3.4 Symmetry reductions and group invariant solutions

We leave out the first operator in (4.28) as it does not yield any invariants.

**Case 2.1** \( X_2 + aX_4 \)

The symmetry operator \( X_2 + aX_4 \) has characteristic equations

\[
\frac{dt}{-e^{-\alpha t}/\lambda} = \frac{dx}{xe^{-\lambda t}} = \frac{du}{(\gamma u e^{-\lambda t}/\lambda) + au}.
\]

(4.29)

Solving equation (4.29), we obtain the invariants \( J_1 = xe^{\lambda t}, \quad J_2 = u e^{\lambda t} + \gamma t \). The invariant solution is given by \( u = f(\zeta) e^{-\alpha \lambda^t} \gamma t \), where \( \zeta = xe^{\lambda t} \) and \( f(\zeta) \) is the solution of

\[
\alpha \zeta f''(\zeta) + \beta \lambda f'(\zeta) - a \lambda f(\zeta) = 0.
\]

We note that when \( a = 0 \), the invariant solution becomes

\[
u(t, x) = \left\{ \frac{c_1 \alpha (xe^{\lambda t})^{(\alpha - \beta \lambda)/\alpha}}{\alpha - \beta \lambda} + c_2 \right\} e^{-\gamma t}.
\]

**Case 2.2** \( X_1 + X_2 + aX_4 \)

This has two subcases.

2.2.1 \( a \neq 0 \)

The operator \( X_1 + X_2 + aX_4 \) gives the characteristic equations

\[
\frac{dt}{(e^{\lambda t} - e^{-\lambda t})/\lambda} = \frac{dx}{xe^{\lambda t} + xe^{\lambda t}} = \frac{du}{-\beta \lambda u e^{\lambda t}/\alpha + \gamma u e^{-\lambda t}/\lambda - \gamma u e^{\lambda t}/\lambda + \lambda u e^{\lambda t}/\alpha + au}
\]
from which we obtain the following two invariants:

\[ J_1 = \frac{xe^{\lambda t}}{1 - e^{2\lambda t}}, \]

\[ J_2 = u \left(1 - e^{\lambda t}\right)^{\frac{\beta - \alpha}{2\alpha}} \left(e^{\lambda t} + 1\right)^{\frac{\alpha + \beta}{2\alpha}} \exp \left\{ \gamma t + \frac{\lambda xe^{2\lambda t}}{\alpha (1 - e^{2\lambda t})} \right\}. \]

Hence, the invariant solution is given by

\[ u = f(z) \exp \left[ \frac{1}{2\alpha} \left\{ \ln \left(1 - e^{\lambda t}\right) (a\alpha - \beta \lambda) - \ln \left(e^{\lambda t} + 1\right) (a\alpha + \beta \lambda) - 2\alpha \gamma t + \frac{2\lambda xe^{2\lambda t}}{e^{2\lambda t} - 1} \right\} \right], \]

where \( z = xe^{\lambda t}/(1 - e^{2\lambda t}) \) and \( f(z) \) satisfies

\[ \alpha^2 z f''(z) + \alpha \beta \lambda f'(z) - a\alpha \lambda f(z) - \lambda^2 z f(z) = 0. \]

2.2.2 \( a = 0 \)

The symmetry operator \( X_1 + X_2 \) gives two invariants

\[ J_1 = \frac{xe^{\lambda t}}{1 - e^{2\lambda t}}, \]

\[ J_2 = u \left(1 - e^{2\lambda t}\right)^{\frac{\beta}{2\alpha}} \exp \left\{ \gamma t - \frac{\lambda xe^{2\lambda t}}{\alpha (e^{2\lambda t} - 1)} \right\} \]

and hence the invariant solution is given by

\[ u = \exp \left\{ -\frac{1}{\alpha} \left( \alpha \gamma t + \frac{1}{2} \beta \lambda \ln \left(1 - e^{2\lambda t}\right) - \frac{\lambda xe^{2\lambda t}}{e^{2\lambda t} - 1} \right) \right\} f(z), \quad z = xe^{\lambda t}/(1 - e^{2\lambda t}) \]

