Potchefstroomse Universiteit
vir Christelike Hoër Onderwys

Ontstaan 1869
50 Jaar selfstandig 2001

WETENSKAPLIKE BYDRAES
REEKS H: INOUGURELE REDE NR. 162

OM DIE AARDE TE MEET /
TO MEASURE THE EARTH

Prof ML Baart

Inougurele rede gehou op 25 Augustus 2000

Publikasiebeheerkomitee
Potchefstroomse Universiteit vir Christelike Hoër Onderwys
Potchefstroom
2520

Navrae in verband met *Wetenskaplike Bydraes* moet gerig word aan:

Die Publikasiebeheerkomitee
Potchefstroomse Universiteit vir Christelike Hoër Onderwys
2520 POTCHEFSTROOM

ISBN 1-86822-376-0

Om Die Aarde Te Meet

Louisa Baart

25 Augustus 2000

Met dank aan Robin McLeod, vir wie "having fun" en "being confused" die twee normale geestestoestande van meetkundiges is.

1 Inleiding

Meetkunde, oftewel die kennis van hoe om die aarde te meet (geometrie: *geo-* ≡ aard- en *metron* ≡ 'n maat) is die oudste vertakking van die wiskundige wetenskappe. Belangstelling in hierdie, tot 'n groot mate verlore, kuns is weer aan die opflikker sedert die koms van rekenaars wat grafiese vermoëns het en dus 'n kennis van berekeningsmeetkunde noodsaak. Ou vergete handboeke oor sintetiese meetkunde word in stowwerige biblioteekargiewe ontdek, en herdruk, herskryf of vertaal om hulle toeganklik te maak vir die moderne gebruiker. Meetkundige probleme word ingesluit in wiskunde-olimpiades. Nuwe teksboeke verskyn, konferensies oor meetkunde en die toepassings van meetkunde word gehou, en kursusse oor verskillende vertakkings van meetkunde word aangebied op tersiêre vlak — maar nie op enige noemenswaardige skaal in Suid-Afrika nie.

Kom ons bestee 'n tydjie daaraan om deur die eeue te loop in die spore van meetkunde, om die plek van meetkunde in moderne wiskunde te probeer evalueer, en om te bespiegel oor 'n toekoms met of sonder meetkunde.

2 Van Die Antieke Tot Die Moderne

2.1 Voor Die Grieke

Die geboorte van meetkunde was verlos van al die gesukkel met stellings en bewyse. Die geleerdes van die ou beskawings in Babilon, Egipte en China het, onder andere, geweet hoe om lengtes en afstande te vergelyk, en wat 'n regte hoek is. Hulle was inderdaad gemoeid met die meet van die aarde op 'n baie praktiese (toegepaste) vlak. Sommige van die bestaande kleitablette en papirusrolle met meetkundige inligting dateer vanaf ongeveer twintig eeue v.C. en bevat resepte vir die berekening van oppervlaktes en volumes van sommige reëlmatige voorwerpe. Die sirkel is reeds verdeel in 360 segmente (waarskynlik na aanleiding van die lengte van die Babiloniese jaar), en sommige van die drietalle wat later (ca. 500 v.C.) aan Pythagoras gekoppel sou word was reeds bekend, en is in driehoeke gebruik ten einde regte hoeke te konstrueer. Inderdaad, die Babeloniërs het Pythagoras se stelling geken lank voor sy geboorte, en die bouers van Salomo se tempel het geweet dat die omtrek van 'n sirkel ongeveer drie maal solank as sy deursnee is. Daar was egter weinige, indien enige, poging om bestaande kennis te klassifiseer en stelselmatig uit te brei.

