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# Stochastic optimization of subprime residential mortgage loan funding and its risks

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By

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degree Magister Scientiae in Applied Mathematics at the Potchefstroom  
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## Preface

One of the contributions made by the NWU-PC to the activities of the stochastic analysis community has been the establishment of an active Finance, Risk and Banking Research Group (FRBRG) that has an interest in institutional finance. In particular, FRBRG has made contributions about modeling, optimization, regulation and risk management in insurance and banking. Students who have participated in projects in this programme under Prof. Petersen's supervision are listed below.

Level	Student	Graduation	Title
MSc	T Bosch	May 2003 Cum Laude	Controllability of HJMM Interest Rate Models
MSc	CH Fouche	May 2006 Cum Laude	Continuous-Time Stochastic Modelling of Capital Adequacy Ratios for Banks
MSc	MP Mulaudzi	May 2008 Cum Laude	A Decision Making Problem in the Banking Industry
PhD	CH Fouche	May 2008	Dynamic Modeling of Banking Activities
PhD	F Gideon	Sept. 2008	Optimal Provisioning for Deposit Withdrawals and Loan Losses in the Banking Industry
MSc	MC Senosi	May 2009 S2A3 Winner for NWU-PC	Discrete Dynamics of Bank Credit and Capital and their Cyclicity
PhD	T Bosch	May 2009	Management and Auditing of Bank Assets and Capital
PhD	BA Tau	May 2009	Bank Loan Pricing and Profitability and Their Connections with Basel II and the Subprime Mortgage Crisis
PhD	MP Mulaudzi	May 2010	The Subprime Mortgage Crisis: Asset Securitization & Interbank Lending
MSc	B De Waal	May 2011 Cum Laude	Stochastic Optimization of Subprime Residential Mortgage Loan Funding and its Risks
PhD	MC Senosi	Current	Discrete-Time Modeling of Subprime Mortgage Credit
PhD	S Thomas	Current	Residential Mortgage Loan Securitization And The Subprime Crisis
Postdoc	J Mukuddem-Petersen	2006-9	Finance, Risk and Banking
Postdoc	T Bosch	2010	Finance, Risk and Banking

# Declaration

I declare that, apart from the assistance acknowledged, the research presented in this dissertation is my own unaided work. It is being submitted in partial fulfilment of the requirements for the degree Magister Scientiae in Applied Mathematics at the Potchefstroom Campus of the North West University. It has not been submitted before for any degree or examination to any other University.

Nobody, including Prof. Mark A. Petersen, Dr. Janine Mukuddem-Petersen, Dr. Mm-boniseni P. Mulaudzi, but myself is responsible for the final version of this dissertation.

Signature.....

Date.....

# Executive Summary

The subprime mortgage crisis (SMC) is an ongoing housing and financial crisis that was triggered by a marked increase in mortgage delinquencies and foreclosures in the U.S. It has had major adverse consequences for banks and financial markets around the globe since it became apparent in 2007. In our research, we examine an originator's (OR's) nonlinear stochastic optimal control problem related to choices regarding deposit inflow rates and marketable securities allocation. Here, the primary aim is to minimize liquidity risk, more specifically, funding and credit crunch risk. In this regard, we consider two reference processes, namely, the deposit reference process and the residential mortgage loan (RML) reference process. This enables us to specify optimal deposit inflows as well as optimal marketable securities allocation by using actuarial cost methods to establish an ideal level of subprime RML extension. In our research, relationships are established in order to construct a stochastic continuous-time banking model to determine a solution for this optimal control problem which is driven by geometric Brownian motion.

In this regard, the main issues to be addressed in this dissertation are discussed in Chapters 2 and 3.

In Chapter 2, we investigate uncertain banking behavior. In this regard, we consider continuous-time stochastic models for OR's assets, liabilities, capital, balance sheet as well as its reference processes and give a description of their dynamics for each stochastic model as well as the dynamics of OR's stylized balance sheet. In this chapter, we consider RML and deposit reference processes which will serve as leading indicators in order to establish a desirable level of subprime RMLs to be extended at the end of the risk horizon.

Chapter 3 states the main results that pertain to the role of stochastic optimal control in OR's risk management in Theorem 2.5.1 and Corollary 2.5.2. Prior to the stochastic control problem, we discuss an OR's risk factors, the stochastic dynamics of marketable securities as well as the RML financing spread method regarding an OR. Optimal portfolio choices are made regarding deposit and marketable securities inflow rates given by Theorem 3.4.1 in order to obtain the ideal RML extension level. We construct the stochastic continuous-time model to determine a solution for this optimal control problem to obtain the optimal marketable securities allocation and deposit inflow rate to ensure OR's stability and security. According to this, a spread method of RML financing is imposed with an existence condition given by Lemma 3.3.2. A numerical example is given in Section 3.5 to illustrates the main issues raised in our research.

Finally, Chapter 4 provides an analysis of the main issues of our results in the previous Chapters as well as their connection with the SMC. The most important aspects of the stochastic model are considered as well as the optimal policies of Theorem 3.4.1 and connections between our results and other related studies. We also show how the weaknesses in banking systems and regulatory frameworks were exposed by the SMC, globally.

The work presented in this dissertation is based on 5 peer-reviewed chapters in books (see [36], [37], [39], [41] and [44]), 1 peer-reviewed journal article (see [20]) and 5 peer-reviewed conference proceedings papers (see [14], [32], [33], [42] and [43]). Moreover, the articles [13] and [38] are already submitted to ISI accredited journals.

**Key Words:** Originator (OR), Subprime Residential Mortgage Loans (RMLs), Marketable Securities, Deposits, Liquidity Risk, Credit Crunch Risk, Funding Risk, Stochastic Model, RML Reference Process, Deposit Reference Process

# Samevatting

Die subprima leningskrisis is 'n deurlopende behuising en finansiële krisis wat deur 'n merkbare toename in verband mislukkings en terugneming van kollateraal in die VSA veroorsaak is wat groot negatiewe gevolge vir banke en finansiële markte globaal veroorsaak het na 2007. In ons navorsing ondersoek ons banke se nie-lineêre stogastiese optimale beheerprobleem wat verband hou met keuses rakende deposito invloei en allokasie van bemarkbare sekuriteite. Die primêre doel is om likiditeitsrisiko, meer spesifiek, befondsing-en kredietkrisisrisiko te verminder. In hierdie verband, twee verwysingsprosesse, naamlik die deposito verwysingsproses en die residensiële verbandlening-verwysingsproses word ondersoek. Dus kan die optimale deposito invloei, asook die optimale allokasie van bemarkbare sekuriteite verkry word deur gebruik te maak van aktuariële metodes om 'n ideale kostevlak vir die uitbreiding van subprima lenings te kry. Verwantskappe word verkry om ten einde 'n stogastiese bankmodel in kontinue-tyd te bou om 'n oplossing te bepaal vir die optimale beheerprobleem wat deur die Brownse beweging gedryf word.

Die belangrikste aspekte wat aangespreek word in die verhandeling, word bespreek in hoofstuk 2 en 3.

In hoofstuk 2 word die onsekerheid van banke se gedrag ondersoek. Stogastiese modelle in kontinue-tyd word beskou rakende die bates, laste, kapitaal, balansstaat asook die verwysingsprosesse van banke. 'n Beskrywing van die dinamika van elke stogastiese model word gegee sowel as die dinamika van die bank se gestileerde balansstaat. Die deposito-en residensiële verbandlening-verwysingsprosesse word in hierdie hoofstuk beskou wat sal dien as leidende aanwysers om 'n gewenste vlak van uitbreiding van subprima lenings aan die einde van die risiko horison te kry.

Hoofstuk 3 bevat die belangrikste resultate wat betrekking het op die rol wat stogastiese optimale beheer op bankrisikobestuur het in Stelling 2.5.1 en 2.5.2. Die bank se risikofaktore, die stogastiese dinamika van bemarkbare sekuriteite, sowel as die finansiële verspreidingsmetode word hier bespreek. Optimale portefeulje keuses word gemaak ten opsigte van deposito invloei en allokasie van bemarkbare sekuriteite in Stelling 3.4.1 om die ideale uitreikingsvlak van lenings te vind. Die stogastiese model word gebruik om 'n oplossing vir hierdie optimale beheerprobleem te kry om die optimale allokasie van bemarkbare sekuriteite en deposito invloei te verkry wat die stabiliteit en veiligheid van banke verseker. Hiervolgens bestaan 'n finansiële verspreidingsmetode wat deur Lemma 3.3.2 voorgestel word. 'n Numeriese voorbeeld word gegee in afdeling 3.5 wat die belangrikste kwessies in die verhandeling illustreer.



Ten slotte, Hoofstuk 4 bied 'n ontleding van die belangrikste kwessies van ons resultate, asook hul verband met die subprima leningskrisis. Die belangrikste aspekte van die stogastiese model word beskou sowel as die optimale beleid van Stelling 3.4.1 en konnektasies tussen ons resultate en ander verwante studies. Ons wys ook hoe die swak punte in die bankstelsels en regulatoriese raamwerke blootgestel word deur die subprima leningskrisis, wêreldwyd.

Hierdie verhandeling is gebaseer op 5 hoofstukke in boeke (sien [36], [37], [39], [41] en [44]), 1 joernaal-artikel (sien [20]) en 5 konferensieverrigtinge (sien [14], [32], [33], [42] en [43]). Verder is die artikels [13] en [38] reeds deurgestuur na ISI geakkrediteerde tydskrifte.

**Sleutelwoorde:** Subprima Residensiële Vebandlenings, Bemerkbare Sekuriteite, Deposito's, Likiditeitsrisiko, Kredietkrisis-*risiko*, Befondsingsrisiko, Stogastiese Model, Residensiële Verbandlening Verwysingsproses, Deposito Verwysingsproses.

# Glossary

*Credit crunch* is a term used to describe the inability or difficulty to obtain loans (credit) from ORs because of a shortage of money supply.

*Deposits* include both demand and time deposits. Demand deposits are the larger part of an originator's (OR's) money supply which are payable immediately on request where time deposits are also a money deposit of ORs which can only be withdrawn after a preset fixed time period.

*Liquidity risk* arises from situations in which a banking agent interested in selling (buying) residential mortgage products (RMPs) cannot do it because nobody in the market wants to buy (sell) those RMPs. Such risk includes funding and credit crunch risk.

*Funding risk* refers to the lack of funds or deposits to finance RMLs.

*Credit crunch risk* refers to the risk of tightened mortgage supply and increased credit standards.

*Securitization* is a structured finance process or the transformation of a non-tradable financial asset or liability with various levels of seniority, which are then sold to investors or third parties.

*Cost of Mortgages* refers to the interest cost banks should pay for the use of funds to extend mortgages.

*Subprime residential mortgage lending* is the practice of extending RMLs to MRs who do not qualify for market interest rates because of their poor credit history. The term *subprime* refers to MRs who are less likely to repay mortgages and who do not qualify for prime interest rates, therefore high interest rates are charged. A subprime RML is worse from an OR's view because it is in the riskiest category of mortgages with high default rates. In general, a RML is subprime if

1. the MR has a poor credit history;
2. it is extended by an OR who specializes in high-cost RMLs;
3. it is part of a reference subprime RML portfolio which is traded on secondary markets;  
or
4. it is issued to a MR with a prime credit history but is a subprime-only contract type, for example a 2/28 hybrid mortgage.

## Abbreviations

ABS - Asset-Backed Security;  
ABCP - Asset-Backed Commercial Paper;  
CDO - Collateralized Debt Obligation;  
CRA - Credit Rating Agency;  
DPE - Dynamic Programming Equation;  
HJBE - Hamilton-Jacobi-Bellman Equation;  
IB - Investing Bank;  
MLI - Monoline Insurer;  
MR - Mortgagor;  
OR - Originator;  
RMBS - Residential Mortgage-Backed Security;  
RML - Residential Mortgage Loan;  
RMP - Residential Mortgage Product;  
SDE - Stochastic Differential Equation;  
SMC - Subprime Mortgage Crisis;  
SPV - Special Purpose Vehicle.

## Basic Notations

$B$  - Marketable Securities;  
 $B^*$  - Optimal Marketable Securities;  
 $B$  - Borrowings;  
 $\tilde{B}$  - Risky Marketable Securities;  
 $b^i$  - Total earnings of the  $i$ -th Marketable Securities class;  
 $c$  - Deposit Inflow Rate;  
 $c^*$  - Optimal Deposit Inflow Rate;  
 $c^{ME}$  - Drift Coefficient of the Stipulated RML Extension Rate;  
 $D$  - Deposits;  
 $D^r$  - Deposit Reference Process;  
 $D^a$  - Additional Deposits;  
 $\mathbf{E}$  - Conditional Expectation;  
 $e$  - Volatility of the Stipulated Level of Reserves and Subprime RMLs to be Extended;  
 $\mathcal{F}$  - Right Continuous Filtration;  
 $\{\mathcal{G}\}_{t \geq 0}$  - Completion of Filtration;  
 $\mathcal{G}_{B_0, M^r_0}$  - Class of Admissible Control Laws;

$g$  - Control Law;  
 $K$  - Capital;  
 $\kappa$  - Weighting Factor;  
 $l$  - RML Extension Rate;  
 $M$  - Subprime Residential Mortgage Loans;  
 $M^u$  - Unfunded Subprime RMLs to be Extended;  
 $M^r$  - RML Reference Process;  
 $\mathbf{P}$  - Probability;  
 $P$  - Distribution Function;  
 $p$  - Density Function;  
 $\pi$  - Marketable Securities Allocation;  
 $\pi^i$  - Value of the Marketable Securities Invested in the  $i$ -th Securities;  
 $\tilde{\pi}$  - Portfolio Process or Trading Strategy;  
 $\tilde{\pi}^*$  - Optimal Marketable Securities Allocation Strategy;  
 $R$  - Reserves;  
 $\rho$  - Risk Premium;  
 $r^a$  - Rate of Actualization;  
 $r^{\tilde{B}}$  - Rate of Return from Risky Marketable Securities;  
 $r^d$  - Discounted Rate of Interest;  
 $r^i$  - Rate of Impatience of the OR;  
 $r^{DD}$  - Rate of Demand Deposit;  
 $r^{TD}$  - Rate of Time Deposit;  
 $r^T$  - Rate of Return from Treasuries;  
 $\sigma^{ME}$  - Volatility of the Positive Value of RML Extension Rate;  
 $T$  - Riskless Treasuries;  
 $t_0$  - Beginning of the period;  
 $t_1$  - End of the Period;  
 $V$  - Objective Function;  
 $V^*$  - Optimal Objective Function;  
 $v$  - Arbitrary instant of time;  
 $W^l$  - Brownian Motion;  
 $\tilde{\xi}$  - Market Price of Risk;

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# Chapter 1

## INTRODUCTION

”US sub-prime is just the leading edge of a financial hurricane.”