with

\[ f(z) = z^{\frac{\alpha - \beta \lambda}{2\alpha}} \left[ \text{BesselJ} \left( \frac{\alpha - \beta \lambda}{2\alpha}, -\frac{i\lambda z}{\alpha} \right) c_1 + \text{BesselY} \left( \frac{\alpha - \beta \lambda}{2\alpha}, -\frac{i\lambda z}{\alpha} \right) c_2 \right], \]

where \( \text{BesselJ}[n, z] \) denotes the Bessel function of the first kind \( J_n(z) \), \( \text{BesselY}[n, z] \) denotes the Bessel function of the second kind \( Y_n(z) \) as can be seen in [1] and \( c_1, c_2 \) are arbitrary constants.

Case 2.3 \( X_3 + aX_4 \)

The operator \( X_3 + aX_4 \) results in characteristic equations

\[ \frac{dt}{1} = \frac{dx}{0} = \frac{du}{au}, \]

which yields invariants \( J_1 = x \) and \( J_2 = ue^{-at} \). This gives the invariant solution \( u = e^{at} f(x) \), where \( f \) solves

\[ \alpha x f''(x) + \lambda (\beta - x) f'(x) + (\alpha + \gamma) f(x) = 0. \]
4.4 Conclusions

In the literature only one solution of the Vasicek equation (Case 1.2) is known [54]. However, in this chapter we obtained optimal systems of one-dimensional subalgebras for the Vasicek equation and bond pricing equation (Case 2.1.1) and subsequently we constructed group-invariant solutions using each element of the optimal systems. It is well known that the majority of closed-form solutions that have real world applications are group-invariant solutions. Thus, we have obtained many new solutions for the Vasicek equation and the bond pricing equation. The solutions obtained in this chapter may be interpreted as price processes of some interest rate derivatives.
Chapter 5

Concluding remarks

In this research project we first recalled some important definitions and results from Lie group analysis, which were later used in the dissertation.

In Chapter two, Lie point symmetries of Black-Scholes equation were determined and then used to obtain some group invariant solutions.

In Chapter three, the group classification of the bond-option pricing PDE (3.1) was performed. It was shown that the principal Lie algebra of equation (3.1) consisted of the symmetries $X_1 = \partial_t$, $X_2 = u \partial_u$ and the solution symmetry $X_B = B(t, x) \partial_u$. Additional symmetries of (3.1) were found to be possible for $p \in \{0, 1, \frac{3}{2}, \frac{4}{3}, \frac{6}{5}, 2, \frac{8}{7}, \frac{8}{3}, 3, 4, 5\}$. However, in this dissertation we only considered the three values of $p$, namely, $p = 0, 1$ and $2$ as these values of $p$ provided us with important models in mathematics of finance. For each of the cases in Chapter three, Lie point symmetries were computed. In Case 1, four subcases were found which extended the principal Lie algebra, while Case 2 produced eleven subcases that extended the principal Lie algebra. In Case 3, only one subcase was found to extend the principal Lie algebra. It was noted that through the use of the method of group classification three well known models in mathematics of finance were recovered, namely,

- the Vasicek model [57], which was Case 1.2, i.e., (3.1) with $p = 0, q = 1$,
- the CIR equation [11], Case 2.3.1, i.e., (3.1) with $p = 1, q = 1$,
- the Black-Scholes equation [7], Case 3, i.e., (3.1) with $p = 2, q = 0$. 
In Chapter four, the optimal systems of one-dimensional subalgebras for each of the Cases 1.2 and 2.1.1 were determined and were used to obtain symmetry reductions and construction of group invariant solutions of the corresponding equations. Hence, we have obtained many new solutions for the Vasicek equation and the bond pricing equation and these solutions may be interpreted as price processes of some interest rate derivatives.

In future work, I intend to extend this research to all values of $p$ and also consider solutions of (3.1) under terminal conditions.
Bibliography