2.2 Die Immergroen Euklides

Die Grieke het die wiskundige onwikkeling oorheers vir ongeveer 'n millennium (ca. 500 v.C. tot 500 n.C.). Een van die gevolge van die uitbreiding van die Griekse ryk was die totstandkoming van die universiteit in Alexandrië, met sy biblioteek en museum, wat vir ongeveer drie eeue (vanaf ca. 300 v.C.) die middelpunt van wetenskaplike ontwikkeling was. Dit is nie seker wanneer Euklides hier sy "Elemente", bestaande uit dertien boekdele, die lig laat sien het nie, maar vermoedelik was dit aan die begin van hierdie tydperk. Euklides het die reusagtige taak onderneem om die bestaande wiskundige kennis, veral van sintetiese meetkunde, te versamel, te sif, te orden en, waar nodig, bewyse te verskaf van stellings. Hierdie werk het meetkunde verhef tot die status van eerste deduktiewe wetenskap, en dit sou ongeveer twintig eeue duur voordat algebra en analise op 'n soortgelyke manier behandel kon word. Hoe goed Euklides in sy taak geslaag het kan ons oordeel aan die feit dat Euklidiese meetkunde vandag nog 'n inherente deel van wiskundige onderrig op skoolvlak uitmaak. Een van die besware wat teen Euklides se werk ingebring kan word is die gebruik van ongedefinieerde terme en begrippe.

Waarom speel definisies so 'n oorheersende rol in vak soos meetkunde? Definisies van konsepte stel ons in staat om minstens te weet dat ons oor dieselfde goed praat. Dink maar aan die Babelse verwarring hier ter plaatse as gevolg van onvolledige definisies rakende basiese begrippe in verband met herkurrikulering, of speel gou "vish" (vir *vicious circle*, volgens die uitdinker J.L. Synge). Dit werk so: kies 'n woord, soek sy definisie op in 'n verklarende woordeboek, en soek dan die betekenis(se) van die woorde in die definisie, en so aan, totdat 'n sirkel voltooi is — 'n woord is gedefinieer in terme van homself. Hoe klink hierdie een:

reguit — in 'n regte lyn; nie krom nie (regte lyn: ongedefinieer)
krom — nie **reguit** nie

"Vish" in een stap! En hier is 'n mooi Engelse een:

point — that which has no dimension
dimension — number of mutually orthogonal directions
direction — in straight line away from **point**

Dis darem nie altyd so erg nie, maar dit illustreer die feit dat aannames noodsaaklik is om te verhoed dat ons al in die rondte redeneer. Euklides het begin met sewe inherent ongedefinieerde begrippe (*point — that which has no part*, ens.), 'n groep daaropvolgende definisies, vyf aksiomas of vanselfsprekende (universele) waarhede, en vyf sogenaamde postulate, wat vandag bekend stäan as Euklides se aksiomas. Hieruit is, sistematies en logies, die stellings van Euklidiese meetkunde afgelei. Kom ons aanvaar voorlopig dat ons alles weet van punte en lyne, dat ons weet hoe om afstande en hoeke te meet, en dat ons weet wat dit beteken as 'n punt en 'n lyn insideer (die punt is op die lyn, of die lyn gaan deur die punt). Dan kan ons Euklides se vyf aksiomas as volg opsom.

1. Twee verskillende punte bepaal 'n segment van 'n (unieke) reguit lyn.

2. Hierdie segment kan onbepaald (reguit) voortgesit word aan albei endpunte.

3. 'n Sirkel met 'n gegewe middelpunt en afstand (straal) kan gekonstrueer word.

4. Alle regte hoeke is gelyk.

5. As twee reguit lyne 'n gemeenskaplike reguit snylyn het, sodanig dat die som van die twee binnehoeke aan een kant van die snylyn kleiner is as twee regte hoeke, dan sal die twee lyne ontmoet aan daardie selfde kant van die snylyn.

Hierdie laaste aksioma is die beroemde (of dalk berugte?) *vyfde aksioma van Euklides*, wat die meetkundige wêreld vir amper twee millennia aan die gons gehad het. Euklides self het sy eerste 28 proposisies (stellings) bewys sonder om die vyfde aksioma te gebruik. Vir eeue het meetkundiges probeer om óf te bewys dat dit afgelei kan word van die ander vier (en dus 'n stelling is, en nie 'n aksioma nie), óf dat dit nie waar is nie (teenstrydig is met een of meer van die ander vier). In dié proses is wel vele alternatiewe formulerings vir die vyfde aksioma ontdek, maar nie een van die twee denkrigtings kon hond haaraf maak nie, en ons moet wag tot die negentiende eeu vir die beslegting van hierdie besondere geskil.