– Bernard Connolly (AIG), 2007.

”As calamitous as the sub-prime blowup seems, it is only the beginning. The credit bubble spawned abuses throughout the system. Sub-prime lending just happened to be the most egregious of the lot, and thus the first to have the cockroaches scurrying out in plain view. The housing market will collapse. New-home construction will collapse. Consumer pocketbooks will be pinched. The consumer spending binge will be over. The U.S. economy will enter a recession.”

– Eric Sprott (Sprott Asset Management), 2007.

”These days America is looking like the Bernie Madoff of economies: For many years it was held in respect, even awe, but it turns out to have been a fraud all along.”

– Prof. Paul Krugman (2008 Nobel Memorial Prize Laureate in Economic Sciences, Princeton University, U.S.), 2009.

Before the subprime mortgage crisis (SMC), large amounts of deposits flowed into the U.S. which funded residential mortgage loans (RMLs) in a low interest rate environment. Consequently, credit was easy to obtain thus boosting the housing and credit market for a number of years. The SMC started in the subprime mortgage market. Originators (ORs) securitized RMLs to shift credit risk to investors who invested in the resulting residential mortgage-backed securities (RMBSs). As a consequence, the amount of RMBSs rose dramatically (see [47]). ORs extended large amounts of subprime RMLs to higher risk

mortgagors (MRs) during 2004-2007 thanks to the booming house market in the U.S. As a consequence, large profits were made while house prices continued to rise.

The proceeds of subprime RML extension were invested in marketable securities for higher yields than earnings on Treasuries and other asset-backed securities (ABSs). In turn, the funds from deposits and marketable securities were then used for financing new subprime RMLs. In Figure 1.1 below, we show the interaction between deposits, ORs, RMLs and RMBSs for a clearer overview of RML financing via deposits and marketable securities.

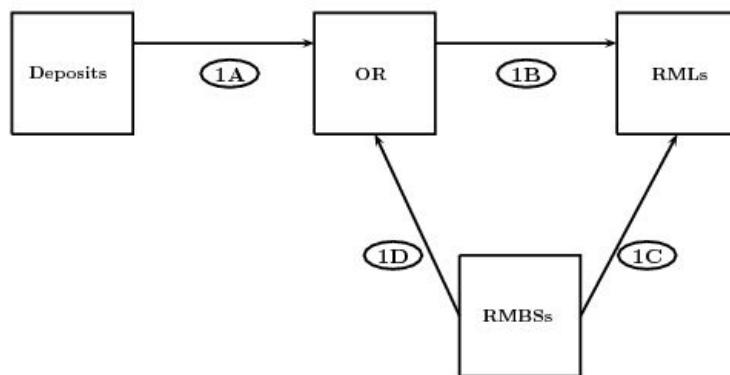


Figure 1.1: Diagrammatic Overview of RML Funding

In Figure 1.1, **1A** is the continued deposit inflows from investors to the OR. This enables the OR to extend RMLs in **1B**. In this regard, **1C** represents the securitization of RMLs via RMBSs. ORs use earnings received from RMBSs, **1D**, to fund new RMLs. This cycle repeats itself, thus, subprime RMLs are funded by deposits and marketable securities (in this regard, subprime RMBSs). This cycle collapsed when MRs started to default and RMBSs lost their value. Banks had suffered greatly from illiquidity with losses reaching US\$300 billion as of June 2008 (see [10]).

OR's aim is to minimize a combination of funding and credit crunch risk. In this regard, funding risk represents the size of the deviation of deposits from the normal cost and is related to the stability of RML extension. On the other hand, credit crunch risk is an indicator of deviations of extension from its actuarially determined level.

In our research, we consider a continuous-time banking model to determine an optimal RML extension rate. In this model, such extension is stochastic and driven by geometric Brownian motion. This set-up specifically leads to OR's nonlinear stochastic optimization

problem that entails minimizing the liquidity risks involving deposits,  $D$ , and marketable securities,  $B$ , respectively. In this regard, we make optimal choices for  $D$  and  $B$ , in order to obtain an ideal level of RML extension. This involves OR's assets that include reserves,  $R$ , subprime RMLs,  $M$ , risky marketable securities,  $\tilde{B}$ , and riskless Treasuries,  $T$ , as well as liabilities such as deposits,  $D$ , and borrowings,  $B$ . OR's capital balances its assets and liabilities and acts as a buffer against losses. The levels of subprime RML extension as well as deposit inflow and marketable securities allocation can be evaluated by an actuarial cost method.

## 1.1 Literature Review

In this section, we consider the association between our contribution and previous literature.

Before the SMC, low credit risk resulted in an increase in RML extension and therefore also lead to more RMBS issuing (see, for instance, [40]). This increase in liquidity and competition in the credit market caused ORs to extend RMLs to risky subprime MRs and securitized these mortgages to fund more subprime RMLs. Subprime lending became a larger, more important sector of the credit market. The paper [9] investigates an OR's risk management for subprime lending and considers default correlation to measure an OR's risk. They conclude that ORs and regulators will profit by measuring risks of subprime lending by using default correlation.

According to [46], the whole financial system was brought down after the housing bubble burst, because ORs had evaded regulatory capital requirements to finance mortgages. ORs placed securitized mortgages in off-balance sheets entities to spread their risk to other players in the financial market which reduced the ORs' capital requirements. Thus, originate-to-distribute operations were used to reduce the specified capital requirement and to spread risks among many financial roleplayers.

Securitization is the main source for funding mortgage extension. According to [3], securitization has altered the ORs' liquidity, credit and maturity transformation duties during the SMC. Therefore, securitization influenced ORs' lending channels and mortgage extensions. They found that securitization of mortgages sheltered the ORs' mortgage supply from monetary policy and strengthened new mortgage extension which also depends on ORs' risks.

The paper [26] shows that converting illiquid mortgages into liquid marketable securities cause OR lending to be less sensitive to cost of fund shocks, because securitization provides

additional funds. Thus, securitization causes an increase in mortgage supply and a decrease in liquid securities holdings. In this regard, mortgage supply is less sensitive to external sources of funding, therefore reducing the influence the monetary authority has on an OR's lending. Also, securitization is a substitute for an OR's on-balance sheet liquidity and therefore also cause increasing liquidity of OR's mortgages. In this regard, an OR's mortgage supply does not depend on an OR's funding availability alone, but also on their willingness to extend mortgages. Therefore, greater liquid mortgage portfolios of ORs lessen the impact of shocks in the costs of external funds.

Niinimäki investigated in [34] how hiding mortgage losses can affect an OR's mortgage interest income, deposits inflows, liquidity and moral hazard. Therefore, hiding mortgage losses by rolling over defaulted mortgages or refinancing of mortgages contributed to the recent SMC. Also, ORs appeared to have high profits and attractive balance sheets. OR's deposit inflows continued to increase yearly, from 1997 to 2008. Therefore, deposit inflows were strong even when ORs faced insolvency and bankruptcy. The ORs' true financial conditions were revealed in the banking crises and may even be the source of the SMC. As a consequence, depositors withdrew funds and potential depositors didn't deposit into these ORs which resulted in bank runs, illiquidity and insolvency of some ORs.

Altunbasa *et al.* found evidence of how an OR's lending channel operates with respect to an OR's risks (see, for instance, [2]). They analysed how an OR's risks influence their mortgage supply and credit crunch risk. Moreover, they investigated how an OR transfers credit risk to diminish their risks and associated indicators of an OR's balance sheets. Bank risk conditions need to be considered as well as other indicators when an OR's ability and willingness to RML extension is investigated. They found that ORs with lower expected default rates have the ability to extend more mortgages and therefore increase their mortgage supply.

We note that [11] and [12] solved optimal control problems from continuous-time models by means of portfolio selection and capital requirements. Improved risk management strategies are of great importance to ORs. We discuss an OR's optimal behavior with respect to risks related to deposit inflows for RML funding and tightened mortgage supply, namely, *funding risk* and *credit crunch risk*, respectively. We minimize these risks by optimal portfolio choices and optimal deposit inflows which are important for an OR's risk management.

In our research, we solve a stochastic control problem that depends on verifying the transversality condition (see, for instance, [8], [21], [22], [25] and more generally [5] and [16]). However, the solution to this control problem and the original value function should correspond, therefore, the transversality discussed in [5] and [16] are of great interest for this dissertation.

The paper [7] investigates the anatomy of the SMC that involves mortgage extension and securitization with operational risk as the main issue. The quantity of mortgages was more important than the quality of mortgages issued. More and more subprime mortgages were extended that contained resets. The underwriting of new subprime RMLs embeds credit and operational risk. House prices started to decline and default rates increased dramatically. On the other hand, credit risk was outsourced via securitization of mortgage mortgages which funded new subprime mortgages. Securitization of subprime RMLs involves operational, tranching and liquidity risk. During the SMC, the value of these securities decreased as default rates increased dramatically. The RMBS market froze and returns from these securities were cut off with RML extension no longer be funded. Financial markets became unstable with a commensurate increase in market risk which led to a collapse of the whole financial system.

## 1.2 Preliminaries

In this section, we discuss key definitions and notation, the funding of RMLs and subprime risks as well as identifying OR's balance sheet.

### 1.2.1 Definitions and Notation

*Subprime residential mortgage lending* involves the extension of subprime RMLs to mortgagors (MRs) who do not qualify for market interest rates due to factors such as income level, size of the down payment made, credit history and employment status.

Many ORs evaluate their levels of RMLs extended and both deposits and marketable securities required at regular time intervals. The *actuarial cost method* may be used for this valuation. This method enables us to specify an optimal rate for deposit inflows to fund RMLs, referred to as the *deposit reference process*,  $D^r$ . Also, a desired rate of RML extension, called the *RML reference process*, will be denoted by  $M^r$ .

These levels are optimally chosen in order for RML extension rates to be reached at the end of a specific time period. Excessive or a shortage of RML extension will develop because economic calculations don't always resemble reality. Therefore, an appropriate provision should be made in adjusting the rate of deposit and marketable securities inflow so that the shortfall can be removed or that the RMLs can be extended optimally. A *spread method of OR RML financing* is a well-known method for handling this adjustment that allows for the *unfunded RML extension* to be spread over a given period of time, namely a *spread period*.

### 1.2.2 RML Financing and Subprime Risks

The main risks that arise when dealing with subprime residential mortgage products (RMPs) are credit (including counterparty), market (including interest rate, basis, prepayment, liquidity and price), tranching (including maturity mismatch and synthetic), operational and systemic risks. For sake of argument, risks falling in the categories described above are cumulatively known as *subprime risks* (see, for instance, Subsections 4.1 and 4.2).

In Figure 1.2 below, we provide a diagrammatic overview of the aforementioned subprime risks.

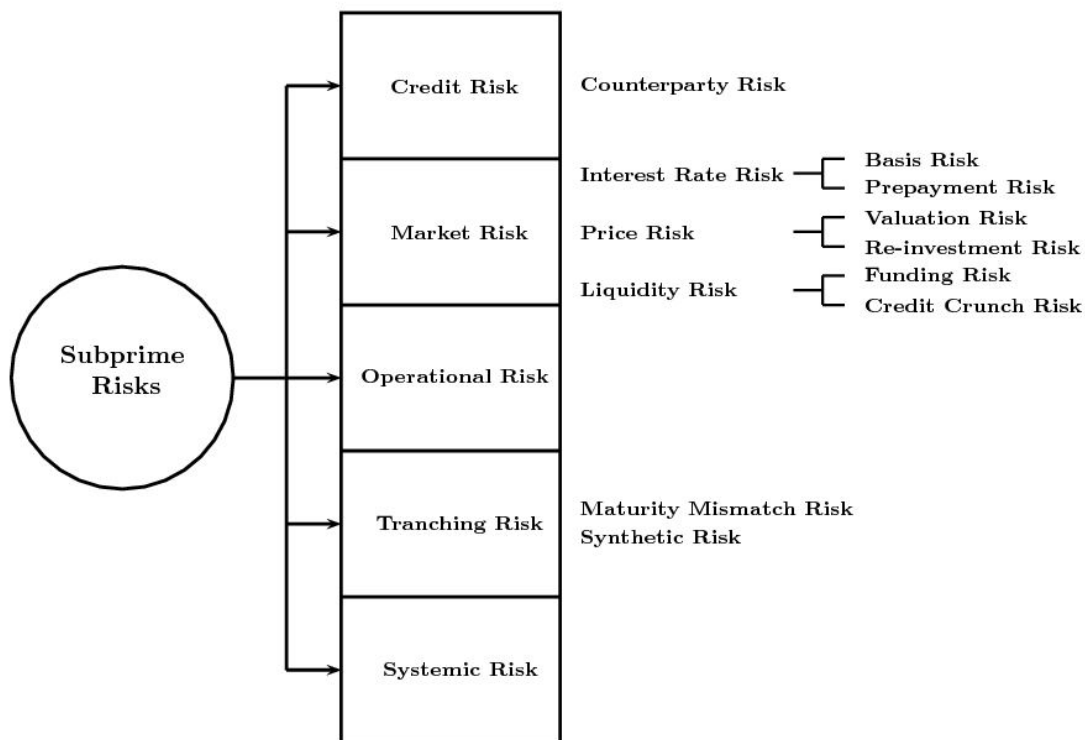


Figure 1.2: Diagrammatic Overview of Subprime Risks

The most fundamental of the above risks is *credit* and *market risk*. The former involves OR's risk of loss from a MR who does not make scheduled payments. This risk category generally includes *counterparty risk* that, in our case, is the risk that a banking agent does not pay out on a bond, credit derivative or credit insurance contract. It refers to the ability of banking agents – such as ORs, MRs, servicers, IBs, SPVs, trustees, underwriters and

depositors – to fulfill their obligations towards each other. During the SMC, even banking agents who thought they had hedged their bets by buying insurance – via credit default swap contracts or monoline insurance (MLI) – still faced the risk that the insurer will be unable to pay.