2.3 Twintig Eeue Later

Die skatkis van Euklidiese meetkunde was voldoende om meetkundiges redelik gelukkig te hou tot ongeveer aan die begin van die sewentiende eeu. In 1637 het die Franse filosoof René Descartes, sommer as so 'n soort nagedagte in 'n aanhangsel van sy boek oor suiwere denke, die idee genoem om die posisie van 'n punt in 'n vlak, relatief tot twee vaste ortogonale asse, te beskryf met behulp van (x, y)-koördinate, oftewel Cartesiese koördinate — dié keer is die regte naam darem op die regte plek — en wiskunde was nooit weer dieselfde nie. Analitiese meetkunde is gebore, en die verband tussen punte op 'n vlakkromme kon nou beskryf word deur middel van vergelykings soos $x^2 + y^2 = 1$ of $xy = 1$. Nog 'n Fransman Pierre de Fermat het amper terselfdertyd ontdek hoe nuttig koördinate kan wees. Terloops, die Franse is nog steeds onder die beste meetkundiges. Een van die eerste toepassings van analitiese meetkunde was die berekening van raaklyne aan krommes by gekose punte, en analise sien die eerste lig. Hierdie suigeling het spoedig die dominante rol van meetkunde in onderrig oorgeneem — geen eerstejaarkursus in wiskunde is sonder 'n goeie dosis analise nie, terwyl dit lyk asof die meeste tersiêre inrigtings heel goed sonder meetkunde kan klaarkom.

Alhoewel hierdie ryk nuwe dimensie in wiskunde die middelpunt van belangstelling was vir die volgende klompie dekades, het die nuuskierigheid oor die raaiselagtighede in Euklidiese meetkunde nooit heeltemal doodgeloop nie. Nie-Euklidiese meetkundige resultate is reeds ontdek en gebruik deur Pappus — ook van Alexandrië — gedurende die vierde eeu, en in die vroeë sewentiende eeu het die idee van "punte by oneindig" afsonderlik opgekom by Johann Kepler en Girard Desargues, maar dit was eers twee eeue later dat die geredekawel oor Euklides se vyfde aksioma op 'n verrassende wyse tot 'n punt gekom het. Aangesien dit blykbaar nòg teenstrydig nòg afhanklik was, kon dit dalk vervang word met iets anders? Selfs daarsonder, net met die eerste vier aksiomas, kan 'n mens (soos Euklides self) al 'n hele end vorder — terloops, hierdie meetkunde staan bekend as *absolute meetkunde* of, deesdae, as *neutrale meetkunde*.

Een van die bekendste alternatiewe formulerings van die vyfde aksioma is *Playfair se aksioma*: deur 'n punt nie op 'n gegewe lyn nie is daar ...

> ... **presies een** lyn wat geen punt in gemeen het met die gegewe lyn nie,

oftewel, wat *ewewydig* is aan die gegewe lyn. Die twee vanselfsprekende kandidate vir die posisie van vyfde aksioma is: deur 'n punt nie op 'n gegewe lyn nie is daar ...

> ... **geen** lyn deur die punt en ewewydig aan die gegewe lyn nie,
> ... **meer as een** lyn deur die punt en ewewydig aan die gegewe lyn.

Die meetkundes wat ontstaan uit hierdie alternatiewe stelsels aksiomas is *nie-Euklidies* en staan onderskeidelik bekend as *elliptiese* en *hiperboliese* meetkunde, terwyl Euklidiese meetkunde dan ooreenkomstig *paraboliese* meetkunde genoem kan word, en elkeen het sy plekkie in die son.

Die vyfde aksioma van elliptiese meetkunde, naamlik dat twee verskillende lyne altyd 'n gemeenskaplike punt het, pas baie mooi by die eerste Euklidiese aksioma, dat twee verskillende punte altyd 'n gemeenskaplike lyn het. Na alles, wat maak 'n "punt" meer spesiaal as 'n "lyn", veral aangesien nie een van die twee regtig gedefinieer is nie? As jy van Beaufort-Wes na Laingsburg ry lyk dit in elk geval asof die pad vorentoe al nouer word, en as jy op jou bed lê en die vloerteëls bekyk lyk hulle nie meer vierkantig nie — ons leef nie altyd in 'n Euklidiese wêreld nie. Dit is dus nie so vreemd nie om te glo dat ewewydige lyne ook êrens ontmoet — ons kan net nie ver genoeg sien nie. As ons nou al hierdie onsigbare punte-by-oneindig — een vir elke rigting in 'n vlak — byvoeg by die gewone (Euklidiese) punte, en hulle saam op een lyn-by-oneindig laat lê, en daarna demokraties verklaar dat daar nie teen hierdie snaakse punte en hulle lyn gediskrimineer mag word nie, het ons 'n voorbeeld van 'n *projektiewe* vlak. Let op dat afstande, hoeke en oppervlaktes nie meer enige betekenis het sodra berekenings met oneindig gedoen mag word nie.

Na 'n redelik chaotiese tydperk gedurende die laat agtiende en vroeë negentiende eeue waarin meetkundes by die dosyne geskep en ondersoek is, is mettertyd orde geskep. Met die 23-jarige Felix Klein (van Klein-bottel faam) se intreerede van 1872, bekend as die *Erlanger Programme*, bereik ons die tydperk van **moderne meetkunde**. Meetkundes is nou volledig geklassifiseer, en dit kon aangetoon word dat, om Arthur Caley aan te haal: " ... projektiewe meetkunde is alle meetkunde."

Gebruik 'n mens ooit al die snaakse meetkundes? Ja — meetkunde op 'n sfeer is ellipties, en Albert Einstein sou nooit sy relatiwiteitsteorie kon ontwikkel het sonder die idees van nie-Euklidiese meetkunde nie (sy meetkunde is baie meer ingewikkeld as ellipties of hiperbolies). Projektiewe meetkunde vorm die wêreld waarin rekenaar-prentjies geskep word. En 'n mens kan baie pret hê met allerhande soorte meetkunde — kom ons speel 'n bietjie!

3 Pret Met Meetkunde

Gestel ons het 'n ballasmandjie vol springtoue en ringtennis ringe, en die speletjie is dat jy die toue en ringe moet gebruik om 'n konstruksie te voltooi wat die volgende reëls gehoorsaam:

1. Begin deur 'n tou te neem.
2. Jy moet presies drie ringe aan elke tou vasmaak.
3. Wanneer jy die drie ring wat vas is aan 'n tou vashou, moet daar minstens een ring wees wat nie aan daardie tou vas is nie.
4. As jy enige twee ringe vashou, moet daar presies een (nie meer nie, en ook nie minder nie) tou vas wees aan albei hierdie ringe.
5. As jy enige twee toue vashou, moet daar ten minste een ring wees wat vas is aan albei.

Hoeveel ringe en toue het jy gebruik?

'n Kollega van my het hierdie speletjie met graad 5/6 kinders gespeel. Dit was vir hulle groot pret, en heeltemal doenbaar, en hulle was salig onbewus daarvan dat hulle die kleinste voorbeeld van 'n eindige projektiewe meetkunde (of Fano meetkunde) opgebou het. Hulle kon selfs gevolgtrekkings maak (stellings bewys), soos dat die "ten minste" in die laaste reël eintlik ook maar 'n "presies" kan wees, omdat 'n mens anders die voorlaaste reël verbreek. Met 'n bietjie hulp het hulle ook opgelet dat, indien die woorde ring en tou in al die reëls omgeruil word, die nuwe konstruksie wat daaruit volg identies is aan die vorige. Kom ons herskryf nou die reëls van die speletjie soos dit vir 'n derdejaarklas in moderne meetkunde sal lyk (onthou "insideer met" is maar net 'n onpartydige manier om te sê "gaan deur" of "lê op"):

1. Daar bestaan 'n lyn.
2. Elke lyn insideer met presies drie punte.
3. Nie alle punte insideer met een lyn nie.
4. Twee verskillende punte insideer met presies een lyn.
5. Twee verskillende lyne insideer met minstens een punt.

Hoeveel punte en lyne is daar?

Die normale reaksie is iets soos: "Natuurlik bestaan daar 'n lyn! Dis seker 'n grap — dit is klaarblyklik onsin om te eis dat daar net drie punte op 'n lyn is." Net omdat ons **dink** ons weet wat met punt en lyn bedoel word, is dit meteens nie meer 'n spel nie, maar moeilike wiskunde.

Kom ons noem net, vir die lekkerte, 'n paar van die stellings wat in hierdie klein meetkunde geld. Daar is presies sewe punte en sewe lyne. Die woorde "punt" en "lyn" kan omgeruil word sonder om enige reël te verbreek (ons sê die meetkunde is *duaal*), wat beteken dat ons met die bewys van enige stelling, onmiddellik nog een present kry. Elke punt insideer dus met presies drie lyne. En vir enige driehoek (definisie: drie lyne sonder 'n gemeenskaplike punt) is daar presies een punt wat nie met die driehoek insideer nie. 'n Hele dubbelperiode se werk, en dit met net sewe punte en lyne!