In addition, *market risk* is the risk that the value of the RMP portfolio will decrease mainly due to changes in value of securities prices and interest rates. *Interest rate risk* arises from the possibility that subprime RMP interest rates will change. Subcategories of interest rate risk are basis and prepayment risk. The former is the risk associated with yields on RMPs and costs on deposits which are based on different bases with different rates and assumptions. *Prepayment risk* results from the ability of subprime MRs to voluntarily (refinancing) and involuntarily (default) prepay their RMLs under a given interest rate regime.

*Liquidity risk* arises from situations in which a banking agent interested in selling (buying) RMPs cannot do it because nobody in the market wants to buy (sell) those RMPs. Such risk includes funding and credit crunch risk. *Funding risk* refers to the lack of funds or deposits to finance RMLs and *credit crunch risk* refers to the risk of tightened mortgage supply and increased credit standards.

We consider *price risk* to be the risk that RMPs will depreciate in value, resulting in financial losses, markdowns and possibly margin calls. Subcategories of price risk are valuation risk (resulting from the valuation of long-term RMP investments) and re-investment risk (resulting from the valuation of short-term RMP investments).

*Tranching risk* is the risk that arises from the complexity associated with the slicing of securitized RMLs into tranches in securitization deals. Another tranching risk that is of issue for RMPs is *maturity mismatch risk* that results from the discrepancy between the economic lifetimes of RMPs and the investment horizons of IBs. *Synthetic risk* can be traded via credit derivatives – like credit default swaps – referencing individual subprime RMBS bonds, synthetic CDOs or via an index linked to a basket of such bonds. *Operational risk* is the risk of incurring losses resulting from insufficient or inadequate procedures, processes, systems or improper actions taken.

In banking, *systemic risk* is the risk of collapse of the entire banking system or RMP market as opposed to risk associated with a single component of that system or market. It refers to the risks imposed by interlinkages and interdependencies in the system where the failure of a single entity or cluster of entities can cause a cascading effect which could potentially bankrupt the banking system or market.

In this dissertation, we specifically investigate the subcategories of liquidity risk, namely, *funding risk* and *credit crunch risk*. The most important goal is to minimize the funding and credit crunch risk by modelling the liabilities and the allocation of assets of an OR. A measure of funding risk is the deviation size of deposit inflow rate from  $D^r$  which is associated with the stability of funding RML extension. The unfunded subprime RMLs' size to be extended is a measurement for credit crunch risk where the value of these unfunded mortgages indicates how secure the OR is. The eruption of the SMC significantly tightened the mortgage supply and increased the credit standards to preserve regulatory capital reserves and liquidity problems which affected the availability of mortgage loans. ORs couldn't obtain capital or deposit inflows which lead to illiquidity of ORs.

During the SMC, credit risk was spread in the financial system and not concentrated in ORs. Therefore, ORs were able to separate credit from market risk and thereby reducing the risks that ORs have to bear. This reduced risk caused lower asset price volatility which increased the financial asset prices. Thus, the risk of failure and financial instability were reduced and more subprime RMLs were extended in later years of the housing boom with a high risk of defaulting. ORs transferred these risks from their books to the broader capital market.

Converting illiquid mortgages to marketable securities has enabled ORs to increase their mortgage turnover as ORs sold these securitized mortgages to move these assets off their balance sheets. According to [30], the amount of RMBSs that were originated and traded reached US\$3 trillion in 2005 in the residential mortgage sector of US\$10 trillion. Therefore, more and more RMLs were extended by ORs as the securitized RMLs were moved from their balance sheets. Moreover, default risk was transferred to RMBS investors. Capital and deposit inflow as well as profits from securities gushed out as fast as they rushed in to meet the constant demand for subprime RMLs, subprime RMBSs and to maintain profits for the brokers (see, for instance, [30]). According to [40], the risk shifted from the securitization of RMLs, have not been eliminated or reduced. It only reduced the exposure to subprime RMLs of ORs. However, systematic risk associated with declining house prices shifted from MRs to ORs, from ORs to investors and investors to guarantors, but the risk remained and on the end brought down the whole financial system.

Liquidity is associated with confidence which is the most difficult to pin down of all risks. We notice in [30] that the effect of a crisis is contagious and leads to a crisis of confidence. When investors lose confidence, the whole structure will collapse. Northern Rock Mortgage Company illustrated how a confidence crisis can be contagious. This OR depended on large institutions for cheaper funding, rather than small depositors while large funds are more



volatile and risky, which caused, in this case, a liquidity crunch and resulted in a run on the OR by small depositors.

### 1.2.3 Balance Sheet

OR's behavior is consistent with the uncertainty associated with reserves, RMLs and marketable securities (assets) and deposits and borrowings (liabilities) appearing on the balance sheet. The aforementioned items are balanced by OR's capital according to the popular relation

$$\text{Total Assets} = \text{Total Liabilities} + \text{OR's Capital}.$$

Therefore, a stylized balance sheet of an OR can be represented at time  $t$  as

$$R_t + M_t + B_t = D_t + \mathbf{B}_t + K_t, \quad (1.1)$$

with  $B_t = \tilde{B} + \mathbf{T}$  where

$R : \Omega \times T \rightarrow \mathbb{R}_+$  - Reserves,  $M : \Omega \times T \rightarrow \mathbb{R}_+$  - RMLs,  $B : \Omega \times T \rightarrow \mathbb{R}_+$  - Marketable Securities,  
 $D : \Omega \times T \rightarrow \mathbb{R}_+$  - Deposits,  $\mathbf{B} : \Omega \times T \rightarrow \mathbb{R}_+$  - Borrowings,  $K : \Omega \times T \rightarrow \mathbb{R}_+$  - OR's Capital,  
 $\tilde{B} : \Omega \times T \rightarrow \mathbb{R}_+$  - Risky Marketable Securities,  $\mathbf{T} : \Omega \times T \rightarrow \mathbb{R}_+$  - Riskless Treasuries.

## 1.3 Main Problems and Outline of the Dissertation

In this section, we state the main problems and provide an outline of the dissertation.

### 1.3.1 Main Problems

Our general objective is to investigate aspects of subprime residential mortgage loan funding in a stochastic manner and its risks as well as its connections with the subprime mortgage crisis in order to construct a stochastic continuous-time model to determine a solution for the control problems below. In this regard, specific research objectives are listed below.

**Problem 1.3.1 (Optimal Control Problem):** *What should the optimal levels of OR's deposit inflows and marketable security returns be in order to reach the optimal specified level for RML extension via the financing spread method ?*

The deposit inflow rate and the returns from marketable securities may be used to achieve the control objectives mentioned in the above problem by means of portfolio choice. Both OR's deposits and marketable securities and subprime RML extension have a positive correlation, (see, for instance, [45]). We interpret our stochastic banking model by means of an infinite time horizon and a positive discount rate. Thus, OR's operations don't stop and therefore short-term behaviour of the marketable securities of an OR is more of a concern.

**Problem 1.3.2 (Optimal Control Problem for Subprime Risk Management):**

*Suppose that  $\mathcal{G}_{B_0, m^{r_0}} \neq \emptyset$ . The system*

$$dB_t = (r^T B_t + \sum_{i=1}^n \pi_t^i (b^i - r^T)) dt + (d_t^a - (c^{ME} - r^a) m_t^r) dt + \sum_{i=1}^n \sum_{j=1}^n \pi_t^i \sigma^{ij} dW_t^j,$$

*for managing OR's risk with a class of admissible control laws,  $\mathcal{G}_{B_0, m^{r_0}}$  and  $V : \mathcal{G}_{B_0, m^{r_0}} \rightarrow \mathbb{R}_+$ , the objective function,  $V$  represented by*

$$V(d^a, \tilde{\pi}) = \mathbf{E} \left[ \int_{t_0}^{t_1} \kappa (d_s^a)^2 + (1 - \kappa) [m_s^r - B_s]^2 ds \right],$$

*should be considered. We want to solve*

$$\min_{(d^a, \tilde{\pi}) \in \mathcal{G}_{B_0, m^{r_0}}} V(d^a, \tilde{\pi}),$$

*in order to determine the objective function  $V^*$ ,*

$$V^* = \min_{(d^a, \tilde{\pi}) \in \mathcal{G}_{B_0, m^{r_0}}} V(d^a, \tilde{\pi}),$$

*and the optimal control law  $(d^a, \tilde{\pi})^*$ , if its existence is known,*

$$(d^a, \tilde{\pi})^* = \arg \min_{(d^a, \tilde{\pi}) \in \mathcal{G}_{B_0, m^r0}} V(d^a, \tilde{\pi}) \in \mathcal{G}_{B_0, m^r0}.$$

### 1.3.2 Outline of the Dissertation

The current chapter is introductory in nature. The remaining chapters of this dissertation are structured as follows.

In this dissertation, we discuss subprime RMLs with our focus being on an OR's subprime RML design (see Chapter 2). Also, we discuss an OR's optimal liquidity risk management associated with OR's subprime RML financing via marketable securities and deposits (see Chapter 3) followed by stochastic examples involving subprime RML financing. In Chapter 4, discussions on subprime RML funding and connections with the SMC is given. Finally, Chapter 5 presents a few concluding remarks and highlights some possible topics for future research, followed by the bibliography in Chapter 6.



## Chapter 2

# SUBPRIME STOCHASTIC MODEL

”Certainly the underwriting standards for a large proportion of the U.S. home mortgages originated in 2005 and 2006 would give most people a pause. The no-down payment, no-documents and no-stated income-or-assets loans were unprecedented in the history of mortgage finance and clearly ripe for abuse.”

– Prof. Linus Wilson (Louisiana at Lafayette), 2008.

”If a guy has a good investment opportunity and he can’t get funding, he won’t do it. And that’s when the economy collapses.”

– Prof. Frederic Mishkin, 2008.

In this Chapter, we investigate uncertain banking behavior. In this regard, we consider continuous-time stochastic models for OR’s assets, liabilities, capital, balance sheet as well as its reference processes.

### 2.1 Assets

In this section, we focus specifically on an OR’s subprime RMLs and marketable securities. An OR is allowed to invest in both Treasuries,  $T$ , and *risky* marketable securities,  $\tilde{B}$ , including, for example, subprime residential mortgage-backed securities (RMBSs), collateralized debt obligations (CDOs) and asset backed commercial paper (ABCP). We consider the probability space,  $(\Omega, \mathcal{F}, \mathbf{P})$  on a time period  $T = [t_0, t_1]$ . The one-dimensional Brownian

motion  $\{W_t^l\}_{t \geq 0}$ , can be associated with the right continuous filtration,  $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$  with probability  $\mathbf{P}$  which is measured on  $\Omega$ .

### 2.1.1 Subprime RMLs

RMLs can be modelled as a stochastic process because of the uncertainty related with them. The OR can choose how to vary the RML extension. In our discussion, we examine how the stipulated levels of RMLs to be extended may be achieved which are targeted at the beginning of a time period. The stochastic process  $l : \Omega \times T \rightarrow \mathbb{R}$  represents the RML extension rate, with value  $l_t$  at time  $t$ . We represent the random  $l$  by a Brownian motion to simplify the problem. Here,  $l$  will imitate reality by, for instance, having positive values with increments being lognormal distributions. It is acceptable for  $l$  to be modelled as a path-continuous scalar Itô process defined on  $(\Omega, \mathcal{F}, \mathbf{P})$ . Thus,  $l$  can be represented by the stochastic integral formula

$$l_t = l_0 + \int_{t_0}^{t_1} c^{ME}(l_s, s) ds + \int_{t_0}^{t_1} \sigma^{ME}(l_s, s) dW_s^l, \quad t \geq 0, \quad l_0 = l_{t_0}, \quad (2.1)$$

where  $c^{ME} \in \mathbb{R}$  is the drift coefficient of the stipulated RML extension rate and  $\sigma^{ME} \in \mathbb{R}_+$  is the volatility of the positive value of RML extension rate. Therefore,  $dW^l$  is the increment of  $W^l$  which represents shocks RML extension can encounter. Thus, management can deal with unexpected events that influence the evolution of RML extension by looking at the form of  $l$  in (2.1). A description of the dynamics of the stipulated RMLs to be extended by means of the stochastic differential equation (SDE) below, can be given by

$$dl_t = l_t [c^{ME} dt + \sigma^{ME} dW_t^l], \quad l_0 = l_{t_0}, \quad dM_t = l_t dt, \quad (2.2)$$

by looking at the representation in (2.1). Thus, the RML extension rate,  $l$ , which is the solution to (2.2), may be given by

$$l_t = l_0 \exp\left\{ \left( c^{ME} - \frac{(\sigma^{ME})^2}{2} \right) t + \sigma^{ME} W_t^l \right\}. \quad (2.3)$$

Therefore, the occurrence in the change in RML extension rate is a constant exponential rate. Also, it is clear that  $l_0$  is  $\mathcal{F}_0$ -measurable and that

$$\exp\left\{-\left(\frac{(\sigma^{ME})^2}{2}\right)s + \sigma^{ME}W_s^l\right\},$$

is an  $\mathcal{F}$ -martingale (see, for instance, [35]).

### 2.1.2 Marketable Securities

Marketable securities,  $B$ , are available for funding the extension of subprime RMLs. An OR usually has  $n + 1$  of these marketable securities. Suppose that the  $n + 1$ -th securities class is riskless namely Treasuries, denoted by  $\mathbf{T}(t)$  and the risky marketable securities 1, 2,  $\dots$ ,  $n$ , denoted by  $\tilde{B}_t^1, \dots, \tilde{B}_t^n$ . Treasuries is deterministic, therefore we write the time instant next to it and not as a subscript. The following differential equation represents the dynamics of the riskless Treasuries given by

$$d\mathbf{T}(t) = \mathbf{T}(t)r^{\mathbf{T}}dt, \quad \mathbf{T}(t) = 1, \quad (2.4)$$

where  $r^{\mathbf{T}}$  is the rate of return from Treasuries described in (2.4). Next, we study the stochastic case regarding marketable securities by looking at the set of depended RMBSs with the evolution of the risky RMBSs given by the stochastic differential equation (SDE)

$$d\tilde{B}_t^i = \tilde{B}_t^i[b^i dt + \sum_{j=1}^n \sigma^{ij} dW_t^j], \quad \tilde{B}_0^i = \mathbf{T}, \quad 1 \leq i \leq n. \quad (2.5)$$

with  $b^i \rightarrow \mathbb{R}_+$ ,  $\sigma^{ij} \rightarrow \mathbb{R}_+$  and  $1 \leq i, j \leq n$ .  $(W_t^0, W_t^1, \dots, W_t^n)^T$  is a  $n + 1$ -dimensional vector on  $(\Omega, \mathcal{G}, \mathbf{P})$ , where the completion of the filtration  $\sigma\{(W_t^0, W_t^1, \dots, W_t^n)^T\}$  with condition  $0 \leq s \leq t$ , is represented by  $\{\mathcal{G}_t\}_{t \geq 0}$ . Suppose there exist a correlation of  $q^i \in [-1, 1]$  between  $W^l$  and  $W^i$  for  $1 \leq i \leq n$ . As a consequence, we have that  $\mathbf{E}(W_t^l W_s^i) = q^i \min(t, s)$  for  $i = 1, \dots, n$  and

$$W_t^l = \sqrt{1 - \tilde{q}^T \tilde{q}} W_t^0 + \tilde{q}^T \tilde{W}_t,$$

where

$$\tilde{q}_t = (q_t^1, \dots, q_t^n)^T \text{ and } \tilde{W}_t = (W_t^1, \dots, W_t^n)^T. \quad (2.6)$$

When  $\tilde{q}^T \tilde{q} \neq 1$ , then the risk associated with subprime RML extension cannot disappear by trading in the financial market. Also, the *market price of risk*,  $\tilde{\xi}$ , is given by

$$\sigma \tilde{\xi} = \tilde{b} - r^T \bar{1} \quad (2.7)$$

with  $\tilde{b} = (b^1, \dots, b^n)^T$ , a column vector,  $\bar{1}$ , consisting of 1's and an invertible matrix,  $\sigma$ . Note that investors bear the risk regarding the market price of the risky marketable securities that is a measure of the risk premium. It is generally described by the *reward-to-risk ratio* of the market portfolio. Furthermore, in the situation described above, we can compute the *risk premium*,  $\rho^i$ , on risky marketable securities  $i$  as

$$\rho^i = \sum_{j=1}^n \sigma^{ij} \tilde{\phi}^j, \quad (2.8)$$

where  $\tilde{\phi} = (\tilde{\phi}^1, \dots, \tilde{\phi}^n)^T$ . We deduce from (2.7) and (2.8) that

$$b^i = r^T + \sum_{j=1}^n \sigma^{ij} \tilde{\phi}^j. \quad (2.9)$$

Equation (2.9) reflects the fact that returns from risky marketable securities are usually higher than returns from Treasuries so that it is realistic to have  $b^i > r^T$  for each  $i = 1, 2, \dots, n$ . In addition, the SDE (2.5) will become

$$d\tilde{B}_t^i = \tilde{B}_t^i \left[ (r^T + \sum_{j=1}^n \sigma^{ij} \tilde{\phi}^j) dt + \sum_{j=1}^n \sigma^{ij} dW_t^j \right], \quad \tilde{B}_0^i = \mathbf{T}, \quad 1 \leq i \leq n.$$

Thus, we note that investing in the  $i$ -th marketable securities class yields a total earning of  $b^i$ .

In the following discussion,  $\sigma$  denotes the invertible matrix  $(\sigma^{ij})$  which results in the symmetric matrix  $C = \sigma \sigma^T$  being positive definite. Furthermore,  $\pi_t^i$  denotes the value of the



marketable securities invested in securities  $i$  at time  $t$ , for  $i = 0, \dots, n$ . In this case,

$$B - \sum_{i=1}^n \pi^i,$$

represents the value of the riskless Treasuries investment. These variables are unbounded with a negative value of  $\pi^i$  implying that the OR may take part in the short selling of the corresponding securities. On the other hand, if

$$B - \sum_{i=1}^n \pi^i < 0,$$

then the OR may borrow money to invest in securities at rate  $r^T$ .

Finally, we have the *portfolio process* or *trading strategy*,  $\tilde{\pi}_t$ , given by  $\tilde{\pi}_t = (\pi_t^1, \dots, \pi_t^n)^T$ . Here  $\tilde{\pi}_t$  is taken to be a  $\mathbb{R}^n$ -measurable process adapted to  $\{\mathcal{G}_t\}_{t \geq 0}$  such that

$$\int_0^\infty (\tilde{\pi}_s)^T \tilde{\pi}_s ds < \infty, \text{ a.s.} \quad (2.10)$$

## 2.2 Liabilities

Liabilities represent the OR's sources of funds. Marketable securities are purchased with these funds. The value of OR's liabilities relies on, for instance, deposits and borrowings that are both associated with randomness, thus the dynamics of liabilities is stochastic.

### 2.2.1 Deposits

In our research, the term deposits includes both demand and time deposits. Deposits,  $D$ , have uncertainty associated with them and thus can be modeled as a stochastic process.

The stochastic process,  $D : \Omega \times T \rightarrow \mathbb{R}_+$ , is taken to be the *deposits*, whose value at time  $t$  is denoted by  $D_t$ . The dynamics of deposits can be written as a diffusion process (see, for instance, [12], [17] and [24]) in the form

$$dD_t = c_t dt + \sigma^1 dW_t^1, \quad D_0 = D_{t_0}, \quad (2.11)$$

with the *deposit inflow rate* denoted by  $c = r^{DD} + r^{TD}$ ,  $c_0 = c_{t_0}$ .

$r^{DD} : T \rightarrow \mathbb{R}_+$  denotes the rate of demand deposit which is payable on demand and  $r^{TD} : T \rightarrow \mathbb{R}_+$  denotes the rate of time deposit which is payable only after a fixed interval of time.

In reality, withdrawals are usually paid on demand when time deposits are small.  $c$  is assumed to be a measurable adapted process with respect to the filtration  $\{\mathcal{F}_t\}$  which satisfies

$$\int_0^\infty |c_s| ds < \infty, \text{ a.s.} \quad (2.12)$$

### 2.2.2 Borrowings

In this dissertation,  $B_t$  represents borrowing from other ORs and the federal reserve bank at time  $t$ . The evolution of borrowings, in fact, are closely related to OR's assets. An assumption is made regarding changes in  $B$  and  $B$  where equal changes in  $B$  from other ORs cause equal changes in riskless and risky marketable securities according to

$$dB_t = \sum_{i=1}^n \pi_t^i \frac{d\tilde{B}_t^i}{\tilde{B}_t^i} + (B_t - \sum_{i=1}^n \pi_t^i) \frac{d\mathbb{T}(t)}{\mathbb{T}(t)}. \quad (2.13)$$

## 2.3 Capital

An OR's available capital consists of share capital reserves and hybrid capital instruments. In particular, *equity capital* is the most important which consists of extended and paid ordinary shares and non-cumulative continued preferred stock. The reason for the importance of equity capital is that it is common to all G-10 countries and it should be reported in an OR's published statements. Furthermore, equity capital information is indispensable when computing profit margins and determining the competitiveness of an OR. According to [15], *core capital* consists of common equity capital, noncumulative continued preferred stock as well as minority interest in consolidated subsidiaries without certain deductions. Core capital describes the OR's capital adequacy which acts as a buffer against losses.

## 2.4 Balance Sheet

We obtain from (1.1) that

$$dR_t + dM_t + dB_t = dD_t + dB_t + dK_t \quad (2.14)$$

to describe the dynamics of the stylized balance sheet. Suppose that an OR's capital gains (losses) are equal to the amount of reserves gained (lost), (see also, Chapter 9 of [31]), then we have that  $dR_t = dK_t$ ,  $t_0 < t < t_1$ , in (2.14) and

$$dB_t = dB_t + dD_t - dM_t, \quad (2.15)$$

where  $t_0$  and  $t_1$  represent the initial and end period, respectively.

In following discussions, we consider the case where  $\sigma^1 = 0$  in (2.11). Choosing and considering  $dM$  in (2.2) and  $dB$  in (2.13), allow us to rewrite  $B$ , represented in (2.15) as

$$dB_t = \sum_{i=1}^n \pi_t^i \frac{d\tilde{B}_t^i}{\tilde{B}_t^i} + (B_t - \sum_{i=1}^n \pi_t^i) \frac{d\mathbb{T}(t)}{\mathbb{T}(t)} + [c_t - l_t]dt. \quad (2.16)$$

## 2.5 Reference Processes

Actuarial cost methods are used to determine charges against annual operating techniques and also as a measure for the required levels of assets and liabilities of an OR at any given time, but, to our knowledge, haven't been used to determine subprime RML extension levels previously. The unfunded subprime RMLs to be extended,  $M^u$ , for the individual cost method, is determined by the difference between the OR's RML reference process,  $M^r$ , and the marketable securities available for RML extension,  $B$ . Symbolically, we have that

$$M_t^u = M_t^r - B_t. \quad (2.17)$$

Thus,  $M^u$  is dealt with by reference processes  $M^r$  and  $D^r$ . *Additional deposits*,  $D^a$ , are used to achieve this adjustment. Thus, we have that

$$D_t^a = c_t - D_t^r. \quad (2.18)$$

In this section, in order to establish an ideal level of subprime RMLs to be extended at the end of the risk horizon, we consider the two reference processes mentioned above, which will serve as leading indicators. The risk horizon time instances which are involved in our analysis are  $t_0$  which represents the beginning of the period,  $t_1$ , which represents the end and  $v$  which is an arbitrary instant of time. The *discounted rate of interest* denoted by  $r^d$  is related to OR valuation. In this regard, when the future value is assumed,  $r^d$  refers to the annual growth rate of securities when the required present value should be found.

### 2.5.1 Description of Reference Processes

Throughout this section,  $\mathbf{E}(\cdot|\mathcal{F}_t)$  represents the conditional expectation with respect to  $\mathcal{F}_t$  associated with  $\{W_t^l\}_{t \geq 0}$ . Below, we present integral formulas for the RML and D reference processes, denoted by  $M^r$  and  $D^r$ , respectively. The *RML reference process* describes the value which is required for an OR to extend subprime RMLs by using actuarial methods and assumptions to determine this value. The *D reference process* is the amount sourced from deposits which intends to fund subprime RML extension for a specific period which is actuarially determined.

Next, we provide formulas for  $M^r$  and  $D^r$ , respectively.

$$M_t^r = \int_{t_0}^{t_1} \exp\{-r_{t_1-v}^d\} P_v \mathbf{E}(l_{t+t_1-v}|\mathcal{F}_t) dv, \quad P_{t_0} = 0 \quad (2.19)$$

for  $t > 0$ .  $P$  is a distribution function which is the stochastic analogue of functions discussed in [6]. The distribution of the accumulation of deposits during a RML financing period is done according to the distribution function  $P$ , with  $p$  as the density function. The value  $P_v$  includes the percentage of the subprime RMLs to be extended, the accumulation of and also the time instant,  $v$ , which occurred during this time of RML financing. Also, we have that

$$D_t^r = \int_{t_0}^{t_1} \exp\{-r_{t_1-v}^d\} p_v \mathbf{E}(l_{t+t_1-v}|\mathcal{F}_t) dv, \quad p_{t_0} = 0 \quad (2.20)$$

represents the D reference process for  $t > 0$ . We observe that  $p_v$  is zero for  $v \leq t_0$  or  $v \geq t_1$  since  $[t_0, t_1]$  is the support of  $p$ . In addition, it follows that

$$P' = p. \tag{2.21}$$

To calculate the reference processes at time  $t$ , given by (2.19) and (2.20), information available up to time  $t$  is used with respect to the conditional expectation. The corresponding element of the filtration,  $\mathcal{F}_t$ , contains the information. The Markov property is satisfied since we made the assumption for  $l$  to be a diffusion process. Therefore,  $\mathbf{E}(\cdot|\mathcal{F}_t)$  equals the conditional expectation at period  $t$  in terms of the current value of  $l$ .

### 2.5.2 Relationship between the Deposit and RML Reference Processes

From here on,  $r^a$  represents the rate of actualization where actualization is defined as the present value of a stipulated process level in the future. When  $r^a$  is high, then the OR will be more concerned about the present value rather than future values. Relationships between the RML and deposit reference processes,  $M^r$  and  $D^r$ , respectively as well as the the specified rate subprime RMLs to be extended,  $l$ , can be explained by Theorem 2.5.1 below.

**Theorem 2.5.1 (Relationship between  $M^r$ ,  $D^r$  and  $l$ ):** *In the case where (2.2) holds, there exist constants  $Q$  and  $Z$  such that  $D^r = Ql$  and  $M^r = Zl$ . Also, identities*

$$Q = 1 + (c^{ME} - r^a)Z \tag{2.22}$$

and

$$(r^a - c^{ME})M_t^r + D_t^r - l_t = 0 \tag{2.23}$$

hold for every  $t \geq 0$ .