Die Fano-meetkunde hier bo is nie die kleinste eindige meetkunde nie — daar is 'n duale meetkunde met slegs drie punte en drie lyne, 'n nie-duale een met vier punte en ses lyne, en sy maatjie met ses punte en vier lyne. Kan hierdie snaakse eindige meetkundes enige nut hê? Weereens is die antwoord ja: hulle word onder andere gebruik vir foutkorrigeerkodes vir rekenaardata.

4 En Wat Nou?

Met rasionalisasie wat steeds 'n groter rol speel in onderrig op alle vlakke, is daar 'n drastiese afname in die meetkundige inhoud van wiskundeleerplanne te bespeur. Terwyl op skoolvlak die tradisionale Euklidiese meetkunde nog vasskop — en genadiglik is daar tekens te bespeur dat die belang van stelling–bewys–toepassing in die ontwikkeling van kognitiewe vermoëns nie oral verwerp word wanneer uitkomste die doelwit word nie — word meetkunde selde op tersiêre vlak as volwaardige kursusse aangebied, en dan feitlik altyd onder die vaandel van wiskunde. In toegepaste wiskunde, asook in rekenaarwetenskap, is daar gewoonlik geen meetkundige inhoud nie. En indien daar wel in wiskunde so iets aangebied word, is dit gewoonlik nie gedurende die eerste twee studiejare nie, wat beteken dat studente in rekenaargrafika meesal geen blootstelling kry aan projektiewe meetkunde nie, aangesien baie min van hulle vir meer as een jaar wiskunde neem. Omgekeerd, studente van moderne meetkunde is baie selde bewus van toepassings in rekenaargrafika, en nie in staat om met gemak tussen teorie en praktyk te beweeg nie.

'n Verdere probleem is dat studente dit besonder moeilik vind om, selfs op derdejaarsvlak of nagraads, 'n kursus in meetkunde suksesvol te voltooi. Waarom? Dit is algemeen bekend onder wiskunde-dosente dat studente nie van "woordsomme" hou nie, en moderne meetkunde is meer taalgebonde as die meeste ander vertakkings van die wiskunde. Prof. T. Erasmus, Viserektor, Universiteit van Pretoria, sê in die Des. 1999 uitgawe van die *S.A. Tydskrif vir Natuurwetenskappe en Tegnologie* 18, dat " — slegs ongeveer 30 % van die studente wat wel matrikulasievrystelling het, oor voldoende taalvaardigheid beskik om 'n eerstejaarslesing te kan volg en 'n skriftelike werkstuk te kan voorberei". Ons kan miskien hier byvoeg: " — en waarskynlik 'n begripstoets oor hulle teksboeke sal kan slaag". Dit is dus nie vreemd dat meetkunde met sy baie woorde as "te moeilik" afgemaak word nie. Moet ons studente nie liewer aanmoedig om dit te sien as 'n uitstekende geleentheid om hulle logiese en taalvaardige vermoëns te verbeter, wat tot voordeel strek van enige ander kursusse wat hulle aanpak, en waardeur hulle terselfdertyd vertroud raak met die streng dissipline van abstrakte denkpatrone. Dan sal hulle nie praat, soos een van myne 'n paar jaar gelede, van "twee punte wat ewe ver van mekaar af is" nie — en hy het doodgewone punte bedoel, nie nuwe esoteries-gedefinieerde punte nie. Een van die beste leermeesters vir logiese denke is ook die oudste: ons moet leer, en ons studente leer, hoe om die aarde te meet, sodat ons nie gemeet en te kort bevind word nie.

To Measure The Earth

Louisa Baart

August 25, 2000

With thanks to Robin McLeod, for whom "having fun" and "being confused" are the two normal states of mind of geometers.