**Proof.** Since (2.3) holds and  $l$  is modeled as a geometric Brownian motion, it follows that

$$\begin{aligned}
\mathbf{E}(l_{t+t_1-v}|\mathcal{F}_t) &= l_0 \mathbf{E}\left(\exp\left\{\left(c^{ME} - \frac{(\sigma^{ME})^2}{2}\right)(t+t_1-v) + \sigma^{ME}W_{t+t_1-v}^l\right\}|\mathcal{F}_t\right) \\
&= l_0 \exp\{c_{t+t_1-v}^{ME}\} \mathbf{E}\left(\exp\left\{-\frac{(\sigma^{ME})^2}{2}(t+t_1-v) + \sigma^{ME}W_{t+t_1-v}^l\right\}|\mathcal{F}_t\right) \\
&= l_0 \exp\{c_{t+t_1-v}^{ME}\} \exp\left\{-\left(\frac{(\sigma^{ME})^2}{2}\right)t + \sigma^{ME}W_t^l\right\} \\
&= l_0 \exp\{c_{t_1-v}^{ME}\} \exp\left\{\left(c^{ME} - \frac{(\sigma^{ME})^2}{2}\right)t + \sigma^{ME}W_t^l\right\} \\
&= \exp\{c_{t_1-v}^{ME}\} l_t
\end{aligned}$$

for every  $v \in [t_0, t_1]$  and  $t \geq 0$ . The RML and deposit reference process constants may be given by

$$Z = \int_{t_0}^{t_1} \exp\{(c^{ME} - r^a)(t_1 - v)\} P_v dv \quad (2.24)$$

and

$$Q = \int_{t_0}^{t_1} \exp\{(c^{ME} - r^a)(t_1 - v)\} p_v dv, \quad (2.25)$$

respectively, in order to complete the first part of this theorem.

Next, in order to show that (2.22) holds, we should integrate (2.25) by parts while bearing (2.21) in mind. Thus, we obtain

$$\begin{aligned}
Q &= \int_{t_0}^{t_1} \exp\{(c^{ME} - r^a)(t_1 - v)\} p_v dv \\
&= \int_{t_0}^{t_1} \exp\{(c^{ME} - r^a)(t_1 - v)\} P'_v dv \\
&= \exp\{(c^{ME} - r^a)(t_1 - v)\} P_v + (c^{ME} - r^a) \int_{t_0}^{t_1} \exp\{(c^{ME} - r^a)(t_1 - v)\} P_v dv \\
&= 1 + (c^{ME} - r^a) Z
\end{aligned}$$

which follows from  $P_{t_0} = 0$  presented in (2.19). As a result, from the first part of this proof and a consideration of the two formulas (2.2) and (2.20), it follows that (2.23) holds.  $\square$

The formulation of Theorem 2.5.1 relates to the discussion in [6] where the deterministic case was the main focuss. As a consequence of the important issues arising from Theorem 2.5.1, the following corollary is given.

**Corollary 2.5.2 (Relationship between  $M^u$  and  $D^a$ ):** *Assume that the hypothesis of this result relates to that of Theorem 2.5.1. Let constants  $Z$  and  $Q$  be non-zero, real values, then*

$$D_t^r = SM_t^r \text{ and } D_t^a = S[M_t^u + B_t] - l_t,$$

for  $t \rightarrow \mathbb{R}$ , where  $S = \frac{Z}{Q}$  represents an interest valuation rate, in terms of  $Q$ ,  $c^{ME}$  and  $r^a$ , by

$$S = \frac{1}{Q} + c^{ME} - r^a.$$

In addition, we have that

$$dM_t^r = M_t^r [c^{ME} dt + \sigma^{ME} dW_t^l], \tag{2.26}$$

where  $M_t^r = M_0^r = Zm_0$ .





## Chapter 3

# OPTIMAL LIQUIDITY RISK MANAGEMENT

”The current credit crisis will come to an end when the overhang of inventories of newly built homes is largely liquidated, and home price deflation comes to an end. That will stabilize the now-uncertain value of the home equity that acts as a buffer for all home mortgages, but most importantly for those held as collateral for residential MBSs. Very large losses will, no doubt, be taken as a consequence of the crisis. But after a period of protracted adjustment, the U.S. economy, and the world economy more generally, will be able to get back to business.”

– Alan Greenspan, 2007.

”It’s now conventional wisdom that a housing bubble has burst. In fact, there were two bubbles, a housing bubble and a financing bubble. Each fueled the other, but they didn’t follow the same course.”

– Wall Street Journal, 2007.

In this Chapter, the main results are stated that pertain to the role of stochastic optimal control in OR’s risk management in Theorem 2.5.1 and Corollary 2.5.2. Prior to the stochastic control problem, the following items will be discussed: an OR’s risk factors, the stochastic dynamics of marketable securities as well as the RML financing spread method regarding an OR.

### 3.1 Stochastic Credit Portfolio Dynamics

The following SDE represents the dynamics of  $B$ , by using (2.4), (2.5) and (2.16).

$$\begin{aligned} dB_t &= \sum_{i=1}^n \pi_t^i [b^i dt + \sum_{j=1}^n \sigma^{ij} dW_t^j] + (c_t - l_t)dt + (B_t - \sum_{i=1}^n \pi_t^i) r^T dt \\ &= (r^T B_t + \sum_{i=1}^n \pi_t^i (b^i - r^T))dt + (c_t - l_t)dt + \sum_{i=1}^n \sum_{j=1}^n \pi_t^i \sigma^{ij} dW_t^j, \end{aligned} \quad (3.1)$$

with initial condition  $B_t = B_0$ . We rewrite the SDE (3.1) by applying Theorem 2.5.1 as follows

$$dB_t = (r^T B_t + \sum_{i=1}^n \pi_t^i (b^i - r^T))dt + (D_t^a - (c^{ME} - r^a)M_t^r)dt + \sum_{i=1}^n \sum_{j=1}^n \pi_t^i \sigma^{ij} dW_t^j. \quad (3.2)$$

Here, the expected value of the unfunded subprime RMLs to be extended is elevated to zero by choosing additional deposits,  $D^a$ , carefully. This is accomplished via borrowings,  $B$ , or deposits,  $D$ , and retained earnings from marketable securities,  $B$ . One cannot trade the stipulated subprime RMLs to be extended, therefore the OR is unable to hedge the risk inherent by these mortgages, thus our model represents an incomplete market. Also, the state variables in (3.2) are the value of the  $B$  available for RML extension and  $M^r$ . It is also known that marketable securities allocation,  $\pi$ , and additional deposits,  $D^a$ , are the control variables.

### 3.2 Stochastic Optimal Control Problem

It is necessary to consider a well-defined objective function,  $V$ , with appropriate constraints in order for an optimal deposit inflow rate,  $c^*$ , and an optimal marketable securities allocation strategy denoted by  $\tilde{\pi}^*$ , to be determined. It is important to make the right choice regarding an objective function and appropriate constraints for solutions to our stochastic optimal control problem not to be ambiguous. Here, we choose that a control law,  $g$ , should be determined to minimize the objective function  $V : \mathcal{G}_{B_0, M^r} \rightarrow \mathbb{R}_+$ , where  $\mathcal{G}_{B_0, M^r}$  is the class of admissible control laws where

$$\begin{aligned}
 \mathcal{G}_{B_0, M^{r_0}} &= \{\text{Markovian stationary } (D^a, \tilde{\pi}) \\
 &\text{adapted to filtration } \{\mathcal{G}_t\}_{t \geq 0} \text{ where both (2.10) and (2.12) hold} \\
 &\text{with (2.26) and (3.2) is a } \mathcal{G}_t \text{-measurable continuous unique solution}\}. \quad (3.3)
 \end{aligned}$$

Therefore,  $(D^a, \tilde{\pi}) \in \mathcal{G}_{B_0, M^{r_0}}$  is closed that is shown in (3.2). Also, the objective function,  $V : \mathcal{G}_{B_0, M^{r_0}} \rightarrow \mathbb{R}_+$ , is defined by

$$V(D^a, \tilde{\pi}) = \mathbf{E} \left[ \int_{t_0}^{t_1} \kappa (D^a)_s^2 + (1 - \kappa) [M_s^r - B_s]^2 ds \right]. \quad (3.4)$$

The parameter,  $\kappa$ , is a weighting factor for  $0 < \kappa \leq 1$ , which reflects the relative importance of risks associated with deposit inflows and equity. This objective function (3.4) is formulated to minimize the combined risk, namely funding and credit crunch risk. Thus, we want to minimize the risk of inefficient deposit inflows which is reflected by the cost on  $D^a$  and credit crunch risk which is reflected by costs on investing in the marketable securities,  $\pi$ , that depends on factors associated with RML extension.

The stochastic optimal control problem can now be stated as follows.

**Problem 3.2.1 (Optimal Control Problem for Subprime Risk Management):**

*Suppose that  $\mathcal{G}_{B_0, M^{r_0}} \neq \emptyset$ . The system (3.2) for managing OR's risk with control laws,  $\mathcal{G}_{B_0, M^{r_0}}$ , stated in (3.3) and  $V : \mathcal{G}_{B_0, M^{r_0}} \rightarrow \mathbb{R}_+$ , the objective function,  $V$  represented by (3.4) should be considered. We want to solve*

$$\min_{(D^a, \tilde{\pi}) \in \mathcal{G}_{B_0, M^{r_0}}} V(D^a, \tilde{\pi}),$$

*in order to determine the objective function  $V^*$ ,*

$$V^* = \min_{(D^a, \tilde{\pi}) \in \mathcal{G}_{B_0, M^{r_0}}} V(D^a, \tilde{\pi}),$$

*and the optimal control law  $(D^a, \tilde{\pi})^*$ , if its existence is known,*

$$(D^a, \tilde{\pi})^* = \arg \min_{(D^a, \tilde{\pi}) \in \mathcal{G}_{B_0, M^{r^0}}} V(D^a, \tilde{\pi}) \in \mathcal{G}_{B_0, M^{r^0}}.$$

### 3.3 Spread Method of RML Financing

In this section, we discuss the spread method of OR RML financing that corresponds to the unfunded subprime RMLs to be extended which are spreading over the spread period,  $[t_0, t_1]$ . This method is used to provide appropriate provision for the development of excessive or shortage of reserves regarding an OR's assets, wherefore the rate of return on riskless Treasury securities,  $r^T$ , is constructed for sources of random investments in risky marketable securities,  $W^i$ ,  $i = 1, 2, \dots, n$ , to be eliminated. We would like the spread method of OR RML financing to be imposed on an OR's operations. A formal definition for the implementation of this spread method follows below.

**Definition 3.3.1 (Implementation of a Spread Method for an OR's RMLs):** *Suppose that  $D^a$  and  $M^u$  are the OR's additional deposits and unfunded subprime RMLs, respectively. The spread method of OR RML financing can be implemented if there exist a constant,  $k$ , such that*

$$D^a = kM^u.$$

Lemma 3.3.2 below provide a condition regarding the existence of a spread method of RML financing.

**Lemma 3.3.2 (Existence of a Spread Method of OR RML financing):** *Suppose that  $\sigma^{ME}$ ,  $r^T$ ,  $\tilde{q}^T$  and  $\tilde{\phi}$  are characterized by (2.2), (2.4), (2.6) and (2.7), respectively. Then a rate of actualization,  $r^a$ , exists of the form*

$$r^a = r^T + \sigma^{ME} \tilde{q}^T \tilde{\phi} \quad (3.5)$$

*that allows a spread method of OR RML financing to be implemented.*

**Proof.** We want to obtain the stipulated level of subprime RMLs to be extended,  $l_{t_1}$ , where  $u = t_1$  is the end of the risk period. To be able to achieve this,  $l$  should be valued as in

Theorem 2.5.1 such that

$$\mathbf{E}(l_{t+t_1-u}|\mathcal{F}_t) = \exp\{(c^{ME} - r^a)(t_1 - u)\}l_t.$$

This valuation should be insensitive to risk associated with RML extension levels. Therefore, we require that

$$c^{ME} - r^a = -(r^T - \alpha^*), \tag{3.6}$$

where  $\alpha^* = c^{ME} - \sigma^{ME}\tilde{q}^T\tilde{\phi}$  in order to correct the growth rate of the stipulated RML extension rate,  $l$ , and the expected value calculation in a market where risk is neutral. The expression (3.5) follows from (3.6).  $\square$

We use lemma 3.3.2 to prove our main results in the next section.

### 3.4 Main Results

Transversality can be an obstacle to overcome when solving infinite horizon optimal control problems (see the discussion in Chapter 1 of [16] and page 49 of [5]). In our optimal control problem, the transversality condition with actualized form

$$\lim_{t \rightarrow \infty} \exp\{k^a t\}h(B_t^*, M_t^T, t) = 0, \tag{3.7}$$

is considered for the candidate solution and feasible state trajectory given by  $B^*$  and  $M^T$ , respectively, with constant  $k^a$  which is chosen later (see, for instance [8], [21], [22] and [25]). Solutions are therefore limited to an infinite horizon dynamic optimization problem. Thus, those solutions involving accumulating, for example, infinite debt, are immediately ruled out. We consider a finite  $T$ -period horizon of the problem of maximizing the present value, we obtain the first-order condition for  $nt + T$ , then we take the limit of this condition as  $T$  goes to infinity in order to obtain the transversality condition. Our main optimal stochastic control result related to an OR's risk management is stated in this section.

**Theorem 3.4.1 (Optimal Allocation of Marketable Securities and the Rate of Deposit Inflows):** *Suppose that the following conditions (2.2) and (3.5) in Lemma 3.3.2*

hold. Also, assume that  $r^i$  represents the strict upper bound for  $2c^{ME} + (\sigma^l)^2$  that relates to the rate of impatience of the OR, i.e.,

$$2c^{ME} + (\sigma^l)^2 < r^i. \quad (3.8)$$

Then the optimal allocation of risky marketable securities is given by

$$\tilde{\pi}_t^* = [C^{-1}\sigma\tilde{\xi} + \sigma^l\sigma^{-T}\tilde{q}](M_t^u + B_t) - C^{-1}\sigma\tilde{\xi}B_t, \quad (3.9)$$

with  $M^u$ , the unfunded subprime RMLs to be extended. Also, the optimal deposit inflow rate is given by

$$c_t^* = D_t^r + \frac{\delta^{aa}}{\kappa}M_t^u, \quad (3.10)$$

where  $\delta^{aa}$  is the only, non-negative solution to

$$(\delta^{aa})^2 + \kappa(r^i - 2r^T + \tilde{\xi}^T\tilde{\xi})\delta^{aa} - \kappa(1 - \kappa) = 0, \quad (3.11)$$

under the assumptions made earlier.