1 Introduction

Geometry, or the knowledge of how to measure the earth (*geo* ≡ earth, and *metron* ≡ a measure) is the oldest branch of the mathematical sciences. Interest in this "lost art" has been on the increase since the advent of computer graphics and the corresponding requirement for programmers with expertise in computational geometry. Old forgotten textbooks on synthetic geometry are being unearthed from dusty library archives, and reprinted, rewritten, or translated in order to make them accessible to the modern user. Problems of geometrical nature are included in mathematics olympiads. New textbooks are appearing, conferences on geometry and the applications of geometry are being organized, and courses on different branches of geometry are offered at tertiary level — but nothing to speak of in South Africa.

Let us spend a little time following the path of geometry through the ages, as well as in trying to evaluate the place of geometry in modern mathematics, and in speculating on a future with or without geometry.

2 From The Antique To The Modern

2.1 Before The Greeks

The birth of geometry was free of all the enforced rigour of theorems and proofs. The scholars of the ancient civilizations in Babylon, Egypt and Greece knew, among other things, how to compare lengths and distances, and the meaning of a right angle. They were involved in measuring the earth on a very practical (applied) level. Some of the existing tablets and scrolls with geometrical content date from approximately twenty centuries B.C. and contain recipes for the calculation of areas and volumes of various regular objects. The circle had already been subdivided into 360 segments (probably as a result of the length of the Babylonian year), and some of the number triples later (ca. 500 B.C.) ascribed to Pythagoras were already being used in the construction of right-angled triangles. In fact, the Babylonians knew Pythagoras's theorem long before his birth. They also knew that the circumference of a circle was approximately thrice the length of its diameter (cf. I Kings 7:23). However, there was little or no effort to classify and expand the existing knowledge in a systematic way.

2.2 The Evergreen Euclid

The Greeks dominated mathematical development for approximately a millennium (ca. 500 B.C. to 500 A.D.). One of the results of the expansion of the Greek empire was the establishment of the university of Alexandria, with its library and museum. This university was, for approximately three centuries (ca. 300 B.C. to 30 A.D.) the hub of scientific development. We don't know exactly when Euclid completed his famous "Elements" consisting of thirteen parts, but it is generally considered to have been at the beginning of this period. Euclid undertook the immense task of collecting the existing knowledge, in particular with respect to synthetic geometry, of sorting, ordering and, where necessary, providing proofs of theorems. His work elevated geometry to the status of first deductive science, and it would take algebra and analysis approximately twenty centuries to achieve the same status. We can perhaps judge the excellence of his work from the fact that even today Euclidean geometry still forms an intrinsic part of school mathematics. One of the major points of dispute about Euclid's work is his use of undefined terms and concepts.

Why do definitions play such a dominant role in subjects such as geometry? Clear definitions of concepts enable us to know that, at least, we are talking about the same things. Just think about the confusion that reigned here recently as a result of incomplete definitions related to basic concepts with respect to our curricula planning. Let's quickly play a game called "vish" (for vicious circle) by its inventor J.L. Synge. It goes like this: select a word, look up its meaning in an explanatory dictionary, and then look up the meaning(s) of the words in the definition, and so on, until a circle has been completed — a word is defined in terms of itself. Here is a nice one:

> **point** — that which has no dimension
> dimension — number of mutually orthogonal directions
> direction — in straight line away from **point**

Thankfully it is not always this bad, but it illustrates the fact that assumptions are necessary to prevent us arguing in circles. Euclid began with seven inherently undefined concepts (*point — that which has no part*, etc.), a set of definitions following from these, five axioms or "common notions" (universal truths), and five so-called postulates, these days commonly referred to as Euclid's axioms. From these were derived, systematically and logically, the propositions (theorems) of Euclidean geometry. Let us assume for the time being that we know everything about points and lines, that we know how to measure distances and angles, and that we know what it means when a point is incident with a line or vice versa (the point is on the line, or the line goes through the point). Then we can summarize Euclid's five axioms as follows.

1. Two distinct points determine a (unique) straight line segment.
2. This segment can be extended indefinitely in a straight line from either extremity.
3. A circle with a given distance (radius) and centre can be constructed.
4. All right angles are equal to one another.
5. If two straight lines have a common intersecting straight line, such that the two interior angles on one side of the intersecting line adds up to less than two right angles, then the two lines will meet on that same side of the intersecting line.