**Proof.** In order to proof this theorem, we use procedures closely related to those suggested in [23], [27], [28] and [29]. In this regard, to obtain the dynamic programming equation (DPE) or Hamilton-Jacobi-Bellman equation (HJBE), we consider the objective function (3.4) and the control system (3.2). Now, in order to obtain the optimal levels of the control variables, we solve the DPE by applying the standard second order partial differential equation theory. In this regard, a solution is chosen to obtain the spread method of OR RML financing. In order to complete the proof, we should verify that the transversality condition holds (see Chapter 1 of [16] and page 49 of [5]) to show that the solution of the control problem and the original value function relate.

The Hamilton-Jacobi-Bellman equation is given by

$$\begin{aligned}
 r^i V = & \min_{(D^a, \tilde{\pi}) \in \tilde{\mathcal{G}}_{B_0, M^r 0}} \left\{ (r^T B + \tilde{\pi}^T (\tilde{b} - r^T \bar{1}) + D^a + (c^{ME} - r^a) M^r) V^a \right. \\
 & + c^{ME} M^r V^{M^r} + \frac{1}{2} \tilde{\pi}^T C \tilde{\pi} V^{aa} + \frac{1}{2} (\sigma^{ME})^2 (M^r)^2 V^{M^r M^r} + \sigma^{ME} M^r \tilde{\pi}^T \sigma \tilde{q} V^{a M^r} \\
 & \left. + \kappa (D^a)^2 + (1 - \kappa) (B - M^r)^2 \right\}, \tag{3.12}
 \end{aligned}$$

for the optimal control problem stated in Problem 3.2.1. Also, observe that the partial differential equation (3.12) can be separated in terms

- (a) that depends on  $D^a$ ;
- (b) that depends on  $\tilde{\pi}$ ;
- (c) that depends on neither  $D^a$  nor  $\tilde{\pi}$ .

We solve the minimization problem in two separate parts. In order to obtain minimization (a), we set the first derivative of

$$D^a (V^a + \kappa D^a)$$

equal to zero with respect to  $D^a$ , thus

$$V^a + 2\kappa D^a = 0. \tag{3.13}$$

Therefore, the optimal deposit inflow rate is

$$c^*(V^a) = -\frac{V^a}{2\kappa}, \tag{3.14}$$

where a smooth, strictly convex solution  $V$  of equation (3.12) exists. Minimization (b) follows under the same conditions as above, from the optimization of

$$\tilde{\pi}^T (\tilde{b} - r^T \bar{1}) V^a + \frac{1}{2} \tilde{\pi}^T C \tilde{\pi} V^{aa} + \sigma^{ME} M^r \tilde{\pi}^T \sigma \tilde{q} V^{a M^r}$$

with respect to  $\tilde{\pi}$ . Here, the optimal allocation of marketable securities is

$$\tilde{\pi}^*(V^a, V^{aa}, V^{MrMr}) = -C^{-1}(\tilde{b} - r^T \bar{1}) \frac{V^a}{V^{aa}} - \sigma^{ME} M^r \sigma^{-T} \tilde{q} \frac{V^{aMr}}{V^{aa}}. \quad (3.15)$$

Now, we substitute the optimal values  $c^*$  and  $\tilde{\pi}^*$  given by (3.14) and (3.15), respectively, into the right hand side of (3.12) to obtain

$$\begin{aligned} & \left\{ r^T B - (\tilde{b}^T - r^T \bar{1}^T) C^{-T} (\tilde{b} - r^T \bar{1}) \frac{V^a}{V^{aa}} - \tilde{q}^T \sigma^{-1} M^r \sigma^{ME} (\tilde{b} - r^T \bar{1}) \frac{V^{aMr}}{V^{aa}} \right. \\ & \quad \left. - \frac{V^a}{2\kappa} + (c^{ME} - r^a) M^r \right\} V^a + c^{ME} M^r V^{Mr} \\ & + \frac{1}{2} \left\{ (\tilde{b}^T - r^T \bar{1}^T) C^{-T} (\tilde{b} - r^T \bar{1}) \left[ \frac{V^a}{V^{aa}} \right]^2 + \tilde{q}^T \sigma^{-1} c^{ME} M^r (\tilde{b} - r^T \bar{1}) \frac{V^{aMr}}{V^{aa}} \frac{V^a}{V^{aa}} \right. \\ & + (\tilde{b}^T - r^T \bar{1}^T) C^{-T} c^{ME} M^r \sigma^{-T} \tilde{q} \frac{V^{aMr}}{V^{aa}} \frac{V^a}{V^{aa}} + \tilde{q}^T \sigma^{-1} (c^{ME})^2 (M^r)^2 \sigma^{-T} \tilde{q} \left[ \frac{V^{aMr}}{V^{aa}} \right]^2 \left. \right\} V^{aa} \\ & + c^{ME} M^r \left\{ -(\tilde{b}^T - r^T \bar{1}^T) C^{-T} \frac{V^a}{V^{aa}} - \tilde{q}^T \sigma^{-1} c^{ME} M^r \frac{V^{aMr}}{V^{aa}} \right\} \sigma \tilde{q} V^{aMr} \\ & + \frac{1}{2} (c^{ME})^2 (M^r)^2 V^{MrMr} + \frac{(V^a)^2}{4\kappa} + (1 - \kappa)(B^2 - 2BM^r + (M^r)^2) \end{aligned} \quad (3.16)$$

The form of (3.16) implies that

$$V(B, M^r) = \delta^{aa} B^2 + \delta^{MrMr} (M^r)^2 + \delta^{aMr} a M^r \quad (3.17)$$

is a quadratic solution of (3.12) by standard second order partial differential equation theory.

$$(\delta^{aa})^2 + \kappa(r^i + \tilde{\phi}^T \tilde{\phi} - 2r^T) \delta^{aa} - \kappa(1 - \kappa) = 0, \quad (3.18)$$

$$\begin{aligned} & 4\kappa(r^i - 2c^{ME} - (\sigma^{ME})^2) \delta^{MrMr} \delta^{aa} + \delta^{aa} (\delta^{aMr})^2 - 4\kappa(c^{ME} - r^a) \delta^{aMr} \delta^{aa} \\ & + \kappa((\sigma^{ME})^2 \tilde{q}^T \tilde{q} + 2\sigma^{ME} \tilde{q}^T \tilde{\phi} - \tilde{\phi}^T \tilde{\phi}) (\delta^{aMr})^2 - 4\kappa(1 - \kappa) \delta^{aa} = 0 \end{aligned} \quad (3.19)$$

and



$$\begin{aligned}
 & -\delta^{aM^r} \delta^{aa} + \kappa(c^{ME} + r^T - r^i - \tilde{\phi}^T \tilde{\phi} - \sigma^{ME} \tilde{q}^T \tilde{\phi}) \delta^{aM^r} \\
 & + 2\kappa(c^{ME} - r^a) \delta^{aa} + 2\kappa(1 - \kappa) = 0
 \end{aligned} \tag{3.20}$$

have solution  $\delta^{aa}$ ,  $\delta^{M^r M^r}$  and  $\delta^{aM^r}$ .

We can solve (3.18), (3.19) and (3.20) by applying standard theory. However, we proceed otherwise in order to obtain a solution that leads to a spread method of OR RML financing. By this approach, we calculate the optimal rate of deposit inflows in order to obtain the rate of actualization,  $r^a$ .

$$D^a = -\frac{1}{2\kappa} \left( 2\delta^{aa} B + \delta^{aM^r} M^r \right) = -\frac{1}{2\kappa} \left( (2\delta^{aa} + \delta^{aM^r}) B + \delta^{aM^r} M^u \right) \tag{3.21}$$

follows from equations (2.17), (3.14) and (3.15). we notice from (3.21) that the additional deposits,  $D^a$ , is directly proportional to the unfunded subprime RMLs to be extended,  $M^u$ , if and only if

$$2\delta^{aa} = -\delta^{aM^r}. \tag{3.22}$$

In order for (3.18) and (3.20) to have a solution, the following condition

$$r^a = r^T + \sigma^{ME} \tilde{q}^T \tilde{\phi} \tag{3.23}$$

should hold. The inequality

$$\delta^{M^r M^r} = \frac{r^i - 2c^{ME} - (\sigma^{ME})^2 \tilde{q}^T \tilde{q}}{r^i - 2c^{ME} - (\sigma^{ME})^2} \delta^{aa} \geq \delta^{aa}$$

is obtained by substituting conditions (3.22) and (3.23) into (3.19) and subtracting from (3.18). We conclude that the value function,  $V$ , is always non-negative.

It is necessary to verify that the transversality condition holds in order to show that the value function from (3.12) is the solution (3.17). Firstly, we should substitute the optimal values  $c^*$  and  $\tilde{\pi}^*$  given by (3.14) and (3.15), respectively, into (3.2). Thus, we represent the

stochastic dynamics of the optimal value,  $B^*$ , of marketable securities by

$$dB_t^* = \left\{ \left( r^\top - \tilde{\phi}^T \tilde{\phi} - \frac{\delta^{aa}}{\kappa} \right) B_t^* + \left( -r^\top + \tilde{\phi}^T \tilde{\phi} + \frac{\delta^{aa}}{\kappa} + c^{ME} \right) M_t^r \right\} dt + \left\{ -\tilde{\phi}^T B_t^* + (\tilde{\phi}^T + \sigma \tilde{q}^T) M_t^r \right\} dW_t.$$

Secondly, we observe that  $V$  in (3.17) is a homogeneous quadratic polynomial in  $B$  and  $M^r$ . Therefore, we have to calculate the conditional expectation of  $(B^*)^2$ ,  $B^* M^r$  and  $(M^r)^2$  with respect to  $(B_0, M^{r\top})$ . The following notation is used in the sequel:

$$f_t = \mathbf{E}_{B_0, M^{r\top}}(B^* M^r)_t, \quad g_t = \mathbf{E}_{B_0, M^{r\top}}(B^*)^2_t \quad \text{and} \quad h_t = \mathbf{E}_{B_0, M^{r\top}}(M^r)^2_t. \quad (3.24)$$

$M^r$  is modelled as a geometric Brownian motion, mentioned earlier, which results in

$$h_t = (M^{r\top})^2 \exp\{(2c^{ME} + (\sigma^{ME})^2)t\} \quad \text{and} \quad \lim_{t \rightarrow \infty} \exp\{-r^\top t\} h_t = 0$$

if and only if (3.8) holds.  $f$  and  $g$  in (3.24) satisfy the linear differential equations (see, for instance, [4])

$$f'_t = \left( r^\top - \tilde{\phi}^T \tilde{\phi} - \frac{\delta^{aa}}{\kappa} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi} \right) f_t + \left( -r^\top + \tilde{\phi}^T \tilde{\phi} + \frac{\delta^{aa}}{\kappa} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi} \right) h_t, \quad f_0 = B_0 M^{r\top}$$

and

$$g'_t = 2 \left( r^\top - \tilde{\phi}^T \tilde{\phi} - \frac{\delta^{aa}}{\kappa} \right) g_t + 2 \left( -r^\top + \tilde{\phi}^T \tilde{\phi} + \frac{\delta^{aa}}{\kappa} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi} \right) f_t, \quad g_0 = B_0^2.$$

In this regard, we conclude that

$$f_t = (B_0 - k^1 M^{r\top}) M^{r\top} \exp \left\{ \left( r^\top - \tilde{\phi}^T \tilde{\phi} - \frac{\delta^{aa}}{\kappa} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi} \right) t \right\} + k^1 h_t, \quad (3.25)$$

with the constant

$$k^1 = \frac{\delta^{aa} + \kappa(-r^T + \tilde{\phi}^T \tilde{\phi} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi})}{\delta^{aa} + \kappa(-r^T + \tilde{\phi}^T \tilde{\phi} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi} + (\sigma^{ME})^2)}.$$

Therefore,

$$\lim_{t \rightarrow \infty} \exp\{-r^i t\} f_t = 0$$

if and only if (3.8) and

$$r^T - \tilde{\phi}^T \tilde{\phi} - \frac{\delta^{aa}}{\kappa} + c^{ME} - \sigma^{ME} \tilde{q}^T \tilde{\phi} < r^i \quad (3.26)$$

both hold simultaneously. Now, we conclude that the positive solution of (3.11) implies (3.26) when we estimate the rate of impatience inequality given by (3.8) and  $\delta^{aa}$  by noticing that

$$\begin{aligned} -\sigma^{ME} \tilde{q}^T \tilde{\phi} &\leq \sigma^{ME} \bar{1}^T \tilde{\phi} = \frac{1}{2} (\tilde{\phi}^T \tilde{\phi} + (\sigma^{ME})^2) - (\tilde{\phi} - \sigma^{ME} \bar{1})^T (\tilde{\phi} - \sigma^{ME} \bar{1}) \\ &\leq \frac{1}{2} \tilde{\phi}^T \tilde{\phi} + \frac{1}{2} (\sigma^{ME})^2. \end{aligned} \quad (3.27)$$

for  $-1 \leq q^i \leq 1$ . We also have that (3.11) implies that

$$r^T - \frac{\tilde{\phi}^T \tilde{\phi}}{2} - \frac{\delta^{aa}}{\kappa} < \frac{r^i}{2}. \quad (3.28)$$

(3.26) holds when we consider the condition (3.8) and inequalities (3.27) and (3.28). Also,

$$\begin{aligned} g_t &= (B_0^2 - 2B_0 M^{r^T} + (2k^1 - k^1 k^2 - k^3)(M^{r^T})^2) \exp \left\{ 2 \left( r^T - \frac{\tilde{\phi}^T \tilde{\phi}}{2} - \frac{\delta^{aa}}{\kappa} \right) t \right\} \\ &\quad + 2f_t - (2k^1 - k^1 k^2 - k^3) h_t, \end{aligned}$$

with constants  $k^2$  and  $k^3$ . The fact that

$$\lim_{t \rightarrow \infty} \exp\{-r^i t\} g_t = 0$$

is analogous regarding the expressions stated above. Finally, the transversality condition holds and therefore the proof of the theorem is complete.  $\square$

The following corollary follows immediately and gives a formula to make the optimal choice when investing in the  $i$ -th risky marketable securities class,  $\tilde{\pi}^{i*}$ .

**Corollary 3.4.2 (Optimal Allocation for the  $i$ -th Marketable Securities Class):**

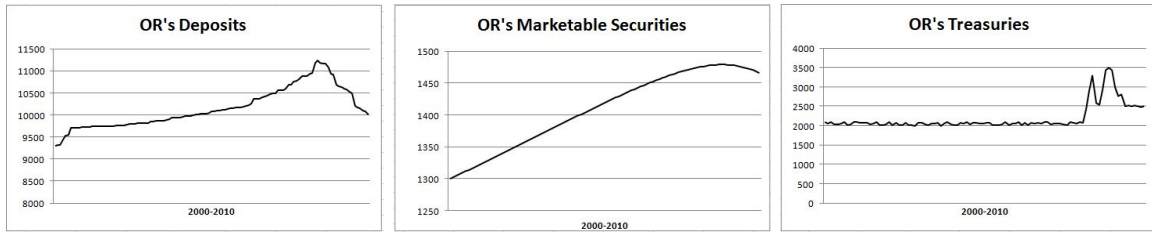
*Assume that Theorem 3.4.1 is true. Then the optimal allocation for the  $i$ -th risky marketable securities class can be represented by*

$$(\tilde{\pi}_t^i)^* = \bar{e}^i [C^{-1} \sigma \tilde{\xi} + \sigma^l \sigma^{-T} \tilde{q}] (M_t^u + B_t) - \bar{e}^i C^{-1} \sigma \tilde{\xi} B_t,$$

where  $\bar{e}^i = (0 \dots 1 \dots 0)$ , the unit vector for all  $i = 1, 2, \dots, n$ , has one in position  $i$ .

### 3.5 Examples Involving Subprime RML Financing

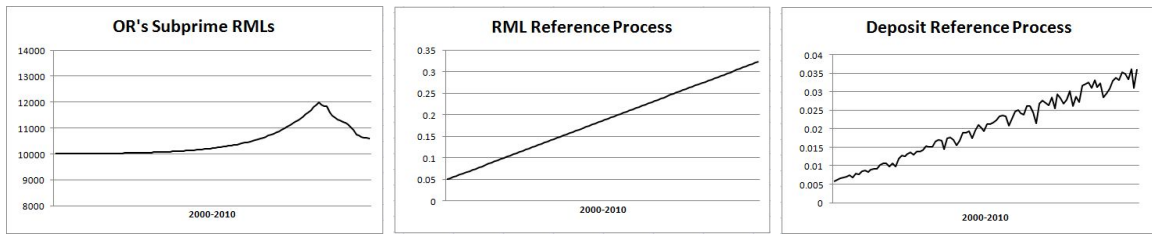
In this section, we provide a numerical example that illustrates the main issues raised in this dissertation. Numeric solutions for the dynamics of an OR's RMLs, Treasuries, deposits and marketable securities are obtained by using the SDEs (2.2), (2.4), (2.11) and (3.2), respectively by means of the numeric Euler-Maruyama method. Solutions for an OR's RML and deposit reference processes in Corollary 2.5.2 in Subsection 2.5.2 are also computed by the use of this method. Sensible values for variables are generated in order to obtain the solutions as shown in the figures below.



(a) Deposits

(b) Marketable Securities

(c) Treasuries



(d) Subprime RMLs

(e) RML Reference Process

(f) Deposit Reference Process

Figure 3.1: Dynamics of RML Funding Components



## Chapter 4

# DISCUSSIONS ABOUT RML FINANCING AND THE SMC

”However, the market for credit risk transfer (CRT) works smoothly only if the quality of loans is public information. If the quality of loans is private information, banks have an incentive to grant unprofitable loans that are then transferred to other parties, leading to an increase in aggregate risk.”

– Proffs. Hakanes and Schnabel, MPI Bonn, Germany, 2010.

”Mortgages were sold to other financial enterprises which tailored a range of saleable financial assets based on the underlying credit claims. The various newly engineered assets differed with respect to their default risk. For several years credit rating agencies granted excellent ratings to the range of these assets privileged regarding default risk. Hence, for a number of years financial decision makers worldwide considered these financial assets to have a very favorable risk-return profile. Today we know better.”

– Prof. Rötheli, Erfurt, Germany, 2010.

In this chapter, we provide an analysis of the main issues of our results in the previous Chapters as well as their relationship with the SMC. We especially consider the most important aspects of our model, optimal policies that Theorem 3.4.1 suggests and connections between our results and other related studies. The SMC is characterized by illiquidity in credit markets world wide as well the opaqueness of risks of financial products. As a result, weaknesses in banking systems and regulatory frameworks were exposed by the SMC, globally.

## 4.1 Subprime Stochastic Model and the SMC

In this section, we discuss what impact the SMC had on an OR's subprime stochastic model. The stylized balance sheet (1.1) in Chapter 1 is used as the foundation for our continuous-time stochastic model. When we apply the condition,  $dR_t = dK_t$ , (see, for instance, Chapter 9 of [31]), a description of the dynamics of OR's portfolio of the form given by (2.15) is obtained. Furthermore, the correlation of borrowings and portfolio aspects of an OR evolve to the SDE given by (3.2).

The model constructed in this dissertation is related to the model in [45] where the capital of an OR and their mortgage extension are aligned. It is also connected with the models of [11] and [12] for portfolio dynamics and management of capital. Also, the stochastic model we constructed relates to [18] where they present an OR's portfolio choice.

To our knowledge, our model doesn't allow for risk-shifting and risk-taking incentives explicitly. A value function which may have some specific convexity features is used to rectify this problem.

### 4.1.1 Assets and the SMC

The stipulated rate of RMLs to be extended is represented by the formula (2.1) in Section 2.1. The dynamics of the RML extension rate is described by means of a SDE (2.2) with solution  $l$  given by (2.3). Before the SMC, low credit risk resulted in an increase in RML extension. An increase in liquidity and competition in the credit market eventually pressured ORs to extend RMLs to subprime MRs which lead to a growing market of subprime marketable securities. The SMC affected the whole economy. Mortgage supply and credit conditions were tightened to preserve regulatory capital reserves and liquidity problems which affected the availability of mortgages as well as marketable securities. Therefore, systemic risk arose that resulted in a credit crunch.

Equation (2.4) represents the dynamics of the riskless Treasuries that becomes (2.4) when  $r^T$  is a deterministic rate of return and the assumption  $\sigma^0 = 0$  holds. The evolution of  $\tilde{B}$  is calculated by (2.5). If  $l_t$  is contingent and  $q^i \in [-1, 1]$  between the Brownian motions  $W^l$  and  $W^i$ , then (2.6) follows with the market price of risk,  $\tilde{\xi}$ , represented by (2.7) when  $\tilde{q}^T \tilde{q} \neq 1$ . We can calculate the risk premium,  $\rho^i$ , on risky marketable securities  $i$  by applying (2.8). From (2.7) and (2.8) we obtain (2.9) that shows that  $\tilde{B}$  returns are usually higher than returns on T, therefore,  $b^i > r^T$ . Finally, we obtain the trading strategy,  $\tilde{\pi}_t$ , for marketable securities with a measurable process (2.10).



Before the SMC, interest rates were low and credit was easily obtainable which boosted the credit market. According to [1], mortgages were securitized for four main reasons, namely, a new source for funding mortgages, transfer of credit risk to reduce their risks and expected losses, new profit and a decrease in capital requirements. This source of funding is not subject to deposit insurance and requirements for reserves. Also, subprime RML extension had risen dramatically as well as the RMBSs as a consequence of the booming house market. Securitization caused ORs to operate with low capital levels and it seemed that ORs had less bad mortgages, less risk and higher profits. ORs relied on credit ratings for this RMBSs which were overoptimistic regarding default risk of the underlying mortgages. According to [19], the flaws in the securitization system were hidden because of the increasing house prices as well as the inflow of funds from these securities to extend more subprime mortgages. The payments investors received from marketable securities are insured by monoline insurers (MLIs). Monolines' credit worthiness caused disturbance in the financial market. These monoline insurers are counterparties to credit derivatives such as RMBSs held by ORs. Therefore, counterparty risk was a great concern which created uncertainty in the marketable securities market. This system collapsed when the housing market had plunged in 2006 and MRs couldn't pay their mortgage payments anymore. Therefore, the incentive to reduce risk by securitization ended up by multiply risks which were spread globally. This way of transferring credit risk to other financial role players decreased an OR's capital requirements to extend more mortgages containing credit- and operational risk. Credit risk in the underlying RMLs became apparent. Also, the decline in ORs' capital tightened the credit around the globe. As a consequence, the declined value of RMBSs due to risks involving operational, tranching and liquidity risk and defaulting RMLs caused large losses globally.

[3] finds that securitization caused the role of the OR's lending channel to decrease. They also find that an OR's risk profile has an enormous effect on an OR's mortgage supply. In this regard, financial intermediaries e.g investors is the buyers of credit and as a result, influenced the lending supply of ORs. According to [26], converting illiquid mortgages into liquid marketable securities lessen the impact of shocks in the costs of external funds. In this regard, an OR's mortgage supply is less sensitive to external sources of funding, thus reducing the affect of monetary authority on an OR's lending.

#### 4.1.2 Liabilities and the SMC

In Section 2.2, the equation (2.11) represents the diffusion process for the dynamics of deposits. The deposit inflow rate,  $c$ , is a measurable process that is a solution to (2.12).

Before the SMC, ORs had a strong deposit inflow that funded mortgages. During this crisis, OR's deposits were used for funding subprime RMLs. Moreover, liquidity is associated with confidence. According to [34], ORs hid risky mortgages and after their true financial conditions were revealed, depositors and investors lost their confidence in these ORs which caused bank runs and resulted in insolvency and illiquidity of these ORs. This caused ORs' deposit levels to eradicate and as a result, ORs' deposit inflow for funding RML was cut off.

We observe that a change in borrowings is directly proportional to risky and riskless marketable securities, according to equation (2.13). ORs borrowed funds to invest in marketable securities and also in RMBSs for higher yields. This investments declined dramatically as a result of the SMC which left ORs with large debt. Losses on risky marketable securities have reduced the capital levels of ORs significantly which left ORs insolvent and with extremely low capital levels. ORs then borrowed from private sources and obtained funds from the government to acquire additional capital.

### 4.1.3 Capital and the SMC

Strong capital levels were favourable conditions before the SMC. During the housing boom, regulations regarding minimum capital ratios were bypassed by moving assets and liabilities off-balance sheet into special purpose vehicles for higher profits and leverage. But this caused increasing losses during the SMC and the large decline in RMBS value significantly reduced the capital reserves of ORs which resulted in insolvency and failure of ORs (see [47]). ORs were recapitalized by selling their toxic assets to the government and other private investors to restore ORs' capital levels. The paper [46] states that the whole financial system crashed as a result of ORs which had evaded regulatory capital requirements. Thus, the ORs' methods and processes failed which embed operational risk. This risk was not fully understood and acknowledged which resulted in the end in illiquidity and failed operational risk management. The effect the SMC had on capital includes dividends cuts and capital injections of more than US\$230 billion as of May 2008.

Hellwig states that there are different purposes for the use of capital reserves (see, for instance, [19]). Capital equity acts as a buffer against insolvency, incentives for risk taking or provides room for intervention. The OR should focus on the total risk when capital acts as a buffer against insolvency risk. When capital acts as an incentive for risk taking, one should be concerned about risk weights of assets and marginal risk. On the other hand, when capital requirements are used for interventions, then there is no need to be concerned about risk weights of assets, the only concern is to prevent manipulation of the

intervention threshold. Thus, it is important to know what purpose the OR's capital has and therefore the OR should investigate the risks associated with the purpose chosen for the capital reserves.

#### 4.1.4 Balance Sheet and the SMC

The dynamics of the stylized balance sheet in Section 2.4 is presented by (2.14). In order to obtain the simplified equation (2.15) for the dynamics of  $B$ , we apply the equality condition,  $dR_t = dK_t$ , to (2.14). Therefore, the SDE (2.15) can be rewritten in the form (2.16). The dynamics of an OR's portfolio depends on their borrowings, deposits, RMLs and marketable securities. Before the SMC, economic conditions were very good, deposit inflows were extremely high as well as the levels of mortgage supply which boosted the credit market. ORs wanted more earnings and turned to the risky subprime market for higher earnings than Treasuries. Deposits and marketable securities funded these risky RMLs associated with high default rates and credit risk. When the economy collapsed, ORs borrowed large funds to prevent bankruptcy. Thus, the variables of the dynamics of an OR's portfolio (2.15) were highly affected by the SMC.

#### 4.1.5 Reference Processes and the SMC

In Section 2.5, unfunded RMLs to be extended,  $M_t^u$ , and additional deposits,  $D_t^a$ , can be calculated by applying (2.17) and (2.18), respectively. Also, RML and deposit reference processes are represented by the formulas (2.19) and (2.20), respectively. The SMC had a great influence on these reference processes. Subprime RML extension was one of the main factors which contributed to the SMC. Deposit inflows were very strong which funded these RMLs. After the collapse of the financial system, deposit inflows plummeted and deposits were withdrawn which caused illiquidity and failure of many ORs.

The relationship between these two reference processes as well as the stipulated RML extension rate are established in Theorem 2.5.1 when identities (2.22) and (2.23) hold where the RML and deposit reference process constants are given by (2.24) and (2.25), respectively. The SDE (2.26) follows from Theorem 2.5.1.

## 4.2 Optimal Liquidity Risk Management and the SMC

In this section, suggestions by the main results involving optimal marketable securities allocation and deposit inflow strategies are considered with connections to the SMC.

Before the SMC, ORs had little risk because of the favourable economic conditions during this period. Strong deposit inflows, high liquidity, low inflation and default rates as well as strong growth were evident in this period which fostered greater risk-taking. Thus, ORs transferred credit risks to the broader capital market by distributing their securitized RMLs to investors. ORs didn't bother to extend subprime RMLs, because mortgage losses and default risk were carried on to the investors. This enabled ORs to increase their mortgage extension. The bursting of the housing bubble caused interest rates, default rates on subprime RMLs and foreclosures to increase dramatically thereafter. Capital and deposit levels plummeted, which resulted in instability and insecurity of ORs and eventually insolvency and bankruptcy.