This last axiom is the famous (or perhaps infamous?) *fifth axiom of Euclid*, that kept the geometrical world buzzing for almost two millennia. Euclid himself proved his first 28 propositions (theorems) without using the fifth axiom. For centuries geometers (and others) tried either to derive it from the other four axioms (thus proving that it is a theorem and not an axiom) or that it is inconsistent (contradicts one or more of the other four axioms). In this process many alternative formulations of the fifth axiom were discovered, but neither of the two opposing factions made much headway, and we have to wait until the nineteenth century for the arbitration of this particular dispute.

2.3 Twenty Centuries After

The treasure chest of Euclidean geometry was sufficient to keep geometers happily occupied until the beginning of the seventeenth century. In 1637 the French philosopher René Descartes added, almost as an afterthought, an appendix to his book on pure thought in which he suggested describing the position of a point in a plane relative to two fixed orthogonal axes by means of (x, y) coordinates, that is, Cartesian coordinates — for once a correct name association — and mathematics was changed forever. Analytic geometry was born, and the relationship among points on planar curves could now be described by means of equations such as $x^2 + y^2 = 1$ or $xy = 1$. Another Frenchman Pierre de Fermat discovered the usefulness of coordinates almost simultaneously with Descartes. By the way, the French are still among the top geometers. One of the first applications of analytic geometry was the calculation of tangent lines to curves at selected points, and calculus arrived on the scene. This juvenile soon took over the dominant role of geometry in education — no first year course in mathematics is complete without an appreciable calculus content, while it seems as if most tertiary institutions can manage quite well without geometry.

Although this rich new dimension in mathematics was the centre of interest for the next few decades, the curiosity over the puzzling aspects of Euclidean geometry never really disappeared completely. Non-Euclidean results had already been discovered and used by Pappus — also of Alexandria — during the fourth century, and the idea of "points at infinity" had occurred independently to both Johann Kepler and Girard Desargues, but it took another two centuries before the controversy over Euclid's fifth axiom was settled in a surprising way. Since it was apparently neither inconsistent nor dependent, could it conceivably be replaced by something else? Even without it, using only the first four axioms, we can make quite a bit of progress (like Euclid himself) — by the way, this geometry is known as *absolute geometry* or, these days, as *neutral geometry*.

One of the best known alternative formulations of the fifth axiom is *Playfair's axiom*: through a point not on a given line there is ...

> ... **exactly one** line with no point in common with the given line,

that is, *parallel* to the given line. The two obvious candidates for the position of fifth axiom is: through a point not on a given line there is ...

> ... **no** line through the point and parallel to the given line,
> ... **more than one** line through the point and parallel to the given line.

The geometries that arise from these alternative sets of axioms are *non-Euclidean* and are known, respectively, as *elliptic* and *hyperbolic* geometry, so that Euclidean geometry can correspondingly be called *parabolic* geometry, and each of these geometries has its own niche.

The fifth axiom of elliptic geometry, namely that two distinct lines always have a common point, fits in very nicely with the first Euclidean axiom, namely that two distinct points always have a common line. After all, why should a "point" be more special than a "line", in particular since neither has been properly defined. When you travel from Beaufort West to Laingsburg it looks as if the road is getting narrower in the distance, and if you consider floor tiles from a prone position they do not appear to be rectangular at all — we do not always live in an

Euclidean world. Therefore, it is not all that strange to believe that parallel lines also meet somewhere — the problem is rather that we cannot see far enough. If we now add all these invisible points-at-infinity — one for each direction in a plane — to the ordinary (Euclidean) points, and stipulate that they all lie on one line-at-infinity, and then declare democratically that no discrimination against these strange points and their line will be allowed, we have an example of a *projective* plane. Note that distances, angles and areas can have no meaning when we are allowed to do calculations with infinity.

After a rather chaotic period during the late eighteenth and early nineteenth century, during which geometries were defined and investigated seemingly for the fun of it, order was gradually imposed. With the inaugural speech at Erlangen of Felix Klein (of Klein bottle fame) in 1872 at the age of 23, we enter the period of *modern geometry.* Geometries could now be completely classified and, to quote Arthur Caley: " ... projective geometry is all geometry."

Are these strange geometries ever used? Yes — geometry on the surface of a sphere is elliptic, and Albert Einstein could never have developed his theory of relativity without the concepts of non-Euclidean geometry (his geometry is far more complicated than merely elliptic or hyperbolic). Projective geometry governs the world in which computer graphics are created. Also, one can have a lot of fun with all kinds of geometries — let's play a little!

3 Fun With Geometry

Suppose that we have a container full of skipping ropes and tenniquoit rings, and the game is that you have to use the ropes and rings to complete a construction that satisfies the following rules:

1. Begin by taking one rope.
2. You must fasten exactly three rings to every rope.
3. When you are holding the three rings attached to a rope, there must be at least one ring not attached to that rope.
4. When you hold any two rings, there must be exactly one rope attached to both of them.
5. When you select any two ropes, there must be at least one ring attached to both these ropes.

How many rings and ropes did you use?

A colleague played this game with grade 5/6 children. They enjoyed it tremendously, found it not too difficult, and were blissfully ignorant of the fact that they had been constructing a model of the smallest finite projective geometry (or Fano geometry). They could even draw some conclusions (prove theorems) such as that the "at least" in the last rule could be changed to "exactly", since otherwise the fourth rule is violated. With a little help they also discovered that, if the words ring and rope are interchanged in all the rules, the resulting construction is identical to the previous one. Let us now rewrite the rules of the game as they would apply to a third year class in modern geometry (remember that to be "incident with" is just an impartial way of saying to "go through" or to "lie on"):

1. There exists a line.

2. Each line is incident with exactly three points.
3. Not all points are incident with one line.
4. Two distinct points are incident with exactly one line.
5. Two distinct lines are incident with at least one point.

How many points and lines are there?

The normal student reaction is something like: "Of course there exists a line! This must be a joke — it is obviously rubbish to insist that there can only be three points on a line." Just because we **think** we know what is meant by point and line, all of a sudden we are no longer playing a game, but have become involved in this terribly difficult geometry stuff.

Let us mention, just for fun, a few of the theorems that can be proved for this tiny geometry. There are exactly seven points and seven lines. The words "point" and "line" can be interchanged without breaking any rules (we say that the geometry is *dual*), which implies that in proving a theorem, we get another one (its dual) for free. Each point is therefore incident with exactly three lines. And for any triangle (definition: three lines without a common point) there is exactly one point not incident with the triangle. Enough work for a whole double-period class, with only seven points and lines!

The Fano geometry discussed above is not the smallest finite geometry — there is a dual geometry with only three points and three lines, a non-dual one with four points and six lines, and its counterpart with six points and four lines. Can these strange finite geometries ever be useful? Once again the answer is yes: among other things they are used to develop error-correcting codes for computer data.

4 And What Now?

With rationalization playing an ever-increasing role in education at all levels, there is a noticeable decrease in the geometrical content of mathematical curricula. While at school level the traditional Euclidean geometry is still digging in its heels — and there are positive signs that the importance of theorem–proof–application in the development of cognitive abilities is not always discarded when outcomes become the aim — geometry is seldom offered at tertiary level as a full-fledged course, and then almost always as a part of mathematics. In applied mathematics, as well as in computer science, any geometrical content is accidental. Even when there is a geometry course in the mathematics curriculum it will not, in general, be offered during the first two years, which implies that students of computer graphics will not be exposed to projective geometry, since few of them do more than one year of mathematics. Conversely, students of modern geometry are seldom aware of applications in graphics, and not able to move with ease between theory and practice.

An additional problem is that students find it extremely difficult, even at third year level, to successfully complete a geometry course. Why? It is generally accepted among lecturers in mathematics that students dislike "word problems", and modern geometry uses more words than most of the other branches of mathematics. In the Dec. 1999 issue of the *S.A. Tydskrif vir Natuurwetenskappe en Tegnologie* **18**, Prof. T. Erasmus, Vice Rector, University of Pretoria, states that approximately only 30 % of those students with matriculation exemption possess sufficient linguistic skills to follow a lecture at first year level, and to prepare a written assignment.

Perhaps we can add that only these students will probably be able to pass a comprehension test based on their prescribed textbooks. It is therefore not strange that geometry with its many words is perceived by students to be "too difficult". Should we not rather encourage our students to see it as an opportunity to improve their logical and linguistic skills, to the benefit of any other courses they may do, while at the same time becoming used to the strict discipline of abstract thought processes. Then they won't talk, as one of my students did a few years ago, of "two points that are the same distance from each other" — and he meant ordinary points, not new esoterically-defined points. One of the best teachers of logical thought is also the oldest: we have to learn, and teach our students, how to measure the earth, so that we won't be measured and found wanting.