It is inefficient when an OR focusses only on size, liquidity and capitalization to accurately assess their ability and willingness to extend additional RMLs according to [2]. The subprime lending portfolio combined with higher risks had significantly grown during the SMC. The SMC made it clear that risk perception is crucial in determining the capability of ORs to extend RMLs and therefore it is important to determine an OR's access to capital to improve liquidity and mortgage supply. Before the SMC, credit standards were gradually lowered as ORs' absence of credit risk management increased. The worsening of risks during the SMC increased the expected capital requirements of ORs, but ORs managed to reduce the capital requirement needed and therefore more RMLs were extended. The eruption of the SMC significantly tightened the mortgage supply and increased the credit standards which resulted in a credit crunch that led to illiquidity of ORs. In short, the SMC showed that the funding of subprime RMLs depends on the OR's risk position, therefore it is very important to investigate the risks that influence the mortgage supply of ORs.

The paper [9] states that the understanding of credit risk of subprime lending portfolios is the most important for ORs and regulators. In particular, they focus specifically on the default correlation of credit risk analysis, because subprime lending portfolios are more sensitive to the volatility in default correlations than higher grade lending portfolios. Thus, managing risks is crucial for subprime RML funding.

#### **4.2.1 Stochastic Credit Portfolio Dynamics and the SMC**

In Section 3.1, we focus on the stochastic dynamics of marketable securities,  $B$ , represented by the SDE (3.1). By applying Theorem 2.5.1, we rewrite (3.1) to obtain the simplified equation (3.2). An OR's marketable securities structure and levels changed dramatically during the SMC. Subprime securities including RMBSs made out an increasing part of the marketable securities sector for larger earnings than Treasuries and other prime securities

portfolios. The declined value of RMBSs caused large losses for ORs. Thus, marketable securities had a great contribution to the SMC.

#### 4.2.2 Stochastic Optimal Control Problem and the SMC

The class of admissible control laws in Section 3.2 is given by (3.3) to obtain a well-defined objective function (3.4) that minimizes the combination of both credit crunch- and funding risk. This is done by modelling the liabilities of an OR and the allocation of assets. An OR's assets and liabilities were highly influenced by the SMC, as seen in the discussion of Subsubsections 4.1.1 and 4.1.2, respectively.

#### 4.2.3 Spread Method of RML Financing and the SMC

A spread method of RML financing in Section 3.3 exists when a rate of actualization,  $r^a$ , given in (3.5), exists. The equality (3.6) is required in order for the valuation of  $l$  to be insensitive to risk related to RML extension levels. Also, a spread method may be implemented when additional deposits are proportional to unfunded RMLs, i.e.  $D^a = kM^u$ , mentioned in Definition 3.3.1 earlier. The SMC also influenced the levels of additional deposits and unfunded RMLs, therefore, the existence and implementation of the spread method were also affected by this crisis.

#### 4.2.4 Main Results and the SMC

The transversality condition, (3.7), in Section 3.4 is used to overcome the difficulty when solving our optimal control problem.

The optimal risky marketable securities allocation,  $\tilde{\pi}^*$ , is represented by (3.9) and has a rate of impatience, (3.8). We notice that we can rewrite the optimal allocation of risky marketable securities represented by (3.9) in Theorem 3.4.1 as

$$\tilde{\pi}^* = C^{-1}(\tilde{b} - r^T \bar{1})M^u + \sigma^{ME} \sigma^{-T} \tilde{q} M^r. \quad (4.1)$$

The solution to the control problem's form implies a direct relationship between  $M^u$  and  $\tilde{\pi}^*$  with a proportional constant which depends on  $\sigma$ ,  $\tilde{b}$  and  $r^T \bar{1}$ . Also, the market price of risk to this proportionality can be introduced by (2.7). The term  $\sigma^l \sigma^{-T} \tilde{q} M^r$  in (4.1) suggests that the optimal allocation strategy of  $B$  depends on uncertain RML extension and  $M^r$ , how subprime RMLs and the returns on marketable securities correlate and constraints of

risks in the model, which were all components of the SMC. In this regard, the optimal allocation of marketable securities was greatly influenced by the SMC.

The following strategies regarding the allocation of  $B$  are discussed which follow from Theorem 3.4.1. These optimal strategies involve borrowing, short selling and overfinancing. In order to facilitate a discussion of the optimal securities allocation strategy, we set

$$J^i = \bar{e}^i C^{-1} \sigma \tilde{\phi} \text{ and } N^i = \sigma^{ME} \bar{e}^i \sigma^{-T} \tilde{q}.$$

**Optimal Allocation Strategy for Marketable Securities (Borrowings):** We notice that money is borrowed by the OR to invest in the risky marketable securities  $i$ , at a rate  $r^{\tilde{B}}$  via the optimal solution (3.9) in Theorem 3.4.1. This implies  $\pi^i > B^*$  when the fund levels of securities is below

$$\left\{ \frac{J^i + N^i}{1 + J^i} \right\} M^r.$$

**Optimal Allocation Strategy for Marketable Securities (Short Selling):** Short selling occurs when the OR expects securities prices to decrease. The OR then sells this securities at a price with the intension to buy it back at a lesser price of its current value in the future. When the marketable securities  $i$  is above the value

$$\left\{ 1 + \frac{N^i}{J^i} \right\} M^r,$$

then the short position in securities  $i$  will be taken by the OR. Thus, we conclude that  $(\pi^i)^* < 0$ .

**Optimal Allocation Strategy for Marketable Securities (Overfinancing):** When overfinancing happens, i.e.,  $B^* > M^r$ , the optimal strategy is still to borrow money to invest in stocks. For this strategy to be reasonable, there should be a positive correlation between the Wiener processes  $W^l$  and  $W^i$ ,  $i = 1, 2, \dots, n$ . Suppose that the fund is invested in just one class of securities and

$$\frac{q\sigma^l}{\sigma} > 1,$$

then the following

$$\frac{J^i + N^i}{1 + J^i} > 1 \text{ and the inequality } M^r < B^* < \left\{ \frac{J^i + N^i}{1 + J^i} \right\} M^r$$

might be true which suggest that borrowing could occur, even where overfinancing occurs, to invest in risky marketable securities. This is a consequence of a higher instantaneous volatility of the stipulated level of reserves and subprime RMLs to be extended,  $e$ , regarding a positive correlation between Wiener processes as well as  $W^i$ ,  $i = 1, 2, \dots, n$ , for the risky marketable securities. Therefore,

$$\mathbf{E}_{B_0, M^{r_0}} B_t^* - \mathbf{E}_{B_0, M^{r_0}} M_t^r = (B_0 - M^{r^T}) \exp\left\{ (r^T - \tilde{\xi}^T \tilde{\xi} - \frac{\delta^{aa}}{\kappa}) t \right\},$$

which follows from the proof of Theorem 3.4.1 and a discussion in [4]. This implies that if

$$\delta^{aa} > \kappa(r^T - \tilde{\xi}^T \tilde{\xi}) \quad (4.2)$$

then  $\mathbf{E}_{B_0, M^{r_0}} B_t^* - \mathbf{E}_{B_0, M^{r_0}} M_t^r$  converges to zero when time approaches infinity. We also notice that inequality (4.2) is automatically satisfied if  $r^T \geq r^i + \tilde{\xi}^T \tilde{\xi}$ . On the other hand, where  $r^T < r^i + \tilde{\xi}^T \tilde{\xi}$ , inequality (4.2) becomes

$$\kappa < \frac{1}{1 + r^T(r^i + \tilde{\xi}^T \tilde{\xi} - r^T)}. \quad (4.3)$$

Since  $\tilde{\xi}^T \tilde{\xi} > 0$ , we conclude that the inequality in (4.3) is more restrictive than in the case where investments are made in fixed rent only.

The optimal deposit inflow rate is given by (3.10). The optimal allocation of marketable securities and deposit inflow rates are associated with risk management which is very important during a crisis period. An OR's risks play an important role during a crises, as discussed earlier. Since (2.18) holds, the additional deposits,  $D^a$ , is directly proportional to the unfunded RML extension,  $M^u$ , which results from (3.10) in the statement of Theorem 3.4.1. Here, the proportionality constant is  $\frac{\delta^{aa}}{\kappa}$  that contains information about deposit and equity risk factors which is consistent with the spread method of RML financing. For

the deposit inflow rate,  $c$ , and the deposit reference process,  $D^r$ , it follows that

$$\mathbf{E}_{c_0 D^r 0} c_t^* - \mathbf{E}_{c_0 D^r 0} D_t^r$$

behave the same when time tends to infinity.

In short, we proof Theorem 3.4.1 as follows: The Hamilton-Jacobi-Bellman equation (3.12) represents the optimal control problem 3.2.1 for the OR's risk management with optimal deposit inflow rate (3.14). The equation (3.15) represents the optimal asset allocation strategy. Substituting the optimal deposit inflow rate and optimal asset allocation in (3.12), results in (3.16). The form of (3.16) suggests that equation (3.17) is a quadratic solution for the Hamilton-Jacobi-Bellman equation (3.12). In this regard,  $\delta^{aa}$ ,  $\delta^{M^r M^r}$  and  $\delta^{a M^r}$ , are solutions of (3.18), (3.19) and (3.20). The equation (3.21) implies that the additional deposits,  $D^a$ , are directly proportional to the unfunded subprime RMLs,  $M^u$ , if and only if (3.22) holds. Furthermore, condition (3.23) should hold in order to obtain solutions for (3.18) and (3.20). From arguments above, it follows that the value function,  $V$ , is always non-negative. Finally, we conclude that the transversality condition (3.7) holds. We find equation (3.25) as a solution to linear differential equations  $f'_t$  and  $g'_t$  by using the notation given in (3.24). Also, a positive solution of (3.11) implies the inequality (3.26) which holds if inequalities (3.27) and (3.28) both hold.

#### 4.2.5 Examples Involving Subprime RML Financing and the SMC

In Section 3.5, we computed solutions for an OR's deposits, marketable securities, Treasuries and RMLs as well as solutions for the RML and deposit reference processes by means of the numeric Euler-Maruyama method. An OR's deposits increased dramatically, as seen in Figure 3.1(a), as large amounts of deposits flowed into the U.S. which funded RMLs in a low interest rate environment. This spurred increases in RML extension and house prices. The securitization of subprime RMLs offered higher returns than standard RMLs, therefore, more and more subprime mortgages were securitized for higher returns which funded new subprime mortgages. That explains the dramatic increase in OR's subprime RMLs and marketable securities as seen in Figures 3.1(d) and 3.1(b), respectively. The SMC affected the mortgage supply of RMLs and credit conditions were tightened. Marketable securities took a downturn when RMLs started to default that caused the value of these securities to decrease. Depositors withdrew their funds and there were no new potential depositors, as illustrated in Figure 3.1(a), which resulted in failure of many ORs. Treasuries are riskless



with very stable Treasury rates that is shown in Figure 3.1(c). More Treasuries were bought after the collapse of the financial market for more stable and riskless securities returns.

An OR's RML and deposit reference processes are shown in Figures 3.1(e) and 3.1(f), respectively.



## Chapter 5

# CONCLUSIONS AND FUTURE DIRECTIONS

”A basic cause of the current financial crisis was the mandate by the U.S. Congress for Fannie Mae to vastly increase its support of low-income housing. This mandate required a lowering of lending standards. These lower standards encouraged people with relatively high incomes to buy more expensive houses than they otherwise would have or to buy speculative second homes with the option of walking away from them if house prices fell. The problem was aggravated by novel, obscure, highly leveraged financial instruments that were not well understood by the companies that used them. These instruments caused an information crisis in which parties refused to enter into transactions with each other whenever doing so involved counterparty risk because no one knew who held bad paper. Part of the cure for the current crisis - which would also remove one potential cause of future crises - is for Congress to stop pressuring Fannie Mae to acquire mortgages with insufficient borrowing standards. On the contrary, any mortgages that Fannie Mae purchases should meet solid, traditional down-payment and documentation requirements. Inducing families to buy houses they could not afford did not benefit them, the U.S. and international financial systems, or the world economy.”

– Prof. Harry Markowitz, University of California, 2010.

In this chapter, we present a few concluding remarks and highlight some possible topics for future research.

## 5.1 Conclusions

ORs securitized RMLs as an incentive to reduce risks and to reduce their capital requirements by moving it off their balance sheet. The lower capital requirements increased their liquidity and as a result, more capital was available to extend new RMLs. Earnings from marketable securities as well as deposit inflows funded RMLs to further increase the OR's mortgage supply. This way of RML extension and the urge for more yield were the main contributing factors of the current SMC. ORs didn't have adequate liquidity risk management structures in place. Therefore, the collapse of the house market and explosion of default rates, caused insolvency and failure of many ORs as a consequence of inefficient capital reserves. The SMC made it clear that RML funding depends on an OR's risk position. Therefore, it is crucial to investigate the risks of an OR. Credit and liquidity risk should be the main focus when dealing with the subprime market, because subprime lending portfolios are more sensitive to defaults than other portfolios. Also, risks caused by liquidity should be recognized when measuring the risk of an OR.

In order to obtain the desirable level of RML extension, optimal portfolio choices should be made regarding deposit inflow rates and marketable securities allocation given by Theorem 3.4.1. We conclude that there is a positive correlation between an OR's deposits, marketable securities and the extension of subprime RMLs. In this regard, the relationship between the RML and deposit reference processes as well as the amount of subprime RML extension is given by Theorem 2.5.1. Also, the unfunded RMLs are directly proportional to additional deposits that is given in Theorem 3.4.1. As a result, this relationships were used in order to construct the stochastic continuous-time model to determine a solution for this optimal control problem. Optimal marketable securities allocation and deposit inflow rates are obtained by means of the RML and deposit reference processes to ensure an OR's stability and security. According to this, a spread method of RML financing is imposed with an existence condition given by Lemma 3.3.2.

## 5.2 Future Directions

In future, instead of constructing a continuous-time stochastic model to determine the optimal control problem, we would like to construct a more sophisticated model e.g. a jump-diffusion process. We would like to include the cyclicity and prudential behavior of assets as well as the OR's financial stability in our model. In this case, a future study to provide a better understanding about relationships and linkages between an OR's risks is also very important for risk management. Also, we would like to remove assumptions made

by collecting and studying data in order to prove the assumptions.



## Chapter 6

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